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CONGRUENCES INVOLVING $g_n(x)=\sum_{k=0}^n\binom nk^2\binom{2k}kx^k$

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Abstract. Define $g_n(x)=\sum_{k=0}^n\binom nk^2\binom{2k}kx^k$ for $n=0,1,2,\ldots$. Those numbers $g_n=g_n(1)$ are closely related to Apéry numbers and Franel numbers. In this paper we establish some fundamental congruences involving $g_n(x)$. For example, for any prime $p>5$ we have

$$\sum_{k=1}^{p-1}\frac{g_k(-1)}{k}\equiv0\pmod{p^2}\quad\text{and}\quad\sum_{k=1}^{p-1}\frac{g_k(-1)}{k^2}\equiv0\pmod{p}.$$

This is similar to Wolstenholme's classical congruences

$$\sum_{k=1}^{p-1}\frac1k\equiv0\pmod{p^2}\quad\text{and}\quad\sum_{k=1}^{p-1}\frac1{k^2}\equiv0\pmod p$$

for any prime $p>3$.

1. Introduction

It is well known that

$$\sum_{k=0}^n\binom nk^2=\binom{2n}n\ \ (n=0,1,2,\ldots)$$

and central binomial coefficients play important roles in mathematics. A famous theorem of J. Wolstenholme [W] asserts that for any prime $p>3$ we have

$$\frac12\binom{2p}p=\binom{2p-1}{p-1}\equiv1\pmod{p^3},$$

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$$H_{p-1} \equiv 0 \pmod{p^2} \quad \text{and} \quad H_{p-1}^{(2)} \equiv 0 \pmod{p},$$

where

$$H_n := \sum_{0<k\leqslant n} \frac{1}{k} \quad \text{and} \quad H_n^{(2)} := \sum_{0<k\leqslant n} \frac{1}{k^2} \quad \text{for } n \in \mathbb{N} = \{0, 1, 2, \dots\};$$

see also [Zh] for some extensions. The reader may consult [S11a], [S11b], [ST1] and [ST2] for recent work on congruences involving central binomial coefficients.

The Franel numbers given by

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \dots)$$

(cf. [Sl, A000172]) were first introduced by J. Franel in 1895 who noted the recurrence relation:

$$(n+1)^2 f_{n+1} = (7n(n+1)+2) f_n + 8n^2 f_{n-1} \ (n = 1, 2, 3, \dots).$$

In 1992 C. Strehl [St92] showed that the Apéry numbers given by

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n+k}{2k}^2 \binom{2k}{k}^2 \ (n = 0, 1, 2, \dots)$$

(arising from Apéry's proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ (cf. [vP])) can be expressed in terms of Franel numbers, namely,

$$A_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} f_k. \tag{1.1}$$

Define

$$g_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \quad \text{for } n \in \mathbb{N}. \tag{1.2}$$

Such numbers are interesting due to Barrucand's identity ([B])

$$\sum_{k=0}^{n} \binom{n}{k} f_k = g_n \quad (n = 0, 1, 2, \dots). \tag{1.3}$$

For a combinatorial interpretation of such numbers, see D. Callan [C]. The sequences $(f_n)_{n\geqslant 0}$ and $(g_n)_{n\geqslant 0}$ are two of the five sporadic sequences (cf. D. Zagier [Z, Section 4]) which are integral solutions of certain Apéry-like recurrence equations and closely related to the theory of modular forms.

In [S12] and [S13b] the author introduced the Apéry polynomials

$$A_n(x):=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}^2x^k\quad(n=0,1,2,\ldots)$$

and the Franel polynomials

$$f_n(x):=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{n}x^k=\sum_{k=0}^n\binom{n}{k}\binom{k}{n-k}\binom{2k}{k}x^k\ (n=0,1,2,\ldots),$$

and deduced various congruences involving such polynomials. (Note that $A_n(1)=A_n$, and $f_n(1)=f_n$ by [St94].) See also [S13a] for connections between primes $p=x^2+3y^2$ and the Franel numbers. Here we introduce the polynomials

$$g_n(x):=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{k}x^k\quad(n=0,1,2,\ldots).$$

Both $f_n(x)$ and $g_n(x)$ play important roles in some kinds of series for $1/\pi$ (cf. Conjecture 3 and the subsequent remark in [S11]).

In this paper we study various congruences involving $g_n(x)$. As usual, for an odd prime p and an integer a, $(\frac{a}{p})$ denotes the Legendre symbol, and $q_p(a)$ stands for the Fermat quotient $(a^{p-1}-1)/p$ if $p\nmid a$. Also, $B_0,B_1,B_2,\ldots$ are the well-known Bernoulli numbers and $E_0,E_1,E_2,\ldots$ are the Euler numbers.

Now we state our main results.

Theorem 1.1. *Let $p>3$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}g_k(x)(1-p^2H_k^{(2)})\equiv\sum_{k=0}^{p-1}\frac{p}{2k+1}\left(1-2p^2H_k^{(2)}\right)x^k\pmod{p^4}.\tag{1.4}$$

Consequently,

$$\sum_{k=1}^{p-1}g_k\equiv p^2\sum_{k=1}^{p-1}g_kH_k^{(2)}+\frac{7}{6}p^3B_{p-3}\pmod{p^4},\tag{1.5}$$

$$\sum_{k=0}^{p-1}g_k(-1)\equiv\left(\frac{-1}{p}\right)+p^2\left(\sum_{k=0}^{p-1}g_k(-1)H_k^{(2)}-E_{p-3}\right)\pmod{p^3},\tag{1.6}$$

$$\sum_{k=0}^{p-1}g_k(-3)\equiv\left(\frac{p}{3}\right)\pmod{p^2}.\tag{1.7}$$

(ii) *We also have*

$$\sum_{k=1}^{p-1}\frac{g_k(x)}{k}\equiv 0 \pmod{p}, \tag{1.8}$$

$$\sum_{k=1}^{p-1}\frac{g_{k-1}}{k}\equiv -\left(\frac{p}{3}\right)2q_p(3) \pmod{p}, \tag{1.9}$$

$$\sum_{k=1}^{p-1}kg_k\equiv -\frac{3}{4} \pmod{p^2}, \tag{1.10}$$

and moreover

$$\frac{1}{3n^2}\sum_{k=0}^{n-1}(4k+3)g_k=\sum_{k=0}^{n-1}\binom{n-1}{k}^2C_k \tag{1.11}$$

for all $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$, *where* C_k *denotes the Catalan number* $\binom{2k}{k}/(k+1)=\binom{2k}{k}-\binom{2k}{k+1}$.

