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PROOF OF A CONJECTURAL SUPERCONGRUENCE

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Abstract. Let $m > 2$ and $q > 0$ be integers with m even or q odd. We show the supercongruence

$$\sum_{k=0}^{p-1}(-1)^{km}\binom{p/m-q}{k}^m \equiv 0 \pmod{p^3}$$

for any prime $p > mq$. This confirms a conjecture of Sun.

1. Introduction

Let p be a prime. A p-adic congruence is called a *supercongruence* if it happens to hold modulo higher powers of p. A classical result due to J. Wolstenholme (cf. [7] or [2]) states that if $p > 3$ then

$$\sum_{k=1}^{p-1}\frac{1}{k} \equiv 0 \pmod{p^2} \quad \text{and} \quad \binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

For some recent supercongruences modulo p^2 involving products of two binomial coefficients, one may consult [4] and [1].

Recently, Z.-W. Sun [5] established some new supercongruences modulo prime powers motivated by the well-known formula

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

For example, he obtained the following theorem.

Theorem 1.1. *For any prime $p > 3$, we have*

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1} \equiv 0 \pmod{p^5}$$

and

$$\sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1} \equiv \frac{2}{3}p^4 B_{p-3} \pmod{p^5},$$

where $B_0, B_1, B_2, \ldots$ are the well-known Bernoulli numbers.

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In this paper we show the following result conjectured by Sun [5].

Theorem 1.2. *Let $m > 2$ and $q > 0$ be integers with m even or q odd. Then, for any prime $p > mq$, we have the supercongruence*

$$\sum_{k=0}^{p-1}(-1)^{km}\binom{p/m-q}{k}^m \equiv 0 \pmod{p^3}. \tag{1.1}$$

We are going to provide few lemmas in the next section, and then show Theorem 1.2 in Section 3.

2. Some lemmas

Lemma 2.1. *For any positive integer n, we have*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^m = 0 \quad \text{for all } m = 0, 1, \ldots, n-1.$$

Remark 2.1. This is a well-known result, see, e.g., [6, pp. 125-126].

Lemma 2.2. *Let q be a positive integer, and let $p > 2q$ be a prime. Then, for any integer k with $q-1 \leqslant k \leqslant p-1$ we have*

$$\sum_{q\leqslant j\leqslant k}\frac{1}{j^2}+\sum_{q\leqslant j<p+q-1-k}\frac{1}{j^2} \equiv \sum_{j=q}^{p-q}\frac{1}{j^2}+\sum_{0\leqslant l<q-1}\frac{1}{(k-l)^2} \pmod{p}. \tag{2.1}$$

Proof. Observe that

$$\begin{aligned}\sum_{q\leqslant j<p+q-1-k}\frac{1}{j^2} &\equiv \sum_{q\leqslant j<p+q-1-k}\frac{1}{(p-j)^2} = \sum_{k+1-q<i\leqslant p-q}\frac{1}{i^2}\\ &= \sum_{0<i<q}\frac{1}{i^2}+\sum_{i=q}^{p-q}\frac{1}{i^2}-\sum_{0<i\leqslant k+1-q}\frac{1}{i^2} \pmod{p}\end{aligned}$$

and

$$\sum_{q\leqslant j\leqslant k}\frac{1}{j^2}+\sum_{0<i<q}\frac{1}{i^2}-\sum_{0<i\leqslant k+1-q}\frac{1}{i^2}=\sum_{k+1-q<j\leqslant k}\frac{1}{j^2}=\sum_{0\leqslant l<q-1}\frac{1}{(k-l)^2}.$$

So the desired (2.1) follows. □

Lemma 2.3. *Let $m > 2$ and $q > 0$ be integers. Then, for any prime $p > mq$ we have*

$$\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m \equiv 0 \pmod{p} \tag{2.2}$$

and

$$\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m\sum_{0\leqslant l<q-1}\frac{1}{(k-l)^2}\equiv 0 \pmod{p}. \tag{2.3}$$

Proof. As $m(q-1)\leqslant mq-1\leqslant p-2$, we can write

$$\binom{x}{q-1}^m=\sum_{j=0}^{p-2}a_jx^j$$

with $a_0,\ldots,a_{p-2}$ p-adic integers. It is well-known that $\sum_{k=0}^{p-1}k^j\equiv 0 \pmod{p}$ for any positive integer $j\not\equiv 0 \pmod{p-1}$(see, e.g., [3, p. 235]). Thus

$$\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m=\sum_{k=0}^{p-1}\sum_{j=0}^{p-2}a_jk^j=\sum_{j=0}^{p-2}a_j\sum_{k=0}^{p-1}k^j\equiv 0 \pmod{p}.$$

Let l be any integer with $0\leqslant l<q-1$. Then

$$\begin{aligned}&\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m\frac{1}{(k-l)^2}\\ &\equiv\sum_{k=l+1}^{p+l-1}\binom{k}{q-1}^m\frac{1}{(k-l)^2}=\sum_{k=1}^{p-1}\binom{k+l}{q-1}^m\frac{1}{k^2} \pmod{p}.\end{aligned}$$

As $m(q-1)<mq\leqslant p-1$ and $\binom{l}{q-1}=0$, we may write

$$\binom{x+l}{q-1}^m=\sum_{m\leqslant j<p-1}c_jx^j$$

with $c_m,\ldots,c_{p-2}$ p-adic integers. Hence

$$\begin{aligned}\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m\frac{1}{(k-l)^2}&\equiv\sum_{k=0}^{p-1}\sum_{m\leqslant j<p-1}c_jk^{j-2}\\ &\equiv\sum_{m\leqslant j<p-1}c_j\sum_{k=0}^{p-1}k^{j-2}\equiv 0 \pmod{p}.\end{aligned}$$

