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THE LEAST MODULUS FOR WHICH CONSECUTIVE POLYNOMIAL VALUES ARE DISTINCT

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Abstract. Let $d \geqslant 4$ and $c \in (-d, d)$ be relatively prime integers. We show that for any sufficiently large integer n (in particular $n > 24310$ suffices for $4 \leqslant d \leqslant 36$), the smallest prime $p \equiv c \pmod d$ with $p \geqslant (2dn - c)/(d-1)$ is the least positive integer m with $2r(d)k(dk-c)$ $(k = 1, \ldots, n)$ pairwise distinct modulo m, where $r(d)$ is the radical of d. We also conjecture that for any integer $n > 4$ the least positive integer m such that $|\{k(k-1)/2 \bmod m : k = 1, \ldots, n\}| = |\{k(k-1)/2 \bmod m+2 : k = 1, \ldots, n\}| = n$ is the least prime $p \geqslant 2n-1$ with $p+2$ also prime.

1. Introduction

To find nontrivial arithmetical functions taking only prime values is a fascinating topic in number theory. In 1947 W. H. Mills [M] showed that there exists a real number A such that $\lfloor A^{3^n} \rfloor$ is prime for every $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$; unfortunately such a constant A cannot be effectively found.

For each integer $h > 1$ and sufficiently large integer n, it was determined in [BSW] the least positive integer m with $1^h, 2^h, \ldots, n^h$ pairwise distinct modulo m, but such integers m are composite infinitely often. In a recent paper [S] the author proved that the smallest integer $m > 1$ such that $2k(k-1) \bmod m$ for $k = 1, \ldots, n$ are pairwise distinct, is precisely the least prime greater than $2n - 2$, and that for $n \in \{4, 5, \ldots\}$ the least positive integer m such that $18k(3k-1)$ $(k = 1, \ldots, n)$ are pairwise distinct modulo m, is the least prime $p > 3n$ with $p \equiv 1 \pmod 3$. When $d \in \{4, 5, 6, \ldots\}$ and $c \in (-d, d)$ are relatively prime, it is natural to ask whether there is a similar result for primes

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in the arithmetic progression $\{c, c+d, c+2d, \dots\}$ since there are infinitely many such primes by Dirichlet's theorem.

Based on our computation we discover the following general result.

Theorem 1.1. *Let $d \geqslant 4$ and $c \in (-d, d)$ be relatively prime integers. Let*

$$f_{d,c}(x) := 2r(d)x(dx - c), \tag{1.1}$$

where $r(d)$ is the radical of d (i.e., the product of all the distinct prime divisors of d). For $n \in \mathbb{Z}^+$ define $m_{d,c}(n)$ as the least positive integer m for which the integers $f_{d,c}(k)$ $(k = 1, \dots, n)$ are pairwise distinct modulo m.

(i) *If $n \in \mathbb{Z}^+$ is sufficiently large, then $m_{d,c}(n)$ is the least prime $p \equiv c \pmod{d}$ with $p \geqslant (2dn - c)/(d - 1)$.*

(ii) *When $4 \leqslant d \leqslant 36$ and $n > M_d$, the required result in the first part holds, where*

$$\begin{gathered} M_4 = 8,\ M_5 = 14,\ M_6 = 9,\ M_7 = 100,\ M_8 = 21,\ M_9 = 315,\ M_{10} = 53, \\ M_{11} = 1067,\ M_{12} = 27,\ M_{13} = 1074,\ M_{14} = 122,\ M_{15} = 809,\ M_{16} = 329, \\ M_{17} = 5115,\ M_{18} = 95,\ M_{19} = 5390,\ M_{20} = 755,\ M_{21} = 3672,\ M_{22} = 640, \\ M_{23} = 11193,\ M_{24} = 220,\ M_{25} = 12810,\ M_{26} = 1207,\ M_{27} = 7087, \\ M_{28} = 2036,\ M_{29} = 13250,\ M_{30} = 177,\ M_{31} = 24310,\ M_{32} = 3678, \\ M_{33} = 12794,\ M_{34} = 5303,\ M_{35} = 15628,\ M_{36} = 551. \end{gathered}$$

