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Note

On disjoint residue classes

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Abstract

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The purpose of this note is to show that, if $n_1, \ldots, n_k$ are positive integers, and for each $d \in \mathbb{Z}^+$ satisfying $f(d) \leqslant k-2$ or a weaker condition $d \leqslant 2^{k-2}$ (where $f(d) = \sum_{i=1}^{r} \alpha_i(p_i - 1)$ if $\prod_{i=1}^{r} p_i^{\alpha_i}$ is the prime factorization of d), the number of pairs $\{i, j\}$ $(1 \leqslant i < j \leqslant k)$ with gcd $(n_i, n_j) = d$ is less than $\sqrt{(d+7)/8}$, then there exist integers $a_1, \ldots, a_k$ such that the residue classes $a_1(\mathrm{mod}\, n_1), \ldots, a_k(\mathrm{mod}\, n_k)$ are pairwise disjoint. We conjecture that $\sqrt{(d+7)/8}$ can be replaced by $2d-1$.

Let $a(n)$ represent the residue class $\{a + nx : x \in \mathbb{Z}\}$. A system $A = \{a_i(n_i)\}_{i=1}^{k}$ of residue classes is said to be disjoint if $a_1(n_1), \ldots, a_k(n_k)$ are pairwise disjoint. If $\bigcup_{i=1}^{k} a_i(n_i) = \mathbb{Z}$ then we call $\{a_i(n_i)\}_{i=1}^{k}$ a covering system. A DCS (disjoint covering system) is a system which is disjoint and covering.

By simple cardinality arguments one can show

Lemma 1. *Let $A = \{a_i(n_i)\}_{i=1}^{k}$ be a system of residue classes.*

(a) *If A is disjoint then $\sum_{i=1}^{k} 1/n_i \leqslant 1$.*

(b) *If A is covering then $\sum_{i=1}^{k} 1/n_i \geqslant 1$.*

(c) *The following conditions are equivalent:*

(i) *A is a DCS.*

(ii) *$\sum_{i=1}^{k} 1/n_i = 1$, and A is disjoint.*

(iii) *$\sum_{i=1}^{k} 1/n_i = 1$, and A is covering.*

Definition. A k-tuple $n_1, \ldots, n_k$ of positive integers is called harmonic if there exist integers $a_1, \ldots, a_k$ such that the system $\{a_i(n_i)\}_{i=1}^{k}$ is disjoint.

It is easy to see that if $n_1, \ldots, n_k$ is harmonic then $(n_i, n_j) > 1$ for all

$i, j = 1, \ldots, k$ with $i \neq j$. (In this note (m, n) denotes the greatest common divisor of m and n.)

In [1], Huhn and Megyesi posed the following interesting problem.

Problem. Characterize the harmonic tuples.

Let $1 < n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ and $\sum_{i=1}^{k} 1/n_i = 1$. By Lemma 1 and the Znám–Newman theorem ([7,3]), if $n_1, \ldots, n_k$ is harmonic then we must have $n_{k-p+1} = \cdots = n_{k-1} = n_k$ where p is the least prime divisor of n_k. From this it seems to me that the above problem is a difficult one.

In [1], Huhn and Megyesi noted that, for $n_1, \ldots, n_k$ to be harmonic, it is necessary that, for any $R \subseteq \{1, 2, \ldots, k\}$ with $|R| \geqslant 2$ the inequality

$$\sum_{i \in R} \frac{1}{\tilde{n}_i(R)} \leqslant 1 \tag{1}$$

holds where $\tilde{n}_i(R) = (n_i, [n_j]_{j \in R, j \neq i}) = [(n_i, n_j)]_{j \in R - \{i\}}$. (In this note the cardinality of a set S is denoted by $|S|$, $[m_i]_{i \in I}$ represents the least common multiple of m_i, $i \in I$.) They then asked whether the above necessary condition is also sufficient. In [5] we showed that this is the case only for $k \leqslant 4$.

In 1982, using a lemma of combinatorial character, Huhn and Megyesi [1] proved that, for $n_1, \ldots, n_k$ to be harmonic, it is sufficient that the pairwise greatest common divisors, $d_{ij} = (n_i, n_j)$, $i \neq j$ are distinct and different from 1. Up to now, no other nontrivial sufficient conditions have been found. In this note we will improve the result, namely we have the following.

Theorem. *Let $n_1, \ldots, n_k$ be positive integers. Suppose that*

$$|\{\{i, j\}: 1 \leqslant i < j \leqslant k \text{ and } (n_i, n_j) = d\}| < \sqrt{\frac{d+7}{8}} \tag{2}$$

holds for all $d \in \mathbb{Z}^+$ with $f(d) \leqslant k - 2$ (or $d \leqslant 2^{k-2}$), where $f(d) = \sum_{i=1}^{r} \alpha_i(p_i - 1)$ if $\prod_{i=1}^{r} p_i^{\alpha_i}$ is the prime factorization of d. Then $n_1, \ldots, n_k$ is harmonic.

