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SOME CONGRUENCES INVOLVING BINOMIAL COEFFICIENTS

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Abstract. Binomial coefficients and central trinomial coefficients play important roles in combinatorics. Let $p > 3$ be a prime. We show that

$$T_{p-1} \equiv \left(\frac{p}{3}\right) 3^{p-1} \pmod{p^2},$$

where the central trinomial coefficient T_n is the constant term in the expansion of $(1 + x + x^{-1})^n$. We also prove three congruences modulo p^3 conjectured by Sun, one of which is

$$\sum_{k=0}^{p-1} \binom{p-1}{k}\binom{2k}{k}((-1)^k - (-3)^{-k}) \equiv \left(\frac{p}{3}\right)(3^{p-1} - 1) \pmod{p^3}.$$

In addition, we get some new combinatorial identities.

1. Introduction

Throughout this paper, we set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

Let $A, B \in \mathbb{Z}$. The Lucas sequences $u_n = u_n(A, B)$ $(n \in \mathbb{N})$ and $v_n = v_n(A, B)$ $(n \in \mathbb{N})$ are defined by

$$u_0 = 0,\ u_1 = 1,\ \text{and } u_{n+1} = Au_n - Bu_{n-1}\ (n \in \mathbb{Z}^+)$$

and

$$v_0 = 2,\ v_1 = A,\ \text{and } v_{n+1} = Av_n - Bv_{n-1}\ (n \in \mathbb{Z}^+).$$

The roots of the characteristic equation $x^2 - Ax + B = 0$ are

$$\alpha = \frac{A + \sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A - \sqrt{\Delta}}{2},$$

where $\Delta = A^2 - 4B$. By induction, one can easily deduce the following known formulae:

$$(\alpha - \beta)u_n = \alpha^n - \beta^n \quad \text{and} \quad v_n = \alpha^n + \beta^n \quad \text{for any } n \in \mathbb{N}.$$

(Note that in the case $\Delta = 0$ we have $v_n = 2(A/2)^n$ for all $n \in \mathbb{N}$.) It is well-known that

$$u_p \equiv \left(\frac{\Delta}{p}\right) \pmod{p} \quad \text{and} \quad u_{p-(\frac{\Delta}{p})} \equiv 0 \pmod{p} \tag{1.1}$$

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for any odd prime p not dividing B (see, e.g., Sun [3]), where $(-)$ denotes the Legendre symbol.

Let $p > 3$ be a prime and let m be an integer not divisible by p. Recently, Sun [3, 4] established the following general congruences involving central binomial coefficients and Lucas sequences:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{\Delta}{p}\right) + u_{p-(\frac{\Delta}{p})}(m-2,1) \pmod{p^2} \tag{1.2}$$

and

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-m)^k} \equiv \left(\frac{\Delta}{p}\right)(m-4)^{p-1}+\left(1-\frac{m}{2}\right)u_{p-(\frac{\Delta}{p})}(m-2,1) \pmod{p^2}, \tag{1.3}$$

where $\Delta = m^2-4m$. Clearly $\binom{p-1}{k} \equiv (-1)^k \pmod p$ for all $k = 0,\ldots,p-1$.

Note that for each $n = 0,1,2,\ldots$ the central binomial coefficient $\binom{2n}{n}$ is the constant term of $(1+x)^{2n}/x^n = (2+x+x^{-1})^n$. For $n \in \mathbb{N}$, the *central trinomial coefficient* T_n is the constant term in the expansion of $(1+x+x^{-1})^n$, i.e.,

$$T_n = \sum_{k=0}^{\lfloor n/2\rfloor}\frac{n!}{k!k!(n-2k)!} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}.$$

Central trinomial coefficients arise naturally in enumerative combinatorics (cf. Sloane [2]), e.g., T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$. As Andrews [1] pointed out, central trinomial coefficients were first studied by L. Euler. Recently, Sun [6] investigated congruence properties of central trinomial coefficients; for example, he proved that $\sum_{k=0}^{p-1}T_k^2 \equiv (\frac{-1}{p}) \pmod p$ for any odd prime p.

Now we state our first theorem.