Remark 1.1. To obtain the effective lower bounds M_d $(4 \leqslant d \leqslant 36)$ in part (ii) of Theorem 1.1, we actually employ some computational results of O. Ramaré and R. Rumely [RR] on primes in arithmetic progressions. Define

$$\begin{gathered} c_4 = -3,\ c_5 = -1,\ c_6 = 1,\ c_7 = -5,\ c_8 = 1,\ c_9 = 2,\ c_{10} = 3, \\ c_{11} = -7,\ c_{12} = 5,\ c_{13} = -5,\ c_{14} = -5,\ c_{15} = -1,\ c_{16} = 11, \\ c_{17} = 15,\ c_{18} = 1,\ c_{19} = 6,\ c_{20} = -9,\ c_{21} = 1,\ c_{22} = 5, \\ c_{23} = 21,\ c_{24} = 1,\ c_{25} = 19,\ c_{26} = -3,\ c_{27} = 23, \\ c_{28} = -9,\ c_{29} = -1,\ c_{30} = 17,\ c_{31} = 3,\ c_{32} = -1, \\ c_{33} = -5,\ c_{34} = 15,\ c_{35} = 12,\ c_{36} = 23. \end{gathered}$$

Then, for every $d = 4, \dots, 36$, the number $m_{d,c_d}(M_d)$ is *not* the least prime $p \equiv c_d \pmod{d}$ with $p \geqslant (2dM_d - c_d)/(d - 1)$.

Theorem 1.1 with $d = 4, 5$ yields the following concrete consequence.

Corollary 1.1. (i) *For each integer $n \geqslant 6$, the least positive integer m such that $4k(4k-1)$ (or $4k(4k+1)$) for $k = 1, \dots, n$ are pairwise distinct modulo m, is the least prime $p \equiv 1 \pmod{4}$ with $p \geqslant (8n - 1)/3$ (resp., $p \equiv -1 \pmod{4}$ with $p \geqslant (8n + 1)/3$).*

(ii) *Let* $C_1 = 8,\ C_2 = 10,\ C_{-1} = 15$ *and* $C_{-2} = 5$. *For any* $r \in \{\pm 1, \pm 2\}$ *and integer* $n \geqslant C_r$, *the least positive integer* m *such that* $10k(5k-r)$ *for* $k = 1, \ldots, n$ *are pairwise distinct modulo* m, *is the least prime* $p \equiv r \pmod 5$ *with* $p \geqslant (10n-r)/4$.

As a supplement to Theorem 1.1, we are able to prove the following result for the cases $d = 2, 3$.

Theorem 1.2. *For* $d \in \{2, 3\}$ *and integer* $c \in (-d, d)$, *let* $S_{d,c}$ *be the set of all primes* $p \equiv c \pmod d$ *and powers of* d. *Then*

$$\begin{aligned} m_{2,1}(n) &= \min\{a \geqslant 4n-1:\ a \in S_{2,1}\} \quad \text{for } n \geqslant 5, \\ m_{2,-1}(n) &= \min\{a \geqslant 4n:\ a \in S_{2,-1}\} \quad \text{for } n \geqslant 7, \\ m_{3,1}(n) &= \min\{a \geqslant 3n:\ a \in S_{3,1}\} \quad \text{for } n \geqslant 4, \\ m_{3,-1}(n) &= \min\{a \geqslant 3n:\ a \in S_{3,-1}\} \quad \text{for } n \geqslant 5, \\ m_{3,2}(n) &= \min\{a \geqslant 3n-1:\ a \in S_{3,2}\} \quad \text{for } n \geqslant 3, \\ m_{3,-2}(n) &= \min\{a \geqslant 3n:\ a \in S_{3,-2}\} \quad \text{for } n \geqslant 8. \end{aligned}$$

Remark 1.2. As the proof of Theorem 1.2 is quite similar to and even easier than that of Theorem 1.1, we omit the details of the proof. Note that if $n > 1$ is a power of two with $4n-1$ composite then $\min\{a \geqslant 4n-1:\ a \in S_{2,1}\} = 4n$ is a power of two. Also, if $n > 1$ is a power of three then $\min\{a \geqslant 3n-1:\ a \in S_{3,2}\} = 3n$ is a power of three.

To conclude this section, we pose some new conjectures.