(iii) *Provided* $p>5$, *we have*

$$\sum_{k=1}^{p-1}\frac{g_k(-1)}{k^2}\equiv 0 \pmod{p}, \tag{1.12}$$

$$\sum_{k=1}^{p-1}\frac{g_k(-1)}{k}\equiv 0 \pmod{p^2}, \tag{1.13}$$

$$\sum_{k=1}^{p-1}\frac{(-1)^kf_k(-1)}{k}H_k\equiv -2\left(\frac{-1}{p}\right)E_{p-3} \pmod{p}. \tag{1.14}$$

Remark 1.1. Let $p>3$ be a prime. By [JV, Lemma 2.7], $g_k\equiv(\frac{p}{3})9^kg_{p-1-k} \pmod{p}$ for all $k=0,\ldots,p-1$. So (1.9) implies that

$$\sum_{k=1}^{p-1}\frac{g_k}{k9^k}\equiv\left(\frac{p}{3}\right)\sum_{k=1}^{p-1}\frac{g_{p-1-k}}{k}=\left(\frac{p}{3}\right)\sum_{k=1}^{p-1}\frac{g_{k-1}}{p-k}\equiv 2q_p(3) \pmod{p}.$$

We conjecture further that

$$\sum_{k=1}^{p-1}\frac{g_{k-1}}{k}\equiv -\left(\frac{p}{3}\right)q_p(9) \pmod{p^2}\quad\text{and}\quad\sum_{k=0}^{p-1}\frac{g_k}{9^k}\equiv\left(\frac{p}{3}\right) \pmod{p^2}.$$

In [S13b] the author showed the following congruences similar to (1.12) and (1.13):

$$\sum_{k=1}^{p-1}\frac{(-1)^kf_k}{k^2}\equiv 0 \pmod{p}\quad\text{and}\quad\sum_{k=1}^{p-1}\frac{(-1)^kf_k}{k}\equiv 0 \pmod{p^2}.$$

Such congruences are interesting in view of Wolstenholme's congruences $H_{p-1}\equiv 0 \pmod{p^2}$ and $H_{p-1}^{(2)}\equiv 0 \pmod p$. Applying the Zeilberger algorithm (cf. [PWZ, pp. 101-119]) via `Mathematica 9` we find the recurrence for $s_n=g_n(-1)$ ($n=0,1,2,\ldots$):

$$\begin{aligned}&(n+3)^2(4n+5)s_{n+3}+(20n^3+125n^2+254n+165)s_{n+2}\\ &\quad+(76n^3+399n^2+678n+375)s_{n+1}-25(n+1)^2(4n+9)s_n=0.\end{aligned}$$

In contrast with (1.11), we are also able to show the congruence

$$\sum_{k=0}^{p-1}(3k+1)\frac{f_k}{8^k}\equiv p^2-2p^3q_p(2)+4p^4q_p(2)^2 \pmod{p^5} \tag{1.15}$$

via the combinatorial identity

$$\frac1{n^2}\sum_{k=0}^{n-1}(3k+1)f_k8^{n-1-k}=\sum_{k=0}^{n-1}\binom{n-1}k^3\left(1-\frac n{k+1}+\frac{n^2}{(k+1)^2}\right) \tag{1.16}$$

which can be shown by the Zeilberger algorithm.

We are going to investigate in the next section connections among the polynomials $A_n(x)$, $f_n(x)$ and $g_n(x)$. Section 3 is devoted to our proof of Theorem 1.1. In Section 4 we shall propose some conjectures for further research.

2. Relations among $A_n(x), f_n(x)$ and $g_n(x)$

Obviously,

$$\frac1n\sum_{k=0}^{n-1}(2k+1)=n\in\mathbb Z\quad\text{and}\quad\frac1n\sum_{k=0}^{n-1}(2k+1)(-1)^k=(-1)^{n-1}\in\mathbb Z$$

for all $n=1,2,3,\ldots$. This is a special case of our following general result.

Theorem 2.1. *Let*

$$X_n=\sum_{k=0}^n\binom nk\binom{n+k}kx_k\quad\textit{and}\quad y_n=\sum_{k=0}^n\binom nk x_k\quad\textit{for all } n\in\mathbb N. \tag{2.1}$$

Then

$$X_n=\sum_{k=0}^n\binom nk\binom{n+k}k(-1)^{n-k}y_k\quad\textit{for every } n\in\mathbb N. \tag{2.2}$$

Also, for any $n \in \mathbb{Z}^+$ we have

$$\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(2k+1)X_k = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(-1)^k y_k \tag{2.3}$$

and

$$\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k X_k = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}x_k. \tag{2.4}$$

Proof. If $n \in \mathbb{N}$, then

$$\begin{aligned}&\sum_{l=0}^{n}\binom{n}{l}\binom{n+l}{l}(-1)^l y_l\\ =&\sum_{l=0}^{n}\binom{n}{l}\binom{-n-1}{l}\sum_{k=0}^{l}\binom{l}{k}x_k\\ =&\sum_{k=0}^{n}\binom{n}{k}x_k\sum_{l=k}^{n}\binom{n-k}{n-l}\binom{-n-1}{l}\\ =&\sum_{k=0}^{n}\binom{n}{k}x_k\binom{-k-1}{n} \quad \text{(by the Chu-Vandermonde identity [G, (2.1)])}\\ =&(-1)^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x_k\end{aligned}$$

and hence (2.2) holds.