Obviously our theorem implies the result of Huhn and Megyesi [1].

To prove the theorem we need several lemmas.

Lemma 2. *If $\{a_i(n_i)\}_{i=1}^{k}$ is a covering system with $1 < n_1 < n_2 \cdots < n_k$, then $k \geqslant 5$.*

Proof. Suppose $k \leqslant 4$. Since $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} < 1$, by Lemma 1 we have $n_1 = 2$. By Krukenberg [2] $n_1 = 2$ implies $k \geqslant 5$. This is a contradiction. □

Lemma 3. *Suppose that $\{a_i(n_i)\}_{i=1}^{k}$ is a covering system but none of its proper subsystems is. Let N be the least common multiple of $n_1, \ldots, n_k$. Then $k - 1 \geqslant f(N) \geqslant f(n_i)$ where f is as in the theorem.*

Proof. See Simpson [4] or Sun [6]. $\square$

Lemma 4. *Let $n_1, \ldots, n_k \in \mathbb{Z}^+$. Suppose that $n_1, \ldots, n_{k-1}$ is harmonic but $n_1, \ldots, n_k$ is not. Then $\{a_i((n_i, n_k))\}_{i=1}^{k-1}$ is covering for some $a_1, \ldots, a_{k-1} \in \mathbb{Z}$.*

Proof. Let $\{a_i(n_i)\}_{i=1}^{k-1}$ be disjoint. If $a_k \in \mathbb{Z} - \bigcup_{i=1}^{k-1} a_i((n_i, n_k))$, then $\{a_i(n_i)\}_{i=1}^{k}$ is disjoint and hence $n_1, \ldots, n_k$ is harmonic. $\square$

Lemma 5. *Let $m, n \in \mathbb{Z}^+$ and $m \leqslant n$. Then*

$$2(\sqrt{n+1} - \sqrt{m}) < \sum_{k=m}^{n} \frac{1}{\sqrt{k}} < 2(\sqrt{n} - \sqrt{m-1}).$$

Proof. It is easy to show that

$$2(\sqrt{k+1} - \sqrt{k}) < 1/\sqrt{k} < 2(\sqrt{k} - \sqrt{k-1}) \quad \text{for } k \in Z^+.$$

So we have

$$2(\sqrt{n+1} - \sqrt{m}) = \sum_{k=m}^{n} 2(\sqrt{k+1} - \sqrt{k}) < \sum_{k=m}^{n} \frac{1}{\sqrt{k}} < \sum_{k=m}^{n} 2(\sqrt{k} - \sqrt{k-1})$$
$$= 2(\sqrt{n} - \sqrt{m-1}). \quad \square$$

Proof of the Theorem. Since any 1-tuple n is harmonic, the theorem holds for $k = 1$. Now let $k > 1$ and assume that the theorem holds for all smaller k. Let $n_1, \ldots, n_k$ be positive integers satisfying the supposition. Clearly (2) holds for $d = 1$, so $(n_i, n_j) > 1$ if $i \neq j$. Now assuming that $n_1, \ldots, n_k$ is not harmonic, we will try to derive a contradiction.

By the inductive hypothesis $n_1, \ldots, n_{s-1}, n_{s+1}, \ldots, n_k$ is harmonic for each $s = 1, \ldots, k$. By Lemma 4 there exist integers $a_1^{(s)}, \ldots, a_{s-1}^{(s)}, a_{s+1}^{(s)}, \ldots, a_k^{(s)}$ such that $A_s = \{a_i^{(s)}((n_i, n_s))\}_{i=1, i\neq s}^{k}$ is a covering system. Let $I_s \subseteq \{1, \ldots, k\} \backslash \{s\}$ be a minimal set such that $B_s = \{a_i^{(s)}((n_i, n_s))\}_{i\in I_s}$ is covering. By Lemma 3 for each $i \in I_s$ we have

$$k - 2 \geqslant |I_s| - 1 \geqslant f((n_i, n_s)) \geqslant \log_2(n_i, n_s). \tag{3}$$

In the case $2 \leqslant k \leqslant 5$, $|I_k|$ is less than 5. By Lemma 2 $(n_i, n_k) = (n_j, n_k)$ for some $i, j \in I_k$, $i \neq j$. Let $d = (n_i, n_k) = (n_j, n_k)$, then $3 \geqslant k - 2 \geqslant f(d) \geqslant \log_2 d$. So we have

$$2 \leqslant |\{\{s, t\}: 1 \leqslant s < t \leqslant k \text{ and } (n_s, n_t) = d\}| < \sqrt{\frac{d+7}{8}} \leqslant \sqrt{\frac{8+7}{8}} < 2$$

which is impossible. This shows that $n_1, \ldots, n_k$ must be harmonic.