Conjecture 1.1. *For any* $d \in \mathbb{Z}^+$ *there is a positive integer* n_d *such that for any integer* $n \geqslant n_d$ *the least positive integer* m *satisfying*

$$\left|\left\{\binom{k}{2} \bmod m:\ k = 1, \ldots, n\right\}\right| = \left|\left\{\binom{k}{2} \bmod m + 2d:\ k = 1, \ldots, n\right\}\right| = n$$

is the smallest prime $p \geqslant 2n-1$ *with* $p + 2d$ *also prime. Moreover, we may take*

$$\begin{gathered} n_1 = 5,\ n_2 = n_3 = 6,\ n_4 = 10,\ n_5 = 9, \\ n_6 = 8,\ n_7 = 9,\ n_8 = 18,\ n_9 = 11,\ n_{10} = 9. \end{gathered}$$

Remark 1.3. A well-known conjecture of de Polignac [P] asserts that for any positive integer d there are infinitely many prime pairs $\{p, q\}$ with $p - q = 2d$.

Conjecture 1.2. *Let n be any positive integer and consider the least positive integer m such that*

$$\left|\left\{\binom{k}{2} \bmod m: \ k=1,\ldots,n\right\}\right| = \left|\left\{\binom{k}{2} \bmod m+1: \ k=1,\ldots,n\right\}\right| = n.$$

Then, each of m and $m+1$ is either a power of two (including $2^0=1$) or a prime times a power of two.

Conjecture 1.3. *Let n be any positive integer. Then the least positive integer m of the form x^2+x+1 (or $4x^2+1$) with $x\in\mathbb{Z}$ such that the coefficients $\binom{k}{2}$ $(k=1,\ldots,n)$ are pairwise distinct modulo m, is the the smallest prime $p\geqslant 2n-1$ of the form x^2+x+1 (resp., $4x^2+1$) with $x\in\mathbb{Z}$.*

Remark 1.4. The conjecture that there are infinitely many primes of the form x^2+x+1 (or $4x^2+1$) is still open. We may also replace $\binom{k}{2}$ in Conjecture 1.3 by $k(k-1)$.

Conjecture 1.4. *For any integer $n>2$, the smallest positive integer m such that the integers $6p_k(p_k-1)$ $(k=1,\ldots,n)$ are pairwise incongruent modulo m is precisely the least prime $p\geqslant p_n$ dividing none of the numbers p_i+p_j-1 $(1\leqslant i<j\leqslant n)$, where p_k denotes the k-th prime.*

Remark 1.5. For any prime $p\geqslant p_n$ dividing none of the numbers p_i+p_j-1 $(1\leqslant i<j\leqslant n)$, clearly $p_j(p_j-1)-p_i(p_i-1)=(p_j-p_i)(p_i+p_j-1)\not\equiv 0 \pmod p$ for all $1\leqslant i<j\leqslant n$.

We also have some other conjectures similar to Conjectures 1.1–1.4.

In the next section we provide some lemmas. Section 3 is devoted to our proof of Theorem 1.1.

2. Some lemmas

Lemma 2.1. *Let c and $d>1$ be relatively prime integers. For any $\varepsilon>0$, if $n\in\mathbb{Z}^+$ is large enough, then there is a prime $p\equiv c \pmod d$ with*

$$\frac{d(2n-1)-c}{d-1} < p \leqslant \frac{d((2+\varepsilon)n-1)-c}{d-1}.$$

Proof. This is an easy consequence of the Prime Number Theorem for arithmetic progressions (cf. (1.5) of [CP, p. 13] or Theorem 4.4.4 of [J, p. 175]) which states that

$$|\{p\leqslant x: \ p \text{ is a prime with } p\equiv c \pmod d\}| \sim \frac{x}{\varphi(d)\log x}$$

as $x\to+\infty$, where φ is Euler's totient function. $\square$

In view of (1.1), for any $c\in\mathbb{Z}$ and $d\in\mathbb{Z}^+$ we have the useful identity

$$f_{d,c}(l)-f_{d,c}(k)=2r(d)(l-k)(d(k+l)-c). \tag{2.1}$$

Lemma 2.2. *Let $d > 2$ and $c \in (-d, d)$ be relatively prime integers. Suppose that p is a prime not exceeding $(d((2+\varepsilon)n-1)-c)/(d-1)$ where $n \geqslant 3d$ and $0 < \varepsilon \leqslant 2/(d-2)$. Then*

$$\begin{aligned} &f_{d,c}(k)\ (k=1,\ldots,n) \text{ are pairwise distinct modulo } p \\ \iff &p \equiv c \pmod d \text{ and } p > (d(2n-1)-c)/(d-1). \end{aligned} \tag{2.2}$$

Proof. If $p \mid 2d$, then $p \mid 2r(d)$ and hence $f_{d,c}(k) \equiv 0 \pmod p$ for all $k = 1, \ldots, n$. Note that $(d(2n-1)-c)/(d-1) \geqslant (3d-c)/(d-1) \geqslant 2d/(d-1) > 2$. If $p \mid d$ then $p \not\equiv c \pmod d$. So (2.2) holds in the case $p \mid 2d$.