Theorem 1.1. *Let $p > 3$ be a prime.*

(i) *We have*

$$T_{p-1} \equiv \left(\frac{p}{3}\right)3^{p-1} \pmod{p^2} \tag{1.4}$$

and

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}((-1)^k-(-3)^{-k}) \equiv \left(\frac{p}{3}\right)(3^{p-1}-1) \pmod{p^3}. \tag{1.5}$$

(ii) *If $p \equiv \pm 1 \pmod{12}$, then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}(-1)^k u_k(4,1) \equiv (-1)^{(p-1)/2}u_{p-1}(4,1) \pmod{p^3}. \tag{1.6}$$

If $p \equiv \pm 1 \pmod 8$, *then*

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k} \frac{u_k(4,2)}{(-2)^k} \equiv (-1)^{(p-1)/2} u_{p-1}(4,2) \pmod{p^3}. \tag{1.7}$$

Remark 1.1. (1.5) and part (ii) of Theorem 1.1 were conjectured by Sun [5, Conj. 1.3].

During our efforts to prove Theorem 1.1, we also obtain some combinatorial identities.

Theorem 1.2. *Let* n *be a positive integer.*

(i) *If* $6 \mid n$, *then*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \frac{\left(\frac{k}{3}\right)}{4^k} = 0. \tag{1.8}$$

If $n \equiv 3 \pmod 6$, *then*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \frac{3[3|k] - 1}{4^k} = 0, \tag{1.9}$$

where $[3 \mid k]$ *is* 1 *or* 0 *according as* $3 \mid k$ *or not.*

(ii) *If* $4 \mid n$, *then*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \frac{u_k(2,2)}{(-4)^k} = 0. \tag{1.10}$$

If $n \equiv 2 \pmod 4$, *then*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \frac{v_k(2,2)}{(-4)^k} = 0. \tag{1.11}$$

(iii) *If* $3 \mid n$, *then*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \frac{u_k(3,3)}{(-4)^k} = 0. \tag{1.12}$$

We will provide two lemmas in the next section and prove Theorems 1.1 and 1.2 in Section 3.

2. Two Lemmas

Lemma 2.1. *Let* $A \in \mathbb{Z}^+$ *and* $B, m \in \mathbb{Z} \setminus \{0\}$ *with* $\Delta = A^2 - 4B \neq 0$. *Let* $\alpha = (A + \sqrt{\Delta})/2$ *and* $\beta = (A - \sqrt{\Delta})/2$. *Then, for every* $n \in \mathbb{N}$ *we have*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \frac{u_k(A,B)}{m^k} = \frac{d^{n/2}(\alpha^n - (-\beta)^n)}{m^n(\alpha - \beta)} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k} d^{-k} \tag{2.1}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \frac{v_k(A,B)}{m^k} = \frac{d^{n/2}(\alpha^n + (-\beta)^n)}{m^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k} d^{-k}, \tag{2.2}$$

where $m = -4B/A$ *and* $d = 4\Delta/A^2$.

Proof. For a polynomial $P(x)$ over the field of complex numbers, we use $[x^n]P(x)$ to denote the coefficient of x^n in $P(x)$. It's easy to see that

$$\begin{aligned}[x^n]((1+\alpha x)^2+mx)^n &= [x^n]\sum_{k=0}^n\binom{n}{k}(1+\alpha x)^{2k}(mx)^{n-k}\\ &= m^n\sum_{k=0}^n\binom{n}{k}\binom{2k}{k}\frac{\alpha^k}{m^k}.\end{aligned}$$

On the other hand,

$$\begin{aligned}[x^n]((1+\alpha x)^2+mx)^n &= [x^n](\alpha^2x^2+(2\alpha+m)x+1)^n\\ &= [x^n]\sum_{\substack{r,s,t\ge0\\ r+s+t=n}}\binom{n}{r,\,s,\,t}\alpha^{2r}(2\alpha+m)^s x^{2r+s}\\ &= \alpha^n\sum_{\substack{r,s\ge0\\ 2r+s=n}}\binom{n}{r,\,s,\,r}\left(2+\frac{m}{\alpha}\right)^s\\ &= \alpha^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\left(2+\frac{m}{\alpha}\right)^{n-2k}.\end{aligned}$$

So we obtain

$$m^n\sum_{k=0}^n\binom{n}{k}\binom{2k}{k}\frac{\alpha^k}{m^k}=\alpha^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\left(2+\frac{m}{\alpha}\right)^{n-2k}. \tag{2.3}$$

Similarly,

$$m^n\sum_{k=0}^n\binom{n}{k}\binom{2k}{k}\frac{\beta^k}{m^k}=\beta^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\left(2+\frac{m}{\beta}\right)^{n-2k}. \tag{2.4}$$