For any given integer $k \geqslant 0$, by induction on n we have

$$\sum_{l=k}^{n-1}(-1)^l(2l+1)\binom{l+k}{2k} = (-1)^{n-1}(n-k)\binom{n+k}{2k} \tag{2.5}$$

for all $n = k+1, k+2, \ldots$. Fix a positive integer n. In view of (2.2) and (2.5),

$$\begin{aligned}\sum_{l=0}^{n-1}(2l+1)X_l =&\sum_{l=0}^{n-1}(2l+1)\sum_{k=0}^{l}\binom{l+k}{2k}\binom{2k}{k}(-1)^{l-k}y_k\\ =&\sum_{k=0}^{n-1}\binom{2k}{k}(-1)^k y_k\sum_{l=k}^{n-1}(-1)^l(2l+1)\binom{l+k}{2k}\\ =&\sum_{k=0}^{n-1}\binom{2k}{k}(-1)^k y_k(-1)^{n-1}(n-k)\binom{n+k}{2k}\\ =&(-1)^{n-1}n\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(-1)^k y_k.\end{aligned}$$

This proves (2.3). Similarly,

$$\begin{aligned}\sum_{l=0}^{n-1}(2l+1)(-1)^lX_l=&\sum_{l=0}^{n-1}(2l+1)(-1)^l\sum_{k=0}^{l}\binom{l+k}{2k}\binom{2k}{k}x_k\\=&\sum_{k=0}^{n-1}\binom{2k}{k}x_k\sum_{l=k}^{n-1}(-1)^l(2l+1)\binom{l+k}{2k}\\=&\sum_{k=0}^{n-1}\binom{2k}{k}x_k(-1)^{n-1}(n-k)\binom{n+k}{2k}\\=&(-1)^{n-1}n\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}x_k.\end{aligned}$$

and hence (2.4) is also valid.

Combining the above, we have completed the proof of Theorem 2.1. □

Lemma 2.1. *For any nonnegative integers m and n we have the combinatorial identity*

$$\sum_{k=0}^{n}\binom{m-x+y}{k}\binom{n+x-y}{n-k}\binom{x+k}{m+n}=\binom{x}{m}\binom{y}{n}. \tag{2.6}$$

Remark 2.1. (2.6) is due to Nanjundiah, see, e.g., (4.17) of [G, p. 53].

The author [S12] proved that $\frac1n\sum_{k=0}^{n-1}(2k+1)A_k(x)\in\mathbb{Z}[x]$ for all $n\in\mathbb{Z}^+$, and conjectured that $\frac1n\sum_{k=0}^{n-1}(2k+1)(-1)^kA_k(x)\in\mathbb{Z}[x]$ for any $n\in\mathbb{Z}^+$, which was confirmed by Guo and Zeng [GZ].

Theorem 2.2. *Let n be any nonnegative integer. Then*

$$\sum_{k=0}^{n}\binom{n}{k}f_k(x)=g_n(x),\quad f_n(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}g_k(x), \tag{2.7}$$

and

$$A_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}f_k(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}g_k(x). \tag{2.8}$$

Also, for any $n\in\mathbb{Z}^+$ we have

$$\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(2k+1)A_k(x)=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(-1)^kg_k(x) \tag{2.9}$$

and

$$\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k(x)=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}f_k(x). \tag{2.10}$$

Proof. By the binomial inversion formula (cf. (5.48) of [GKP, p. 192]), the two identities in (2.7) are equivalent. Observe that

$$\begin{aligned}\sum_{l=0}^{n}\binom{n}{l}f_l(x)=&\sum_{l=0}^{n}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}\binom{k}{l-k}\binom{2k}{k}x^k\\=&\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}x^k\sum_{l=k}^{n}\binom{n-k}{n-l}\binom{k}{l-k}\\=&\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}x^k\binom{n}{n-k}=g_n(x)\end{aligned}$$

with the help of the Chu-Vandermonde identity. Thus (2.7) holds.

Next we show (2.8). Clearly

$$\begin{aligned}\sum_{l=0}^{n}\binom{n}{l}\binom{n+l}{l}f_l(x)=&\sum_{l=0}^{n}\binom{n}{l}\binom{n+l}{l}\sum_{k=0}^{l}\binom{l}{k}\binom{k}{l-k}\binom{2k}{k}x^k\\=&\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}x^k\sum_{l=k}^{n}\binom{n-k}{l-k}\binom{k}{l-k}\binom{n+l}{n}\\=&\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}x^k\sum_{j=0}^{k}\binom{n-k}{j}\binom{k}{k-j}\binom{n+k+j}{n}\\=&\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}x^k\binom{n+k}{n-k}\binom{n+k}{k}\quad\text{(by Lemma 2.1).}\end{aligned}$$

This proves the first identity in (2.8). Applying Theorem 2.1 with $x_n=f_n(x)$ and $X_n=A_n(x)$ for $n\in\mathbb{N}$, we get the identity

$$A_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}g_k(x) \tag{2.11}$$

as well as (2.9) and (2.10), with the help of (2.7).

The proof of Theorem 2.2 is now complete. □

Remark 2.2. (2.7) and (2.8) in the case $x=1$ are well known.

Corollary 2.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}A_k(x)\equiv p\sum_{k=0}^{p-1}\frac{(-1)^kf_k(x)}{2k+1}\pmod{p^2}\tag{2.12}$$

and

$$\sum_{k=0}^{p-1}(-1)^kA_k(x)\equiv p\sum_{k=0}^{p-1}\frac{g_k(x)}{2k+1}\pmod{p^2}.\tag{2.13}$$