From now on we assume that $p \nmid 2d$. Then $jp \equiv -c \pmod d$ for some $1 \leqslant j \leqslant d-1$.

Negating the right-hand side of (2.2), we suppose first that $p \not\equiv c \pmod d$ or $p \leqslant (d(2n-1)-c)/(d-1)$. Write $jp + c = dq$ with $q \in \mathbb{Z}$. If $p \not\equiv c \pmod d$, then $j \leqslant d-2$ and hence

$$\begin{aligned} q \leqslant &\frac{c}{d} + \frac{d-2}{d}p \leqslant \frac{c}{d} + \frac{d-2}{d} \cdot \frac{d((2+\varepsilon)n-1)-c}{d-1} \\ \leqslant &\frac{c-d(d-2)}{d(d-1)} + \frac{d-2}{d-1}\left(2 + \frac{2}{d-2}\right)n < 2n. \end{aligned}$$

If $p \equiv c \pmod d$ and $p \leqslant (d(2n-1)-c)/(d-1)$, then $j = d-1$ and $q \leqslant 2n-1$. When $q > 2$, we have $0 < k := \lfloor (q-1)/2 \rfloor < l := \lfloor (q+2)/2 \rfloor \leqslant n$, also

$$d(k+l) - c = dq - c = jp \equiv 0 \pmod p$$

and hence $f_{d,c}(k) \equiv f_{d,c}(l) \pmod p$ in view of (2.1). If $q \leqslant 2$, then $p \leqslant jp = dq - c \leqslant 2d - c < 3d \leqslant n$ and $f_{d,c}(p+1) \equiv f_{d,c}(1) \pmod p$.

We now assume the right-hand side of (2.2). Then $(d-1)p + c = dq$ for some integer $q \geqslant 2n$. In view of (2.1), we only need to show that $p \nmid (l-k)$ and $d(k+l) \not\equiv c \pmod p$ for any $1 \leqslant k < l \leqslant n$. Note that

$$0 < l - k < n \leqslant \frac{dq}{2d} = \frac{(d-1)p + c}{2d} < \frac{p+1}{2} \leqslant p$$

and also $d(k+l) - c \leqslant d(2n-1) - c < (d-1)p$. If $d(k+l) \equiv c \pmod p$, then for some $t = 1, \ldots, d-2$ we have $d(k+l) - c = tp \equiv tc \not\equiv -c \pmod d$, which leads to a contradiction.

The proof of Lemma 2.2 is now complete. □

Lemma 2.3. *Let $d > 2$ and $c \in (-d, d)$ be relatively prime integers, and let $n \geqslant 6d$ be an integer. Suppose that $m \in [n, (d((2+\varepsilon)n-1)-c)/(d-1)]$ is a power of two or twice an odd prime, where $0 < \varepsilon \leqslant 2/3$. Then, there are $1 \leqslant k < l \leqslant n$ such that $f_{d,c}(k) \equiv f_{d,c}(l) \pmod m$.*

Proof. Note that $m \geqslant n \geqslant 6d > 4$ and

$$\frac{m}{4} \leqslant \frac{d((2+\varepsilon)n-1)-c}{4(d-1)} < \frac{d(2+\varepsilon)}{4(d-1)}n \leqslant \frac{d(2+2/3)}{4(d-1)}n = \frac{8dn}{8d+4(d-3)} \leqslant n.$$

If d is even and m is a power of two, then for $k = 1$ and $l = m/4+1 \leqslant n$ we have $m \mid 2r(d)(l-k)$ and hence $f_{d,c}(k) \equiv f_{d,c}(l) \pmod{m}$ by (2.1). If $m = 2p$ with p an odd prime dividing d, then $m \mid 2r(d)$ and hence $f_{d,c}(k) \equiv 0 \pmod{m}$ for all $k = 1, \ldots, n$.