As $4B=-mA$, we see that

$$2+\frac{2m}{A\pm\sqrt{\Delta}}=2+\frac{2m(A\mp\sqrt{\Delta})}{4B}=\pm\frac{2m}{mA}\sqrt{A^2+mA}=\pm\sqrt{d},$$

i.e., $2+m/\alpha=\sqrt{d}$ and $2+m/\beta=-\sqrt{d}$. Since $u_k=(\alpha^k-\beta^k)/(\alpha-\beta)$ and $v_k=\alpha^k+\beta^k$ for all $k\in\mathbb{N}$, combining (2.3) and (2.4) we get (2.1) and (2.2) immediately. □

Lemma 2.2. *Let $p>3$ be a prime, and let $d\in\mathbb{Z}$ with $p\nmid d$. Then*

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}d^{-k}\\ &\equiv\left(\frac{D}{p}\right)\left(\frac{1-d^{p-1}}{2}+(d-4)^{p-1}\right)-\frac{d}{4}u_{p-(\frac{D}{p})}(d-2,1)\pmod{p^2},\end{aligned}\tag{2.5}$$

where $D=d(d-4)$.

Proof. For every $k=0,1,\ldots,p-1$, we clearly have

$$\binom{p-1}{k}=(-1)^k\prod_{0<j\leqslant k}\left(1-\frac{p}{j}\right)\equiv(-1)^k(1-pH_k)\ (\mathrm{mod}\ p^2),\tag{2.6}$$

where H_k denotes the harmonic number $\sum_{0<j\leqslant k}1/j$. Thus

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}d^{-k}\\ \equiv&\sum_{k=0}^{(p-1)/2}(-1)^k(1-pH_k)\binom{p-1-k}{k}d^{-k}\\ =&\sum_{k=0}^{(p-1)/2}\binom{p-1-k}{k}(-d)^{-k}-p\sum_{k=0}^{(p-1)/2}H_k\binom{p-1-k}{k}(-d)^{-k}\ (\mathrm{mod}\ p^2).\end{aligned}$$

Since $\binom{p-1-k}{k}\equiv\binom{-1-k}{k}=(-1)^k\binom{2k}{k}\ (\mathrm{mod}\ p)$ for all $k=0,\ldots,p-1$, we obtain from the above

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}d^{-k}\\ \equiv&\sum_{k=0}^{(p-1)/2}\binom{p-1-k}{k}(-d)^{-k}-p\sum_{k=0}^{(p-1)/2}H_k\binom{2k}{k}d^{-k}\ (\mathrm{mod}\ p^2).\end{aligned}\tag{2.7}$$

It is known that

$$u_{n+1}(A,B)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}A^{n-2k}(-B)^k\quad\text{for all}\quad n=0,1,2,\ldots$$

which can be easily proved by induction. So we have

$$u_p(d,d)=\sum_{k=0}^{(p-1)/2}\binom{p-1-k}{k}d^{p-1-2k}(-d)^k=d^{p-1}\sum_{k=0}^{(p-1)/2}\binom{p-1-k}{k}(-d)^{-k}.$$

By [3, Lemma 2.4],

$$2u_p(d,d)-\left(\frac{D}{p}\right)d^{p-1}\equiv u_p(d-2,1)+u_{p-(\frac{D}{p})}(d-2,1)\ (\mathrm{mod}\ p^2).$$

In view of [4, (3.6)], if $p\nmid d-4$ then

$$u_p(d-2,1)-\left(\frac{D}{p}\right)\equiv\left(\frac{d}{2}-1\right)u_{p-(\frac{D}{p})}(d-2,1)\ (\mathrm{mod}\ p^2).$$

This also holds when $p\mid d-4$, since $(\frac{D}{p})=0$ and

$$u_p(d-2,1)=u_{p-(\frac{D}{p})}(d-2,1)=u_{p-\left(\frac{(d-2)^2-4\cdot 1}{p}\right)}(d-2,1)\equiv 0\ \ (\mathrm{mod}\ p)$$

by (1.1). Combining the above two congruences we immediately get

$$u_p(d,d)\equiv\left(\frac{D}{p}\right)\frac{d^{p-1}+1}{2}+\frac{d}{4}u_{p-(\frac{D}{p})}(d-2,1)\ \ (\mathrm{mod}\ p^2).$$

Hence

$$\sum_{k=0}^{(p-1)/2}\binom{p-1-k}{k}(-d)^{-k}\equiv\left(\frac{D}{p}\right)\frac{d^{p-1}+1}{2d^{p-1}}+\frac{d}{4}u_{p-(\frac{D}{p})}(d-2,1)\ (\mathrm{mod}\ p^2)\tag{2.8}$$

since $u_{p-(\frac{D}{p})}(d-2,1)\equiv 0\ (\mathrm{mod}\ p)$ and $d^{p-1}\equiv 1\ (\mathrm{mod}\ p)$.