Now it follows from the above that

$$\begin{aligned}&\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m\sum_{0\leqslant l<q-1}\frac{1}{(k-l)^2}\\ &=\sum_{0\leqslant l<q-1}\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m\frac{1}{(k-l)^2}\equiv 0 \pmod{p}.\end{aligned}$$

This concludes the proof. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. For each $k \in \{1, 2, \cdots, p-1\}$, obviously

$$(-1)^{km}\binom{p/m-q}{k}^m = (-1)^{km}\prod_{j=1}^{k}\left(\frac{p/m-q-j+1}{j}\right)^m = \prod_{j=1}^{k}\left(1+\frac{q-1}{j}-\frac{p}{jm}\right)^m$$

is congruent to

$$\begin{aligned}&\prod_{j=1}^{k}\left(\left(1+\frac{q-1}{j}\right)^m-\left(1+\frac{q-1}{j}\right)^{m-1}\frac{pm}{jm}+\frac{m(m-1)}{2}\left(1+\frac{q-1}{j}\right)^{m-2}\frac{p^2}{j^2m^2}\right)\\ =&\prod_{j=1}^{k}\left(\left(1+\frac{q-1}{j}\right)^m-\left(1+\frac{q-1}{j}\right)^{m-1}\frac{p}{j}+\frac{m-1}{2m}\left(1+\frac{q-1}{j}\right)^{m-2}\frac{p^2}{j^2}\right)\end{aligned}$$

module p^3, and hence

$$\begin{aligned}&(-1)^{km}\binom{p/m-q}{k}^m-\prod_{j=1}^{k}\left(\left(1+\frac{q-1}{j}\right)^m-\left(1+\frac{q-1}{j}\right)^{m-1}\frac{p}{j}\right)\\ \equiv&\frac{m-1}{2m}\sum_{j=1}^{k}\left(1+\frac{q-1}{j}\right)^{m-2}\frac{p^2}{j^2}\prod_{\substack{i=1\\ i\neq j}}^{k}\left(1+\frac{q-1}{i}\right)^m\\ =&p^2\frac{m-1}{2m}\sum_{j=1}^{k}\frac{1}{(j+q-1)^2}\prod_{i=1}^{k}\left(1+\frac{q-1}{i}\right)^m\\ =&p^2\frac{m-1}{2m}\binom{k+q-1}{k}^m\sum_{j=1}^{k}\frac{1}{(j+q-1)^2}\pmod{p^3}.\end{aligned}$$

Note that

$$\begin{aligned}&\prod_{j=1}^{k}\left(\left(1+\frac{q-1}{j}\right)^m-\left(1+\frac{q-1}{j}\right)^{m-1}\frac{p}{j}\right)\\ =&\prod_{j=1}^{k}\left(1+\frac{q-1}{j}\right)^{m-1}\prod_{j=1}^{k}\left(1+\frac{q-1}{j}-\frac{p}{j}\right)\\ =&\binom{k+q-1}{k}^{m-1}(-1)^k\binom{p-q}{k}.\end{aligned}$$

So, from the above we obtain

$$\begin{aligned}
&\sum_{k=0}^{p-1}(-1)^{km}\binom{p/m-q}{k}^m-\sum_{k=0}^{p-1}\binom{k+q-1}{k}^{m-1}(-1)^k\binom{p-q}{k}\\
\equiv&\sum_{k=0}^{p-1}p^2\frac{m-1}{2m}\binom{k+q-1}{q-1}^m\sum_{0<j\leqslant k}\frac{1}{(j+q-1)^2}\\
=&p^2\frac{m-1}{2m}\sum_{k=q-1}^{p+q-2}\binom{k}{q-1}^m\sum_{q\leqslant j\leqslant k}\frac{1}{j^2}\\
\equiv&p^2\frac{m-1}{2m}\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m\sum_{q\leqslant j\leqslant k}\frac{1}{j^2}\pmod{p^3}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
&2\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m\sum_{q\leqslant j\leqslant k}\frac{1}{j^2}\\
=&\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m\sum_{q\leqslant j\leqslant k}\frac{1}{j^2}+\sum_{k=q-1}^{p-1}\binom{p+q-2-k}{q-1}^m\sum_{q\leqslant j<p+q-1-k}\frac{1}{j^2},
\end{aligned}$$

and

$$\binom{p+q-2-k}{q-1}^m\equiv\binom{q-2-k}{q-1}^m=(-1)^{m(q-1)}\binom{k}{q-1}^m=\binom{k}{q-1}^m\pmod p$$

for all $k=q-1,q,\ldots,p-1$. Therefore, with the helps of Lemmas 2.2 and 2.3, we get

$$\begin{aligned}
&2\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m\sum_{q\leqslant j\leqslant k}\frac{1}{j^2}\\
\equiv&\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m\left(\sum_{q\leqslant j\leqslant k}\frac{1}{j^2}+\sum_{q\leqslant j<p+q-1-k}\frac{1}{j^2}\right)\\
\equiv&\sum_{k=q-1}^{p-1}\binom{k}{q-1}^m\left(\sum_{j=q}^{p-q}\frac{1}{j^2}+\sum_{0\leqslant l<q-1}\frac{1}{(k-l)^2}\right)\\
\equiv&0\pmod p.
\end{aligned}$$

As the degree of the polynomial $\binom{x+q-1}{q-1}^{m-1}$ is $(m-1)(q-1)<(m-1)q<p-q$, by Lemma 2.1 we have

$$\sum_{k=0}^{p-1}\binom{k+q-1}{k}^{m-1}(-1)^k\binom{p-q}{k}=\sum_{k=0}^{p-q}\binom{p-q}{k}(-1)^k\binom{k+q-1}{q-1}^{m-1}=0.$$

Therefore the desired (1.1) follows from the above. □

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