Proof. In view of (2.8),

$$\begin{aligned}\sum_{l=0}^{p-1}A_l(x)&=\sum_{l=0}^{p-1}\sum_{k=0}^{l}\binom{k+l}{2k}\binom{2k}{k}f_k(x)=\sum_{k=0}^{p-1}\binom{2k}{k}f_k(x)\sum_{l=k}^{p-1}\binom{k+l}{2k}\\&=\sum_{k=0}^{p-1}\binom{2k}{k}f_k(x)\binom{p+k}{2k+1}=\sum_{k=0}^{p-1}\binom{2k}{k}f_k(x)\frac{p}{(2k+1)!}\prod_{0<j\leqslant k}(p^2-j^2)\\&\equiv\sum_{k=0}^{p-1}f_k(x)\frac{p}{2k+1}(-1)^k\pmod{p^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{l=0}^{p-1}(-1)^lA_l(x)&=\sum_{l=0}^{p-1}\sum_{k=0}^{l}\binom{k+l}{2k}\binom{2k}{k}(-1)^kg_k(x)\\&=\sum_{k=0}^{p-1}\binom{2k}{k}(-1)^kg_k(x)\binom{p+k}{2k+1}\\&\equiv\sum_{k=0}^{p-1}g_k(x)\frac{p}{2k+1}\pmod{p^2}.\end{aligned}$$

This concludes the proof of Corollary 2.1. $\square$

Remark 2.3. In [S12] the author investigated $\sum_{k=0}^{p-1}(\pm1)^kA_k(x)\bmod p^2$ (where p is an odd prime) and made some conjectures.

For any $n\in\mathbb{Z}$ we set

$$[n]_q=\frac{1-q^n}{1-q}=\begin{cases}\sum_{0\leqslant k<n}q^k&\text{if }n\geqslant0,\\-q^n\sum_{0\leqslant k<-n}q^k&\text{if }n<0;\end{cases}$$

this is the usual q-analogue of the integer n. Define

$$\begin{bmatrix}n\\0\end{bmatrix}_q=1\quad\text{and}\quad\begin{bmatrix}n\\k\end{bmatrix}_q=\prod_{j=1}^{k}\frac{[n-j+1]_q}{[j]_q}\quad\text{for }k\in\mathbb{Z}^+.$$

Obviously, $\lim_{q\to1}\begin{bmatrix}n\\k\end{bmatrix}_q=\binom nk$.

For $n\in\mathbb{N}$ we define

$$A_n(x;q):=\sum_{k=0}^n q^{2n(n-k)}\begin{bmatrix}n\\k\end{bmatrix}_q^2\begin{bmatrix}n+k\\k\end{bmatrix}_q^2x^k$$

and

$$g_n(x;q):=\sum_{k=0}^n q^{2n(n-k)}\begin{bmatrix}n\\k\end{bmatrix}_q^2\begin{bmatrix}2k\\k\end{bmatrix}_q x^k.$$

Clearly

$$\lim_{q\to1}A_n(x;q)=A_n(x)\quad\text{and}\quad\lim_{q\to1}g_n(x;q)=g_n(x).$$

Those identities in Theorem 2.2 have their q-analogues. For example, the following theorem gives a q-analogue of (2.11).

Theorem 2.3. *Let $n\in\mathbb{N}$. Then we have*

$$A_n(x;q)=\sum_{k=0}^n(-1)^{n-k}q^{(n-k)(5n+3k+1)/2}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n+k\\k\end{bmatrix}_q g_k(x;q). \tag{2.14}$$

Proof. Let $j\in\{0,\ldots,n\}$. By the q-Chu-Vandermonde identity (see, e.g., Ex. 4(b) of [AAR, p. 542]),

$$\sum_{k=j}^n q^{(k-j)^2}\begin{bmatrix}-n-1-j\\k-j\end{bmatrix}_q\begin{bmatrix}n-j\\n-k\end{bmatrix}_q=\begin{bmatrix}-2j-1\\n-j\end{bmatrix}_q.$$

This, together with

$$\begin{bmatrix}-n-1\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q=\begin{bmatrix}-n-1\\j\end{bmatrix}_q\begin{bmatrix}-n-1-j\\k-j\end{bmatrix}_q,$$

yields that

$$\sum_{k=j}^n q^{(k-j)^2}\begin{bmatrix}-n-1\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q\begin{bmatrix}n-j\\k-j\end{bmatrix}_q=\begin{bmatrix}-n-1\\j\end{bmatrix}_q\begin{bmatrix}-2j-1\\n-j\end{bmatrix}_q.$$

It is easy to see that

$$\begin{bmatrix}-m-1\\k\end{bmatrix}_q=(-1)^kq^{-km-k(k+1)/2}\begin{bmatrix}m+k\\k\end{bmatrix}_q.$$

So we are led to the identity

$$\sum_{k=j}^{n}(-1)^{n-k}q^{\binom{n-k+1}{2}+2j(n-k)}\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q\begin{bmatrix}n-j\\k-j\end{bmatrix}_q=\begin{bmatrix}n+j\\j\end{bmatrix}_q\begin{bmatrix}n+j\\2j\end{bmatrix}_q. \tag{2.15}$$

Since

$$\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q=\begin{bmatrix}n\\j\end{bmatrix}_q\begin{bmatrix}n-j\\k-j\end{bmatrix}_q \quad\text{and}\quad \begin{bmatrix}n\\j\end{bmatrix}_q\begin{bmatrix}n+j\\j\end{bmatrix}_q=\begin{bmatrix}n+j\\2j\end{bmatrix}_q\begin{bmatrix}2j\\j\end{bmatrix}_q,$$

multiplying both sides of (2.15) by $\begin{bmatrix}n\\j\end{bmatrix}_q\begin{bmatrix}2j\\j\end{bmatrix}_q x^j$ we get

$$\sum_{k=j}^{n}(-1)^{n-k}q^{\binom{n-k+1}{2}+2j(n-k)}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n+k\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q^2\begin{bmatrix}2j\\j\end{bmatrix}_q x^j=\begin{bmatrix}n\\j\end{bmatrix}_q^2\begin{bmatrix}n+j\\j\end{bmatrix}_q^2 x^j.$$

In view of the last identity we can easily deduce the desired (2.14). □

By applying Theorem 2.2 we obtain the following new result.

