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ON A PAIR OF ZETA FUNCTIONS

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Abstract. Let m be a positive integer, and define

$$\zeta_m(s)=\sum_{n=1}^{\infty}\frac{(-e^{2\pi i/m})^{\omega(n)}}{n^s}\quad\text{and}\quad \zeta_m^*(s)=\sum_{n=1}^{\infty}\frac{(-e^{2\pi i/m})^{\Omega(n)}}{n^s},$$

for $\Re(s)>1$, where $\omega(n)$ denotes the number of distinct prime factors of n, and $\Omega(n)$ represents the total number of prime factors of n (counted with multiplicity). In this paper we study these two zeta functions and related arithmetical functions. We show that

$$\sum_{\substack{n=1\\ n\text{ is squarefree}}}^{\infty}\frac{(-e^{2\pi i/m})^{\omega(n)}}{n}=0\quad\text{if } m>4,$$

which is similar to the known identity $\sum_{n=1}^{\infty}\mu(n)/n=0$ equivalent to the Prime Number Theorem. For $m>4$, we prove that

$$\zeta_m(1):=\sum_{n=1}^{\infty}\frac{(-e^{2\pi i/m})^{\omega(n)}}{n}=0\quad\text{and}\quad \zeta_m^*(1):=\sum_{n=1}^{\infty}\frac{(-e^{2\pi i/m})^{\Omega(n)}}{n}=0.$$

We also raise a hypothesis on the parities of $\Omega(n)-n$ which implies the Riemann Hypothesis.

1. Introduction

The Riemann zeta function $\zeta(s)$, defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\qquad\text{for } \Re(s)>1,$$

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plays a very important role in number theory. As Euler observed,

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\quad\text{for }\Re(s)>1.$$

(In such a product we always let p run over all primes.) It is well-known that $\zeta(s)$ for $\Re(s)>1$ can be continued analytically to a complex function which is holomorphic everywhere except for a simple pole at $s=1$ with residue 1. The famous Riemann Hypothesis asserts that if $0\leqslant\Re(s)\leqslant 1$ and $\zeta(s)=0$ then $\Re(s)=1/2$. The Prime Number Theorem $\pi(x)\sim x/\log x$ (as $x\to+\infty$) is actually equivalent to $\zeta(1+it)\neq 0$ for any nonzero real number t. (See, e.g., R. Crandall and C. Pomerance [CP, pp. 33-37].)

The Möbius fucntion μ defined on $\mathbb{Z}^+=\{1,2,3,\ldots\}$ is given by

$$\mu(n)=\begin{cases}1 & \text{if } n=1,\\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes},\\ 0 & \text{if } p^2\mid n \text{ for some prime } p.\end{cases}$$

It is well known that

$$\zeta(s)\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=1\quad\text{for }\Re(s)>1.$$

Also, either of $\sum_{n=1}^{\infty}\mu(n)/n=0$ and $\sum_{n\leqslant x}\mu(n)=o(x)$ is equivalent to the Prime Number Theorem. (Cf. T. M. Apostol [Ap, §3.9 and §4.1].)

The reader may consult [Ap] and [IR, pp. 18-21] for the basic knowledge of arithmetical functions and the theory of Dirichlet's convolution and Dirichlet series.

If $n\in\mathbb{Z}^+$ is squarefree, then $\mu(n)=(-1)^{\Omega(n)}$ depends on $\Omega(n)$ modulo 2, where $\Omega(n)$ denotes the number of all prime factors of n (counted with multiplicity). For the Liouville function $\lambda(n)=(-1)^{\Omega(n)}$, it is known that

$$\sum_{d\mid n}\lambda(d)=\begin{cases}1 & \text{if } n \text{ is a square},\\ 0 & \text{otherwise}.\end{cases}$$

(See, e.g., [Ap, p. 38].) J. van de Lune and R. E. Dressler [LD] showed that $\sum_{n=1}^{\infty}(-1)^{\omega(n)}/n=0$, where $\omega(n)$ denotes the number of distinct prime factors of n.

Now we give natural extensions of the functions $\mu(n)$, $\lambda(n)$ and $\zeta(s)$.

Definition 1.1. Let m be any positive integer. For $n\in\mathbb{Z}^+$ we set

$$\mu_m(n)=\begin{cases}(-e^{2\pi i/m})^{\omega(n)} & \text{if } n \text{ is squarefree},\\ 0 & \text{otherwise},\end{cases}\tag{1.1}$$

$$\nu_m(n) = (-e^{2\pi i/m})^{\omega(n)} \text{ and } \nu_m^*(n) = (-e^{2\pi i/m})^{\Omega(n)}. \tag{1.2}$$

For $\Re(s) > 1$ we define

$$\zeta_m(s) = \sum_{n=1}^{\infty} \frac{\nu_m(n)}{n^s} = \prod_p \left(1 - \frac{e^{2\pi i/m}}{p^s - 1}\right) \tag{1.3}$$

and

$$\zeta_m^*(s) = \sum_{n=1}^{\infty} \frac{\nu_m^*(n)}{n^s} = \prod_p \left(1 + \frac{e^{2\pi i/m}}{p^s}\right)^{-1}. \tag{1.4}$$

As ν_m^* is completely multiplicative, the second identity in (1.4) is easy and in fact known. Since ν_m is multiplicative, if $\Re(s) > 1$ then

$$\sum_{n=1}^{\infty} \frac{\nu_m(n)}{n^s} = \prod_p \sum_{k=0}^{\infty} \frac{\nu_m(p^k)}{p^{ks}} = \prod_p \left(1 - e^{2\pi i/m} \sum_{k=1}^{\infty} \frac{1}{p^{ks}}\right)$$

and hence the second equality in (1.3) does hold.