In the other cases, d and $m/2$ are relatively prime. Thus $jd \equiv c \pmod{m/2}$ for some $j = 1, \ldots, m/2$. If $j \leqslant 2$, then

$$\frac{m}{2} \leqslant jd - c \leqslant 2d - c < 3d \leqslant \frac{n}{2}$$

which contradicts $m \geqslant n$. So $3 \leqslant j \leqslant m/2$ and hence

$$0 < k := \left\lfloor \frac{j-1}{2} \right\rfloor < l := \left\lfloor \frac{j+2}{2} \right\rfloor \leqslant \frac{m}{4} + 1 < n+1.$$

Since $d(k+l) - c = jd - c \equiv 0 \pmod{m/2}$, by (2.1) we have $f_{d,c}(k) \equiv f_{d,c}(l) \pmod{m}$. This concludes the proof. $\square$

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\varepsilon = 2/(\max\{11, d\} - 2)$. By Lemma 2.1, if $n \in \mathbb{Z}^+$ is large enough then there is at least a prime $p \equiv c \pmod{d}$ with

$$\frac{d(2n-1)-c}{d-1} < p \leqslant \frac{d((2+\varepsilon)n-1)-c}{d-1}. \tag{3.1}$$

(i) Choose an integer $N \geqslant \max\{6d, 243\}$ such that for any integer $n \geqslant N$ there is a prime $p \equiv c \pmod{d}$ satisfying (3.1). Fix an integer $n \geqslant N$ and let $m = m_{d,c}(n)$. Clearly $m \geqslant n$. By Lemma 2.2, $m \leqslant m'$ where m' denotes the least prime $p \equiv c \pmod{d}$ satisfying (3.1).

Assume that $m \neq m'$. We want to reach a contradiction. Clearly m is not a prime by Lemma 2.2. Note that $\varepsilon \leqslant 2/9$. In view of Lemma 2.3, m is neither a power of two nor twice an odd prime. So we have $m = pq$ for some odd prime p and integer $q > 2$. Observe that

$$\frac{m}{3} \leqslant \frac{d((2+\varepsilon)n-1)-c}{3(d-1)} < \frac{d(2+2/9)}{3(d-1)}n = \frac{20d}{27(d-1)}n \leqslant \frac{80}{81}n$$

and hence

$$\frac{m}{3} + 3 < \frac{80}{81}n + \frac{n}{81} = n. \tag{3.2}$$

If $p \mid d$, then for $k := 1$ and $l := q + 1 = m/p + 1 < m/3 + 3 < n$, we have $pq \mid r(d)(l - k)$ and hence $f_{d,c}(k) \equiv f_{d,c}(l) \pmod m$ by (2.1).

Now suppose that $p \nmid d$. Then $2dk \equiv c - dq \pmod p$ for some $1 \leqslant k \leqslant p$. Clearly, $l := k + q \leqslant p + q = m/q + m/p$. Note that

$$(l - k)(d(l + k) - c) = q(d(2k + q) - c) \equiv 0 \pmod{pq}$$

and hence $f_{d,c}(k) \equiv f_{d,c}(l) \pmod m$ by (2.1). If $\min\{p, q\} \leqslant 4$, then

$$l \leqslant p + q = \frac{m}{\min\{p, q\}} + \min\{p, q\} \leqslant \frac{m}{3} + 4 < n + 1$$

by (3.2). If $\min\{p, q\} \geqslant 5$, then

$$l \leqslant \frac{m}{q} + \frac{m}{p} \leqslant \max\left\{\frac{m}{6} + \frac{m}{7},\ \frac{m}{5} + \frac{m}{8}\right\} < \frac{m}{3} < n$$

since $pq = m \geqslant n \geqslant 243 \geqslant 40$. So we get a contradiction as desired.

(ii) Now assume that $4 \leqslant d \leqslant 36$. By Table 1 of [RR, p. 419], we have

$$(1 - \varepsilon_d)\frac{x}{\varphi(d)} \leqslant \theta(x; c, d) \leqslant (1 + \varepsilon_d)\frac{x}{\varphi(d)} \quad \text{for all } x \geqslant 10^{10}, \tag{3.3}$$

where

$$\theta(x; c, d) := \sum_{\substack{p \leqslant x \\ p \equiv c \pmod d}} \log p \quad \text{with } p \text{ prime},$$