Theorem 2.4. *Let n be any positive integer. Then*

$$\sum_{k=0}^{n-1}(-1)^k(6k^3+9k^2+5k+1)A_k\equiv 0 \pmod{n^3}. \tag{2.16}$$

Proof. By induction on n, for each $k=0,\dots,n-1$ we have

$$\sum_{l=k}^{n-1}(-1)^l(6l^3+9l^2+5l+1)\binom{l+k}{2k}=(-1)^{n-1}(n-k)(3n^2-3k-2)\binom{n+k}{2k}.$$

Thus, in view of (2.8),

$$\begin{aligned}&\frac{1}{n}\sum_{l=0}^{n-1}(-1)^{n-l}(6l^3+9l^2+5l+1)A_l(x)\\
=&\frac{(-1)^n}{n}\sum_{l=0}^{n-1}(-1)^l(6l^3+9l^2+5l+1)\sum_{k=0}^{l}\binom{l+k}{2k}\binom{2k}{k}f_k(x)\\
=&\frac{(-1)^n}{n}\sum_{k=0}^{n-1}\binom{2k}{k}f_k(x)\sum_{l=k}^{n-1}(-1)^l(6l^3+9l^2+5l+1)\binom{l+k}{2l}\\
=&\frac{(-1)^n}{n}\sum_{k=0}^{n-1}\binom{2k}{k}f_k(x)(-1)^{n-1}(n-k)(3n^2-3k-2)\binom{n+k}{2k}\\
=&\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(3k+2-3n^2)f_k(x).\end{aligned}$$

Hence we have reduced (2.16) to the congruence

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(3k+2)f_k\equiv 0 \pmod{n^2}. \tag{2.17}$$

The author [S13a, (1.12)] conjectured that

$$a_m:=\frac{1}{m^2}\sum_{k=0}^{m-1}(3k+2)(-1)^kf_k\in\mathbb{Z}\quad\text{for all } m=1,2,3,\ldots,$$

and this was confirmed by V.J.W. Guo [Gu]. Set $a_0=0$. Observe that

$$\begin{aligned}&\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(3k+2)f_k\\
=&\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{-n-1}{k}\left((k+1)^2a_{k+1}-k^2a_k\right)\\
=&\sum_{k=1}^{n}\binom{n-1}{k-1}\binom{-n-1}{k-1}k^2a_k-\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{-n-1}{k}k^2a_k\\
=&\binom{-n-1}{n-1}n^2a_n+\sum_{0<k<n}k^2a_k\left(\binom{n-1}{k-1}\binom{-n-1}{k-1}-\binom{n-1}{k}\binom{-n-1}{k}\right).\end{aligned}$$

As

$$\binom{n-1}{k-1}\binom{-n-1}{k-1}-\binom{n-1}{k}\binom{-n-1}{k}=\frac{n^2}{k^2}\binom{n-1}{k-1}\binom{-n-1}{k-1}$$

for all $k=1,\ldots,n-1$, we have (2.17) by the above, and hence (2.16) holds. $\square$

The author [S12] conjectured that for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}(2k+1)(-1)^kA_k\equiv p\left(\frac{p}{3}\right)\pmod{p^3}, \tag{2.18}$$

and this was confirmed by Guo and Zeng [GZ].

Corollary 2.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(2k+1)^3(-1)^kA_k\equiv -\frac{p}{3}\left(\frac{p}{3}\right)\pmod{p^3}. \tag{2.19}$$

Proof. Clearly
$$3(2k+1)^3=4(6k^3+9k^2+5k+1)-(2k+1).$$
Thus (2.19) follows from (2.16) and (2.18). $\Box$

Remark 2.4. Let $p>3$ be a prime. We are also able to prove that
$$\sum_{k=0}^{p-1}(2k+1)^5(-1)^kA_k\equiv-\frac{13}{27}p\left(\frac{p}{3}\right)\pmod{p^3}\tag{2.20}$$
and
$$\sum_{k=0}^{p-1}(2k+1)^7(-1)^kA_k\equiv\frac{5}{9}p\left(\frac{p}{3}\right)\pmod{p^3}.\tag{2.21}$$
It seems that for each $r=0,1,2,\ldots$ there is a p-adic integer c_r only depending on r such that
$$\sum_{k=0}^{p-1}(2k+1)^{2r+1}(-1)^kA_k\equiv c_rp\left(\frac{p}{3}\right)\pmod{p^3}.$$

3. Proof of Theorem 1.1

Lemma 3.1. *For any odd prime p, we have*
$$\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)A_k(x)\equiv\sum_{k=0}^{p-1}g_k(x)-p^2\sum_{k=0}^{p-1}g_k(x)H_k^{(2)}\pmod{p^4}.\tag{3.1}$$

Proof. Obviously,
$$(-1)^k\binom{p-1}{k}\binom{p+k}{k}=\prod_{0<j\leqslant k}\left(1-\frac{p^2}{j^2}\right)\equiv1-p^2H_k^{(2)}\pmod{p^4}\tag{3.2}$$
for every $k=0,\ldots,p-1$. Thus (3.1) follows from (2.9) with $n=p$. $\Box$

Lemma 3.2. *Let $p>3$ be a prime. Then*
$$g_{p-1}\equiv\left(\frac{p}{3}\right)(1+2p\,q_p(3))\pmod{p^2}.\tag{3.3}$$

Proof. For $k=0,\ldots,p-1$, clearly
$$\binom{p-1}{k}^2=\prod_{0<j\leqslant k}\left(1-\frac{p}{j}\right)^2\equiv\prod_{0<j\leqslant k}\left(1-\frac{2p}{j}\right)=(-1)^k\binom{2p-1}{k}\pmod{p^2}.$$
Thus, with the help of [S12b, Corollary 2.2] we obtain
$$g_{p-1}\equiv\sum_{k=0}^{p-1}\binom{2p-1}{k}(-1)^k\binom{2k}{k}\equiv\left(\frac{p}{3}\right)\left(2\times3^{p-1}-1\right)\pmod{p^2}.$$
and hence (3.3) holds. $\Box$

Lemma 3.3. *For any odd prime p, we have*

$$p\sum_{k=0}^{p-1}\frac{(-3)^k}{2k+1}\equiv\left(\frac{p}{3}\right)\pmod{p^2}. \tag{3.4}$$