Now consider the case $k \geqslant 6$. Since B_s is covering, by Lemma 1 we have

$$\sum_{i\in I_s} \frac{1}{(n_i, n_s)} \geqslant 1 \quad \text{for } s = 1, \ldots, k.$$

Hence

$$\sum_{1\le i<j\le k, i\in I_j \text{ or } j\in I_i} \frac{1}{(n_i, n_j)} \ge \frac{1}{2}\sum_{s=1}^{k}\sum_{i\in I_s}\frac{1}{(n_i, n_s)} \ge \frac{k}{2}.$$

Let $D=\{d\in\mathbb{Z}^+: k-2\ge f(d)$ and $(n_i, n_j)=d$ for some $i, j=1, \ldots, k,\ i\ne j\}$. Obviously $1\notin D$. Note that $(n_i, n_s)\in D$ for every $i\in I_s$. Let $d_1<d_2<\cdots<d_m$ be the list of elements of D in the ascending order. Then $m=|D|\le\binom{k}{2}$ and

$$|\{\{i, j\}: 1\le i<j\le k \text{ and } (n_i, n_j)=d_t\}|\le\left[\sqrt{\frac{d_t+6}{8}}\right]$$

for each $t=1, \ldots, m$. (It is easy to check that $[\sqrt{(d+6)/8}]$ (the integral part of $\sqrt{(d+6)/8}$) is the largest integer less than $\sqrt{(d+7)/8}$.)

For convenience we let $d_{m+i}=d_m+i$ for $i=1, \ldots, \binom{k}{2}-m$. Since $k\ge 6$, $\binom{k}{2}=k(k-1)/2\ge 15$. Note that $\sqrt{(d+6)}/d$ is a decreasing function for $d\in\mathbb{Z}^+$. Since $d_i\ge i+1$ we have

$$\sum_{i=1}^{12}\left[\sqrt{\frac{d_i+6}{8}}\right]\Big/ d_i\le\sum_{i=1}^{12}\frac{1}{i+1}<2.2.$$

(If $d_i\ge 26$ then

$$\sqrt{\frac{d_i+6}{8}}\Big/ d_i\le\sqrt{\frac{26+6}{8}}\Big/ 26=\tfrac{1}{13}.)$$

If $d_{14}<26$, then $[\sqrt{(d_{13}+6)/8}]=[\sqrt{(d_{14}+6)/8}]=1$ and hence

$$\sum_{1\le i<j\le k, i\in I_j \text{ or } j\in I_i}\frac{1}{(n_i, n_j)}\le\sum_{d\in D}\frac{|\{\{i, j\}: 1\le i<j\le k \text{ and } (n_i, n_j)=d\}|}{d}\le\sum_{i=1}^{\binom{k}{2}}\frac{\left[\sqrt{\frac{d_i+6}{8}}\right]}{d_i}$$

$$<2.2+\frac{1}{14}+\frac{1}{15}+\sum_{i=15}^{\binom{k}{2}}\sqrt{\frac{(i+1)+6}{8}}\Big/(i+1)<2.35+\frac{1}{\sqrt{8}}\sum_{i=15}^{\binom{k}{2}}\frac{1}{\sqrt{i+1-6}}$$

$$<2.35+\frac{1}{\sqrt{2}}\left(\sqrt{\frac{k(k-1)}{2}-5}-\sqrt{10-1}\right)<2.35+\frac{1}{\sqrt{2}}\left(\frac{k-\frac{1}{2}}{\sqrt{2}}-3\right)<\frac{k}{2}.$$

If $d_{14}\ge 26$ then

$$\sum_{1\le i<j\le k, i\in I_j \text{ or } j\in I_i}\frac{1}{(n_i, n_j)}\le\sum_{i=1}^{\binom{k}{2}}\left[\sqrt{\frac{d_i+6}{8}}\right]\Big/ d_i<2.2$$

$$+\sqrt{\frac{d_{13}+6}{8}}\Big/ d_{13}+\sum_{i=14}^{\binom{k}{2}}\sqrt{\frac{d_i+6}{8}}\Big/ d_i$$

$$\le 2.2+\frac{\sqrt{20/8}}{14}+\sum_{i=14}^{\binom{k}{2}}\frac{\sqrt{(i+12+6)/8}}{i+12}$$

$$\le 2.2+0.15+\frac{1}{\sqrt{8}}\sum_{i=14}^{\binom{k}{2}}\frac{1}{\sqrt{i+12-6}}$$