As $\mu_1 = \mu$, we call μ_m the generalized Möbius function of order m. Note that $\zeta_2(s) = \zeta_2^*(s) = \zeta(s)$. Also, $\nu_1^*(n) = (-1)^{\Omega(n)}$ is the Liouville function $\lambda(n)$, and

$$\zeta_1^*(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} = \prod_p \left(1 + \frac{1}{p^s}\right)^{-1} \quad \text{for } \Re(s) > 1.$$

(Cf. [Ap, pp. 229-230].) If we replace $-e^{2\pi i/m}$ in the definition of $\zeta_m^*(s)$ by $e^{2\pi i/m}$, the resulting function was shown to have an infinitely many valued analytic continuation into the half plane $\Re(s) > 1/2$ by T. Kubota and M. Yoshida [KY]. (See also [A] and [CD].) It seems that the zeta function $\zeta_m(s)$ introduced here has not been studied before.

Our first theorem is a basic result.

Theorem 1.1. *Let m be any positive integer.*

(i) *The function $\mu_m^*(n) = \mu_m(n)\lambda(n)$ is the inverse of $\nu_m^*(n)$ with respect to the Dirichlet convolution, and hence*

$$\zeta_m^*(s) \sum_{n=1}^{\infty} \frac{\mu_m^*(n)}{n^s} = 1 \qquad \textit{for } \Re(s) > 1. \tag{1.5}$$

For $\Re(s) > 1$ we also have

$$\zeta_m(s) \sum_{n=1}^{\infty} \frac{(1 + e^{2\pi i/m})^{\Omega(n)}}{n^s} = \zeta(s). \tag{1.6}$$

(ii) *If $m>4$, then*

$$\prod_p\left(1+\frac{e^{2\pi i/m}}{p}\right)^{-1}=0. \tag{1.7}$$

On the other hand,

$$\prod_p\left(1+\frac{e^{2\pi i/3}}{p}\right)=0\quad\text{and}\quad\lim_{x\to\infty}\left|\prod_{p\leqslant x}\left(1+\frac{e^{2\pi i/4}}{p}\right)\right|=\frac{\sqrt{15}}{\pi}. \tag{1.8}$$

Remark 1.1. If $\Re(s)>1$, then both $\zeta_m^*(s)$ and $\zeta_m(s)$ are nonzero by (1.5) and (1.6).

Our second theorem is a general result.

Theorem 1.2. *Let z be a complex number with $\Re(z)<1$. For $x\geqslant 2$ we have*

$$\sum_{n\leqslant x}\frac{z^{\omega(n)}}{n}=\mathcal{F}(z)(\log x)^z+c(z)+O((\log x)^{z-1}) \tag{1.9}$$

and

$$\sum_{\substack{n\leqslant x\\ n\text{ is squarefree}}}\frac{z^{\omega(n)}}{n}=\mathcal{G}(z)(\log x)^z+c_*(z)+O((\log x)^{z-1}), \tag{1.10}$$

where $c(z)$ and $c_*(z)$ are constants only depending on z, and

$$\begin{aligned}\mathcal{F}(z)=&\frac{1}{\Gamma(1+z)}\prod_p\left(1+\frac{z}{p-1}\right)\left(1-\frac{1}{p}\right)^z,\\ \mathcal{G}(z)=&\frac{1}{\Gamma(1+z)}\prod_p\left(1+\frac{z}{p}\right)\left(1-\frac{1}{p}\right)^z.\end{aligned}$$

If $|z|<2$, then for $x\geqslant 2$ we have

$$\sum_{n\leqslant x}\frac{z^{\Omega(n)}}{n}=\mathcal{H}(z)(\log x)^z+C(z)+O((\log x)^{z-1}), \tag{1.11}$$

where $C(z)$ is a constant only depending on z, and

$$\mathcal{H}(z)=\frac{1}{\Gamma(1+z)}\prod_p\left(1-\frac{z}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^z.$$

Theorem 1.2 obviously has the following consequence.