Proof. Clearly (3.4) holds for $p=3$. Below we assume $p>3$. Observe that

$$\begin{aligned}\sum_{\substack{k=0\\k\neq(p-1)/2}}^{p-1}\frac{(-3)^k}{2k+1}=&\sum_{k=1}^{(p-1)/2}\left(\frac{(-3)^{(p-1)/2-k}}{2((p-1)/2-k)+1}+\frac{(-3)^{(p-1)/2+k}}{2((p-1)/2+k)+1}\right)\\ \equiv&\left(\frac{-3}{p}\right)\frac{1}{2}\sum_{k=1}^{(p-1)/2}\left(\frac{(-3)^k}{k}-\frac{1}{3}\cdot\frac{(-3)^{p-k}}{p-k}\right)\\ =&\frac{1}{2}\left(\frac{p}{3}\right)\left(\frac{4}{3}\sum_{k=1}^{(p-1)/2}\frac{(-3)^k}{k}-\frac{1}{3}\sum_{k=1}^{p-1}\frac{(-3)^k}{k}\right)\\ =&-2\left(\frac{p}{3}\right)\sum_{k=1}^{(p-1)/2}\frac{(-3)^{k-1}}{k}+\frac{1}{2}\left(\frac{p}{3}\right)\sum_{k=1}^{p-1}\frac{(-3)^{k-1}}{k}\pmod{p}.\end{aligned}$$

Since

$$\frac{1}{p}\binom{p}{k}=\frac{1}{k}\binom{p-1}{k-1}\equiv\frac{(-1)^{k-1}}{k}\pmod{p}\quad\text{for }k=1,\ldots,p-1,$$

we have

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-3)^{k-1}}{k}\equiv&\frac{1}{3p}\sum_{k=1}^{p-1}\binom{p}{k}3^k=\frac{4^p-1-3^p}{3p}=4(2^{p-1}+1)\frac{2^{p-1}-1}{3p}-\frac{3^{p-1}-1}{p}\\ \equiv&\frac{8}{3}q_p(2)-q_p(3)\pmod{p}.\end{aligned}$$

Note also that

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{(-3)^{k-1}}{k}=&\sum_{k=1}^{(p-1)/2}\int_0^1(-3x)^{k-1}dx=\int_0^1\frac{1-(-3x)^{(p-1)/2}}{1+3x}dx\\ =&\int_0^1\sum_{k=1}^{(p-1)/2}\binom{(p-1)/2}{k}(-1-3x)^{k-1}dx\\ =&\sum_{k=1}^{p-1}\binom{(p-1)/2}{k}\frac{(-1-3x)^k}{-3k}\bigg|_{x=0}^{1}\\ \equiv&\sum_{k=1}^{p-1}\binom{-1/2}{k}\frac{(-1)^k-(-4)^k}{3k}=\frac{1}{3}\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k4^k}-\frac{1}{3}\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\\ \equiv&\frac{2}{3}q_p(2)\pmod{p}\end{aligned}$$

since

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k4^k}\equiv 2q_p(2)\pmod p\quad\text{and}\quad\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod{p^2}$$

by [ST1, (1.12) and (1.20)]. Thus, in view of the above, we get

$$\begin{aligned}\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{(-3)^k}{2k+1}\equiv&-2\left(\frac p3\right)\frac23 q_p(2)+\frac12\left(\frac p3\right)\left(\frac83 q_p(2)-q_p(3)\right)\\ =&-\left(\frac p3\right)\frac{q_p(3)}{2}\pmod p.\end{aligned}$$

It follows that

$$\begin{aligned}p\sum_{k=0}^{p-1}\frac{(-3)^k}{2k+1}\equiv&(-3)^{(p-1)/2}-\left(\frac p3\right)\frac{3^{p-1}-1}{2}\\ =&(-3)^{(p-1)/2}-\left(\frac p3\right)\frac{(-3)^{(p-1)/2}+(\frac{-3}{p})}{2}\left((-3)^{(p-1)/2}-\left(\frac{-3}{p}\right)\right)\\ \equiv&(-3)^{(p-1)/2}-\left((-3)^{(p-1)/2}-\left(\frac{-3}{p}\right)\right)=\left(\frac p3\right)\pmod{p^2}.\end{aligned}$$

We are done. □

Lemma 3.4. *For any prime p, we have*

$$k\binom{2k}{k}\sum_{r=0}^{p-1}\binom{-k}{r}\binom{-k-1}{r}\equiv p\pmod{p^2}\quad\text{for all } k=1,\ldots,p-1.\tag{3.5}$$

Proof. Define

$$u_k=\sum_{r=0}^{p-1}\binom{-k}{r}\binom{-k-1}{r}\quad\text{for all } k\in\mathbb{N}.$$

Applying the Zeilberger algorithm via `Mathematica 9`, we find the recurrence

$$\begin{aligned}&k(k+1)^2(2(2k+1)u_{k+1}-ku_k)\\ =&(p+k)(p+k-1)(2kp+p+3k^2+3k+1)\binom{-1-k}{p-1}\binom{-k}{p-1}\\ =&p^2\binom{p+k}{p}\binom{p+k-1}{p}(2kp+p+3k^2+3k+1).\end{aligned}$$

Thus, for each $k=1,\ldots,p-2$, we have

$$2(2k+1)u_{k+1}\equiv ku_k\pmod{p^2}$$

and hence

$$\begin{aligned}(k+1)\binom{2(k+1)}{k+1}u_{k+1} =&2(k+1)\binom{2k+1}{k+1}u_{k+1}\\ =&2(2k+1)\binom{2k}{k}u_{k+1}\equiv k\binom{2k}{k}u_k \pmod{p^2}.\end{aligned}$$

So it remains to prove $\binom{2}{1}u_1 \equiv p \pmod{p^2}$. With the help of the Chu-Vandermonde identity, we actually have

$$\begin{aligned}u_1 =&\sum_{r=0}^{p-1}(-1)^r\binom{-2}{r} = (-1)^{p-1}\sum_{r=0}^{p-1}\binom{-1}{p-1-r}\binom{-2}{r}\\ =&(-1)^{p-1}\binom{-3}{p-1} = \binom{p+1}{p-1} = \frac{p^2+p}{2}.\end{aligned}$$

This concludes the proof. □

Proof of Theorem 1.1. (i) By [S12, (2.13)],

$$\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)A_k(x) \equiv \sum_{k=0}^{p-1}\frac{p}{2k+1}\left(1-2p^2H_k^{(2)}\right)x^k \pmod{p^4}.$$

Combining this with (3.1) we immediately get (1.4).