and

$\varepsilon_4 = 0.002238,\ \varepsilon_5 = 0.002785,\ \varepsilon_6 = 0.002238,\ \varepsilon_7 = 0.003248,\ \varepsilon_8 = 0.002811,$
$\varepsilon_9 = 0.003228,\ \varepsilon_{10} = 0.002785,\ \varepsilon_{11} = 0.004125,\ \varepsilon_{12} = 0.002781,\ \varepsilon_{13} = 0.004560,$
$\varepsilon_{14} = 0.003248,\ \varepsilon_{15} = 0.008634,\ \varepsilon_{16} = 0.008994,\ \varepsilon_{17} = 0.010746,\ \varepsilon_{18} = 0.003228,$
$\varepsilon_{19} = 0.011892,\ \varepsilon_{20} = 0.008501,\ \varepsilon_{21} = 0.009708,\ \varepsilon_{22} = 0.004125,\ \varepsilon_{23} = 0.012682,$
$\varepsilon_{24} = 0.008173,\ \varepsilon_{25} = 0.012214,\ \varepsilon_{26} = 0.004560,\ \varepsilon_{27} = 0.011579,\ \varepsilon_{28} = 0.009908,$
$\varepsilon_{29} = 0.014102,\ \varepsilon_{30} = 0.008634,\ \varepsilon_{31} = 0.014535,\ \varepsilon_{32} = 0.011103,\ \varepsilon_{33} = 0.011685,$
$\varepsilon_{34} = 0.010746,\ \varepsilon_{35} = 0.012809,\ \varepsilon_{36} = 0.009544.$

As $\varepsilon = 2/(\max\{11, d\} - 2)$, we can easily verify that

$$\frac{\varepsilon}{2} - \frac{2}{10^{10}} > \frac{2\varepsilon_d}{1 - \varepsilon_d} = \frac{1 + \varepsilon_d}{1 - \varepsilon_d} - 1.$$

If $n \geqslant 10^{10}/2$, then

$$((2 + \varepsilon)n - 2)\frac{d}{d - 1} \geqslant 2n\frac{d}{d - 1} > 10^{10}$$

and
$$\frac{\varepsilon}{2}-\frac{1}{n}+1\geqslant\frac{\varepsilon}{2}-\frac{2}{10^{10}}+1>\frac{1+\varepsilon_d}{1-\varepsilon_d},$$
hence by (3.3) we have
$$\begin{aligned}&\frac{\theta(((2+\varepsilon)n-2)d/(d-1);c,d)}{\theta(2nd/(d-1);c,d)}\\ \geqslant&\frac{(1-\varepsilon_d)((2+\varepsilon)n-2)d/(d-1)}{(1+\varepsilon_d)2nd/(d-1)}=\frac{1-\varepsilon_d}{1+\varepsilon_d}\left(1+\frac{\varepsilon}{2}-\frac{1}{n}\right)>1\end{aligned}$$
and thus there is a prime $p\equiv c \pmod d$ for which
$$\frac{2dn}{d-1}<p\leqslant\frac{((2+\varepsilon)n-2)d}{d-1}$$
and hence (3.1) holds.

Let N_d be the least positive integer such that for any $n=N_d,\ldots,10^{10}/2$ and any $a\in\mathbb{Z}$ relatively prime to d, the interval $(2dn/(d-1),((2+\varepsilon)n-2)d/(d-1))$ contains a prime congruent to a modulo d. Via a computer we find that
$$\begin{gathered}N_4=79,\ N_5=206,\ N_6=103,\ N_7=471,\ N_8=301,\ N_9=356, N_{10}=232,\\ N_{11}=1079,\ N_{12}=346,\ N_{13}=1166,\ N_{14}=806,\ N_{15}=1310,\ N_{16}=2183,\\ N_{17}=5153,\ N_{18}=1135,\ N_{19}=5402,\ N_{20}=2388,\ N_{21}=4059,\ N_{22}=2934,\\ N_{23}=11246,\ N_{24}=2480,\ N_{25}=13144,\ N_{26}=4775,\ N_{27}=11646,\\ N_{28}=5314,\ N_{29}=13478,\ N_{30}=5215,\ N_{31}=24334,\ N_{32}=8964,\\ N_{33}=15044,\ N_{34}=14748,\ N_{35}=16896,\ N_{36}=9847.\end{gathered}$$
For any integer $n\geqslant N_d$, there is a prime $p\equiv c \pmod d$ satisfying (3.1). Note that $6d\leqslant 6\times 36<243$. So, for $n\geqslant N=\max\{N_d,243\}$ we may apply part (i) to get the desired result. If $M_d<n\leqslant\max\{N_d,243\}$, then we can easily verify the desired result via a computer.

In view of the above, we have completed the proof of Theorem 1.1. □

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