(Note that for $i \geqslant 14$, $d_i \geqslant d_{14} + (i-14) \geqslant 26 + i - 14 = i + 12$.)

$$< 2.35 + \frac{1}{\sqrt{2}}\left(\sqrt{\frac{k(k-1)}{2} + 6} - \sqrt{20-1}\right)$$

$$< 2.35 + \frac{1}{\sqrt{2}}((k+\tfrac{1}{2})/\sqrt{2} - \sqrt{18}) < \frac{k}{2}.$$

(Apply Lemma 5 and observe that $k^2 - k + 12 \leqslant k^2 + k < (k + \frac{1}{2})^2$.)

From the above we see that

$$\sum_{1 \leqslant i<j \leqslant k, i \in I_j \text{ or } j \in I_i} \frac{1}{(n_i, n_j)} \geqslant \frac{k}{2} > \sum_{1 \leqslant i<j \leqslant k, i \in I_j \text{ or } j \in I_i} \frac{1}{(n_i, n_j)}.$$

This contradiction shows that $n_1, \ldots, n_k$ is harmonic.

The proof (by induction) is now completed. □

Remark. One of the referees noticed that the proof of Huhn and Megyesi actually requires a somewhat weaker condition than the one they stated. Namely for $n_1, \ldots, n_k$ to be harmonic, it suffices that for all $d < k$ the number of pairs $\{i, j\}$ $(1 \leqslant i < j \leqslant k)$ with $(n_i, n_j) \leqslant d$ is less than d. By our intuition, this new version seems to be independent of our theorem. Let $k > 26$. Suppose that $(n_i, n_j) = d$ holds for no pair $\{i, j\}$ $(1 \leqslant i < j \leqslant k)$ if $d = 1$, for exactly one pair if $2 \leqslant d \leqslant 25$, and for two pairs if $26 \leqslant d \leqslant 2^{k-2}$. Then the k-tuple $n_1, \ldots, n_k$ satisfies our condition but violates the weakened version of Huhn and Megyesi's condition. However, we are not able to construct such a tuple explicitly.

Corollary 1. *Let $n_1, \ldots, n_k$ be positive integers. If $\sum_{i=1}^{k} 1/n_i > 1$ then*

$$|\{\{i, j\}: (n_i, n_j) = d \text{ and } 1 \leqslant i < j \leqslant k\}| \geqslant \sqrt{\frac{d+7}{8}} \tag{4}$$

holds for some d satisfying $f(d) < k - 1$.

Proof. Note that by Lemma 1 $\sum_{i=1}^{k} 1/n_i > 1$ implies that $n_1, \ldots, n_k$ is not harmonic. □

Corollary 2. *Let $n_1, \ldots, n_k \in \mathbb{Z}^+$ and N be the lcm of $n_1, \ldots, n_k$. If*

$$|\{1 \leqslant i \leqslant k: n_i \nmid d\}| \leqslant f\left(\frac{N}{d}\right) - N\left(1 - \sum_{i=1}^{k} \frac{1}{n_i}\right) \tag{5}$$

holds for some d which divides N and does not equal N, then (4) is true for some $d \leqslant 2^{k-2}$.

Proof. Suppose that (2) holds for all $d \leqslant 2^{k-2}$. By the theorem $n_1, \ldots, n_k$ is harmonic. Let $\{a_i(n_i)\}_{i=1}^k$ be disjoint. Among $0, 1, \ldots, N-1$ there exist exactly $l = N - \sum_{i=1}^k N/n_i$ numbers not in $\bigcup_{i=1}^k a_i(n_i)$, so for some integers $b_1, \ldots, b_l$, the system $A = \{a_1(n_1), \ldots, a_k(n_k), b_1(N), \ldots, b_l(N)\}$ is a DCS. By Simpson [4] or Sun [6],

$$|\{1 \leqslant i \leqslant k : n_i \nmid d\}| + l \geqslant 1 + f\left(\frac{N}{d}\right)$$

where d may be any divisor of N except N. This contradicts the condition.

To end this paper, we make the following conjecture.

Conjecture. Let $n_1, \ldots, n_k$ be positive integers. If

$$|\{\{i, j\} : 1 \leqslant i < j \leqslant k \text{ and } (n_i, n_j) = d\}| < 2d - 1$$

for all $d \in \mathbb{Z}^+$ with $f(d) \leqslant k-2$, then $n_1, \ldots, n_k$ is harmonic.

We remark that we have verified the conjecture for $k \leqslant 6$.

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