By [S12, (1.6)-(1.7)],

$$\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)A_k \equiv 1+\frac{7}{6}p^3B_{p-3} \pmod{p^4}$$

and

$$\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)A_k(-1) \equiv \left(\frac{-1}{p}\right) - p^2E_{p-3} \pmod{p^3}.$$

Combining this with (3.1) we obtain (1.5) and (1.6). In view of (1.4) and (3.4), we get (1.7).

(ii) With the help of (2.7),

$$\begin{aligned}\sum_{l=1}^{p-1}\frac{g_l(x)}{l} =&\sum_{l=1}^{p-1}\frac{1}{l}\sum_{k=0}^{l}\binom{l}{k}f_k(x) = H_{p-1}+\sum_{l=1}^{p-1}\sum_{k=1}^{l}\frac{f_k(x)}{l}\binom{l}{k}\\ \equiv&\sum_{k=1}^{p-1}\frac{f_k(x)}{k}\sum_{l=k}^{p-1}\binom{l-1}{k-1} = \sum_{k=1}^{p-1}\frac{f_k(x)}{k}\binom{p-1}{k}\\ \equiv&\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_k(x)(1-pH_k) \pmod{p^2}.\end{aligned}$$

In view of [S13b, (2.7)], this implies that

$$\sum_{k=1}^{p-1}\frac{g_k(x)}{k}\equiv p\sum_{k=(p+1)/2}^{p-1}\frac{x^k}{k^2}-p\sum_{k=1}^{p-1}\frac{(-1)^kH_k}{k}f_k(x)\pmod{p^2}. \tag{3.6}$$

So (1.8) follows.

By induction, for any integers $m>k\geqslant 0$, we have

$$\sum_{n=k}^{m-1}(2n+1)\binom{n+k}{2k}=\frac{m(m-k)}{k+1}\binom{m+k}{2k}.$$

This, together with (2.8) and (3.2), yields

$$\begin{aligned}\sum_{n=0}^{p-1}(-1)^n(2n+1)A_n=&\sum_{n=0}^{p-1}(2n+1)\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}(-1)^kg_k\\ =&\sum_{k=0}^{p-1}\binom{2k}{k}(-1)^kg_k\sum_{n=k}^{p-1}(2n+1)\binom{n+k}{2k}\\ =&\sum_{k=0}^{p-1}\binom{2k}{k}(-1)^kg_k\frac{p(p-k)}{k+1}\binom{p+k}{2k}\\ =&g_{p-1}\binom{2p-2}{p-1}(2p-1)+p^2\sum_{k=0}^{p-2}\binom{p-1}{k}\binom{p+k}{k}(-1)^k\frac{g_k}{k+1}\\ =&p\,g_{p-1}\binom{2p-1}{p-1}+p^2\sum_{k=1}^{p-1}\frac{g_{k-1}}{k}\\ \equiv&p\,g_{p-1}+p^2\sum_{k=1}^{p-1}\frac{g_{k-1}}{k}\pmod{p^4}\end{aligned}$$

since $\binom{2p-1}{p-1}\equiv 1\pmod{p^3}$ by Wolstenholme's theorem. Combining this with (2.18) and (3.3), we obtain

$$p\left(\frac p3\right)\equiv p\left(\frac p3\right)(1+2p\,q_p(3))+p^2\sum_{k=1}^{p-1}\frac{g_{k-1}}{k}\pmod{p^3}$$

and hence (1.9) follows.

(1.10) follows from a combination of (1.5) and (1.11) in the case $n=p$. If we let u_n denote the left-hand side or the right-hand side of (1.11), then by applying the Zeilberger algorithm via `Mathematica 9` we get the recurrence relation

$$\begin{aligned}&(n+2)(n+3)^2(2n+3)u_{n+3}\\=&(n+2)(22n^3+121n^2+211n+120)u_{n+2}\\&-(n+1)(38n^3+171n^2+229n+102)u_{n+1}+9n^2(n+1)(2n+5)u_n\end{aligned}$$

for $n=1,2,3,\ldots$. Thus (1.11) can be proved by induction.

(iii) Now we show (1.12)-(1.14) provided $p>5$.

Observe that

$$\begin{aligned}\sum_{l=1}^{p-1}\frac{g_l(x)-1}{l^2}=&\sum_{l=1}^{p-1}\frac{1}{l^2}\sum_{k=1}^{l}\binom{l}{k}^2\binom{2k}{k}x^k=\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{l=k}^{p-1}\binom{l-1}{k-1}^2\\=&\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{j=0}^{p-1-k}\binom{k+j-1}{j}^2=\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{j=0}^{p-1-k}\binom{-k}{j}^2\\\equiv&\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{j=0}^{p-1-k}\binom{p-k}{j}^2\pmod{p}.\end{aligned}$$

Recall that $H_{p-1}^{(2)}\equiv 0\pmod{p}$. Also, for any $k=1,\ldots,p-1$ we have

$$\sum_{j=0}^{p-1-k}\binom{p-k}{j}^2=\sum_{j=0}^{p-k}\binom{p-k}{j}\binom{p-k}{p-k-j}-1=\binom{2(p-k)}{p-k}-1$$

by the Chu-Vandermonde identity. Thus

$$\sum_{k=1}^{p-1}\frac{g_k(x)}{k^2}\equiv\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\left(\binom{2(p-k)}{p-k}-1\right)\equiv-\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\pmod{p}$$

(Note that $\binom{2k}{k}\binom{2(p-k)}{p-k}\equiv 0\pmod{p}$ for $k=1,\ldots,p-1$.) It is known that

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}\equiv 0\pmod{p}\tag{3.7}$$

(cf. Tauraso [T]) and moreover

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}\binom{2k}{k}\equiv\frac{56}{15}pB_{p-3}\pmod{p^2}$$

by Sun [S14]. So (1.12) is valid.