Note that $p\mid\binom{2k}{k}$ for $k=(p+1)/2,\ldots,p-1$. With the help of (2.6), we have

$$\begin{aligned}
&p\sum_{k=0}^{(p-1)/2}H_k\binom{2k}{k}d^{-k}\\
\equiv&\sum_{k=0}^{(p-1)/2}\left(1-(-1)^k\binom{p-1}{k}\right)\binom{2k}{k}d^{-k}\\
=&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{d^k}-\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-d)^k}\\
=&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{d^k}+\sum_{k=(p+1)/2}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-d)^k}-\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-d)^k}\\
\equiv&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{d^k}-\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-d)^k}\pmod{p^2}.
\end{aligned}$$

Thus, by applying (1.2) and (1.3) with $m=d$ we find that $p\sum_{k=0}^{(p-1)/2}H_k\binom{2k}{k}d^{-k}$ is congruent to

$$\left(\frac{D}{p}\right)+u_{p-(\frac{D}{p})}(d-2,1)-\left(1-\frac{d}{2}\right)u_{p-(\frac{D}{p})}(d-2,1)-\left(\frac{D}{p}\right)(d-4)^{p-1}$$

modulo p^2. Thus

$$p\sum_{k=0}^{(p-1)/2}H_k\binom{2k}{k}d^{-k}\equiv\left(\frac{D}{p}\right)(1-(d-4)^{p-1})+\frac{d}{2}u_{p-(\frac{D}{p})}(d-2,1)\pmod{p^2}.\tag{2.9}$$

Combining (2.7), (2.8) and (2.9), we finally obtain

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}d^{-k}\\
\equiv&\left(\frac{D}{p}\right)\left(\frac{1-d^{p-1}}{2d^{p-1}}+(d-4)^{p-1}\right)-\frac{d}{4}u_{p-(\frac{D}{p})}(d-2,1)\\
\equiv&\left(\frac{D}{p}\right)\left(\frac{1-d^{p-1}}{2}+(d-4)^{p-1}\right)-\frac{d}{4}u_{p-(\frac{D}{p})}(d-2,1)\ (\mathrm{mod}\ p^2).
\end{aligned}$$

This concludes the proof. □

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1(i). Let ω be the primitive cubic root $(-1+\sqrt{-3})/2$. For each $k=0,1,2,\ldots$, we clearly have

$$u_{3k}(-1,1)=u_{3k}(\omega+\bar\omega,\omega\bar\omega)=\frac{\omega^{3k}-\bar\omega^{3k}}{\omega-\bar\omega}=0.$$

As

$$T_{p-1}=\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k},$$

applying (2.5) with $d=1$ we get

$$T_{p-1}\equiv\left(\frac{-3}{p}\right)(-3)^{p-1}-\frac14 u_{p-(\frac{-3}{p})}(-1,1)=\left(\frac{p}{3}\right)3^{p-1}\pmod{p^2}.$$

This prove (1.4).

Note that $u_k(4,3)=(3^k-1)/(3-1)$ for all $k\in\mathbb{N}$. With the help of Lemma 2.1 and (1.4), we have

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}\frac{u_k(4,3)}{(-3)^k}\\
=&\frac{3^{p-1}-(-1)^{p-1}}{(3-1)(-3)^{p-1}}\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}=\frac{3^{p-1}-1}{2\times 3^{p-1}}T_{p-1}\\
\equiv&\frac{3^{p-1}-1}{2\times 3^{p-1}}\left(\frac{p}{3}\right)3^{p-1}\pmod{p^3}\end{aligned}$$

and hence the desired (1.5) follows. □

Proof of Theorem 1.1(ii). Suppose that $p\equiv\pm1\pmod{12}$. In light of the second congruence in (1.1),

$$u_{p-1}(4,1)=u_{p-(\frac{4^2-4\cdot1}{p})}(4,1)\equiv0\pmod p.$$

By Lemma 2.2,

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}3^{-k}\\
\equiv&\left(\frac{-3}{p}\right)\left(\frac{1-3^{p-1}}{2}+(-1)^{p-1}\right)-\frac34u_{p-(\frac{-3}{p})}(1,1)\equiv\left(\frac{p}{3}\right)\frac{3-3^{p-1}}{2}\pmod{p^2}\end{aligned}$$

since

$$u_{3k}(1,1)=\frac{(-\omega)^{3k}-(-\bar\omega)^{3k}}{-\omega-(-\bar\omega)}=0\quad\text{for all }k\in\mathbb{N}.$$