Note that

$$\begin{aligned}\sum_{l=1}^{p-1}\frac{g_l(x)-1}{l}=&\sum_{l=1}^{p-1}\frac1l\sum_{k=1}^{l}\binom lk^2\binom{2k}{k}x^k=\sum_{k=1}^{p-1}\binom{2k}{k}x^k\sum_{l=k}^{p-1}\frac1k\binom{l-1}{k-1}\binom lk\\ =&\sum_{k=1}^{p-1}\frac{\binom{2k}k}{k}x^k\sum_{j=0}^{p-1-k}\binom{k+j-1}{j}\binom{k+j}{j}.\end{aligned}$$

For $1\leqslant k\leqslant p-1$ and $p-k<j\leqslant p-1$, clearly

$$\binom{k+j-1}{j}\binom{k+j}{j}=\frac{(k+j-1)!(k+j)!}{(k-1)!k!(j!)^2}\equiv 0 \pmod{p^2}.$$

If $j=p-k$ with $1\leqslant k\leqslant p-1$, then

$$\begin{aligned}\binom{k+j-1}{j}\binom{k+j}{j}=&\binom{p-1}{j}\binom pj=\frac pj\binom{p-1}{j-1}\binom{p-1}{j}\\ \equiv&-\frac pj\equiv\frac pk \pmod{p^2}.\end{aligned}$$

Recall that $H_{p-1}\equiv 0 \pmod{p^2}$. So we have

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{g_k(x)}{k}\equiv&\sum_{k=1}^{p-1}\frac{\binom{2k}k}{k}x^k\left(\sum_{j=0}^{p-1}\binom{k+j-1}{j}\binom{k+j}{j}-\frac pk\right)\\ =&\sum_{k=1}^{p-1}\frac{\binom{2k}k}{k}x^k\sum_{j=0}^{p-1}\binom{-k}{j}\binom{-k-1}{j}-p\sum_{k=1}^{p-1}\frac{\binom{2k}k}{k^2}x^k\\ \equiv&\sum_{k=1}^{p-1}\frac{x^k}{k^2}p-p\sum_{k=1}^{p-1}\frac{\binom{2k}k}{k^2}x^k=p\sum_{k=1}^{p-1}\frac{1-\binom{2k}k}{k^2}x^k \pmod{p^2}\end{aligned}$$

with the help of (3.5). Thus, in view of (3.7) we get

$$\sum_{k=1}^{p-1}\frac{g_k(-1)}{k}\equiv p\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}=p\sum_{k=1}^{(p-1)/2}\left(\frac{(-1)^k}{k^2}+\frac{(-1)^{p-k}}{(p-k)^2}\right)\equiv 0 \pmod{p^2}.$$

This proves (1.13). Combining this with (3.6) we obtain

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-1)^kf_k(-1)}{k}H_k\equiv&\sum_{k=(p+1)/2}^{p-1}\frac{(-1)^k}{k^2}\equiv-\sum_{j=1}^{(p-1)/2}\frac{(-1)^j}{j^2}\\ \equiv&-2\left(\frac{-1}{p}\right)E_{p-3} \pmod p\end{aligned}$$

with the help of [S11b, Lemma 2.4]. So (1.14) holds.

In view of the above, we have completed the proof of Theorem 1.1. $\square$

4. Some open conjectural congruences

In this section we pose several related conjectural congruences.

Conjecture 4.1. (i) *For any integer $n > 1$, we have*

$$\sum_{k=0}^{n-1}(9k^2+5k)(-1)^k f_k \equiv 0 \pmod{(n-1)n^2}$$

Also, for each odd prime p we have

$$\sum_{k=0}^{p-1}(9k^2+5k)(-1)^k f_k \equiv 3p^2(p-1) - 16p^3 q_p(2) \pmod{p^4}.$$

(ii) *For every $n = 1, 2, 3, \ldots$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)g_k(x) \in \mathbb{Z}[x]$$

and the number

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(8k^2+12k+5)g_k(-1)$$

is always an odd integer. Also, for any prime p we have

$$\sum_{k=0}^{p-1}(8k^2+12k+5)g_k(-1) \equiv 3p^2 \pmod{p^3}.$$

For any nonzero integer m, the 3-adic valuation $\nu_3(m)$ of m is the largest $a \in \mathbb{N}$ with $3^a \mid m$. For convenience, we also set $\nu_3(0) = +\infty$.

Conjecture 4.2. *Let n be any positive integer. Then*

$$\nu_3\bigg(\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k\bigg) = 3\nu_3(n) \leqslant \nu_3\bigg(\sum_{k=0}^{n-1}(2k+1)^3(-1)^k A_k\bigg).$$

If n is a positive multiple of 3, then

$$\nu_3\bigg(\sum_{k=0}^{n-1}(2k+1)^3(-1)^k A_k\bigg) = 3\nu_3(n) + 2.$$

Conjecture 4.3. *For $n\in\mathbb{N}$ define*

$$F_n:=\sum_{k=0}^n\binom nk^3(-8)^k\quad\text{and}\quad G_n:=\sum_{k=0}^n\binom nk^2(6k+1)C_k.$$

For any $n\in\mathbb{Z}^+$, the number

$$\frac1n\sum_{k=0}^{n-1}(6k+5)(-1)^kF_k$$

is always an odd integer. Also, for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}(-1)^kF_k\equiv\left(\frac p3\right)\pmod{p^2}\quad\text{and}\quad\sum_{k=1}^{p-1}G_k\equiv-\frac43p^3B_{p-3}\pmod{p^4}.$$

Remark 4.1. For any prime $p>3$, the author [S13b, S12] proved that $\sum_{k=0}^{p-1}(-1)^kf_k\equiv(\frac p3)\pmod{p^2}$ and $\sum_{k=1}^{p-1}h_k\equiv0\pmod{p^2}$ with $h_k=\sum_{j=0}^k\binom kj^2C_j$.

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