Combining this with Lemma 2.1 we get

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}(-1)^k u_k(4,1)\\ =&\frac{3^{(p-1)/2}}{(-1)^{p-1}}u_{p-1}(4,1)\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}3^{-k}\\ \equiv&3^{(p-1)/2}u_{p-1}(4,1)\left(\frac{p}{3}\right)\frac{3-3^{p-1}}{2}\pmod{p^3}.\end{aligned}$$

Note that $3^{p-1}\equiv 2\cdot 3^{(p-1)/2}-1\pmod{p^2}$ since $3^{(p-1)/2}\equiv(\frac{3}{p})=1\pmod p$. So we have

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}(-1)^k u_k(4,1)\equiv&3^{(p-1)/2}\left(\frac{-3}{p}\right)\frac{3-3^{p-1}}{2}u_{p-1}(4,1)\\ \equiv&(-1)^{(p-1)/2}3^{(p-1)/2}(2-3^{(p-1)/2})u_{p-1}(4,1)\\ \equiv&(-1)^{(p-1)/2}u_{p-1}(4,1)\pmod{p^3}.\end{aligned}$$

This proves (1.6).

Now assume that $p\equiv\pm1\pmod 8$. In view of the second congruence in (1.1),

$$u_{p-1}(4,2)=u_{p-(\frac{4^2-4\cdot2}{p})}(4,2)\equiv 0\pmod p.$$

By Lemma 2.2,

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}2^{-k}\\ \equiv&\left(\frac{-4}{p}\right)\left(\frac{1-2^{p-1}}{2}+(-2)^{p-1}\right)-\frac{2}{4}u_{p-(\frac{-4}{p})}(0,1)=\left(\frac{-1}{p}\right)\frac{1+2^{p-1}}{2}\pmod{p^2}\end{aligned}$$

since $u_{2k}(0,1)=0$ for all $k\in\mathbb{N}$. Combining this with Lemma 2.1 we get

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}\frac{u_k(4,2)}{(-2)^k}=&\frac{2^{(p-1)/2}}{(-2)^{p-1}}u_{p-1}(4,2)\sum_{k=0}^{(p-1)/2}\binom{p-1}{k}\binom{p-1-k}{k}2^{-k}\\ \equiv&\frac{u_{p-1}(4,2)}{2^{(p-1)/2}}\left(\frac{-1}{p}\right)\frac{1+2^{p-1}}{2}\pmod{p^3}.\end{aligned}$$

This is equivalent to (1.7) since $2^{p-1}+1-2\cdot 2^{(p-1)/2}=(2^{(p-1)/2}-1)^2\equiv 0\pmod{p^2}$.

In view of the above, we have completed the proof of Theorem 1.1(ii). □

Proof of Theorem 1.2. (i) As $-\omega-\bar\omega=1$ and $(-\omega)(-\bar\omega)=1$, for any $k\in\mathbb{Z}$ we have

$$u_k(1,1)=\frac{(-\omega)^k-(-\bar\omega)^k}{-\omega-(-\bar\omega)}=(-1)^{k-1}\left(\frac{k}{3}\right)$$

and

$$v_k(1,1)=(-\omega)^k+(-\bar\omega)^k=(-1)^k(3[3|k]-1).$$

If $6 \mid n$, then $(-\omega)^n = 1 = \bar{\omega}^n$ and hence by (2.1) we have

$$\sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\frac{u_k(1,1)}{(-4)^k} = 0,$$

which is equivalent to (1.8). If $n \equiv 3 \pmod 6$, then $(-\omega)^n = -1 = -\bar{\omega}^n$ and hence by (2.2) we have

$$\sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\frac{v_k(1,1)}{(-4)^k} = 0,$$

which is equivalent to (1.9).

(ii) Clearly $(1+i)+(1-i) = (1+i)(1-i) = 2$. When n is even,

$$(1+i)^n = i^n(1-i)^n = (-1)^{n/2}(1-i)^n = \begin{cases} (i-1)^n & \text{if } 4 \mid n, \\ -(i-1)^n & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

So we get the desired result in Theorem 1.2(ii) by applying Lemma 2.1.

(iii) Let $\alpha = (3+\sqrt{-3})/2$ and $\beta = (3-\sqrt{-3})/2$. Then $\alpha+\beta = \alpha\beta = 3$. Observe that

$$\alpha^2 - \alpha\beta + \beta^2 = (\alpha+\beta)^2 - 3\alpha\beta = 0$$

and hence $\alpha^3 = (-\beta)^3$. If $3 \mid n$, then $\alpha^n = (-\beta)^n$ and hence (1.12) holds by (2.1).

In view of the above, we have finished the proof of Theorem 1.2. □

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