Corollary 1.1. *For any complex number z with $\Re(z) < 0$, we have*

$$\sum_{n=1}^{\infty} \frac{z^{\omega(n)}}{n} = c(z) \quad \text{and} \quad \sum_{\substack{n=1 \\ n \text{ is squarefree}}}^{\infty} \frac{z^{\omega(n)}}{n} = c_*(z). \tag{1.12}$$

If $|z| < 2$ and $\Re(z) < 0$, then

$$\sum_{n=1}^{\infty} \frac{z^{\Omega(n)}}{n} = C(z). \tag{1.13}$$

Theorem 1.3. *We have*

$$\sum_{n=1}^{\infty} \frac{\mu_5(n)}{n} = \sum_{n=1}^{\infty} \frac{\mu_6(n)}{n} = \cdots = 0. \tag{1.14}$$

Moreover, for any positive integer $m \neq 2$ we have

$$(\log x)^{e^{2\pi i/m}} \sum_{n \leqslant x} \frac{\mu_m(n)}{n} = \mathcal{G}(-e^{2\pi i/m}) + O\left(\frac{1}{\log x}\right) \quad (x \geqslant 2), \tag{1.15}$$

where $\mathcal{G}(z)$ is defined as in Theorem 1.2.

Remark 1.2. It is known that

$$\sum_{n \leqslant x} \frac{\mu_2(n)}{n} = \sum_{n \leqslant x} \frac{|\mu(n)|}{n} = \frac{6}{\pi^2} \log x + c + O\left(\frac{1}{\sqrt{x}}\right) \ (x \geqslant 2),$$

where $c = 1.04389\ldots$ (see, e.g., [BS, Lemma 14]). (1.15) with $m = 4$ implies that

$$\lim_{x \to \infty} \left| \sum_{n \leqslant x} \frac{\mu_4(n)}{n} \right| = |\mathcal{G}(-i)|.$$

After reading the first version of this paper, D. Broadhurst simplified $|\mathcal{G}(-i)|$ as $\sqrt{15(\sinh \pi)/\pi^3}$.

Theorem 1.4. *Let*

$$V_m(x) = \sum_{n \leqslant x} \frac{\nu_m(n)}{n} \quad \text{and} \quad V_m^*(x) = \sum_{n \leqslant x} \frac{\nu_m^*(n)}{n}$$

for $m \in \mathbb{Z}^+$ and $x \geqslant 2$. Then

$$\begin{aligned} V_3(x) =& \mathcal{F}(-e^{2\pi i/3})(\log x)^{(1-i\sqrt{3})/2} + c_3 + O\left(\frac{1}{\sqrt{\log x}}\right), \\ V_3^*(x) =& \mathcal{H}(-e^{2\pi i/3})(\log x)^{(1-i\sqrt{3})/2} + C_3 + O\left(\frac{1}{\sqrt{\log x}}\right), \end{aligned} \tag{1.16}$$

and

$$\begin{aligned} V_4(x) =&\mathcal{F}(-i)(\log x)^{-i} + c_4 + O\left(\frac{1}{\log x}\right),\\ V_4^*(x) =&\mathcal{H}(-i)(\log x)^{-i} + C_4 + O\left(\frac{1}{\log x}\right), \end{aligned} \tag{1.17}$$

where c_3, C_3, c_4, C_4 *are suitable constants. Also, for* $m = 5, 6, \ldots$ *we have* $V_m(x) = o(1)$ *and* $V_m^*(x) = o(1)$*, i.e.,*

$$\zeta_m(1) := \sum_{n=1}^{\infty} \frac{\nu_m(n)}{n} = 0 \quad and \quad \zeta_m^*(1) := \sum_{n=1}^{\infty} \frac{\nu_m^*(n)}{n} = 0. \tag{1.18}$$

Moreover, for $m = 1, 5, 6, \ldots$ *we have*

$$V_m(x)(\log x)^{e^{2\pi i/m}} = \mathcal{F}(-e^{2\pi i/m}) + O\left(\frac{1}{\log x}\right) \tag{1.19}$$

and

$$V_m^*(x)(\log x)^{e^{2\pi i/m}} = \mathcal{H}(-e^{2\pi i/m}) + O\left(\frac{1}{\log x}\right). \tag{1.20}$$

Remark 1.3. It seems that c_3 and C_3 are nonzero but $c_4 = 0$ (and probably also $C_4 = 0$). Broadhurst simplified $|\mathcal{H}(-i)|$ as $\sqrt{(\sinh \pi)\pi/15}$.

Theorem 1.1 is not difficult. Our proofs of Theorems 1.2-1.4 depend heavily on some results of A. Selberg [S] (see also H. Delange [D] and Theorem 7.18 of [MV, p. 231]) and the partial summation method via Abel's identity (see, [Ap, p. 77]).

Motivated by Theorem 1.4 we pose the following conjecture for further research.

Conjecture 1.1. *Both* $V_1(x) = \sum_{n\leqslant x}(-1)^{\omega(n)}/n$ *and* $V_1^*(x) = \sum_{n\leqslant x}(-1)^{\Omega(n)}/n$ *are* $O(x^{\varepsilon-1/2})$ *for any* $\varepsilon > 0$*. Also,* $|\sum_{n\leqslant x}(-2)^{\Omega(n)}| < x$ *for all* $x \geqslant 3078$*.*

Remark 1.4. It seems that $V_1(x)$ might be $O(\sqrt{(\log x)/x})$ or even $O(1/\sqrt{x})$. The asymptotic behavior of $\sum_{n\leqslant x} 2^{\Omega(n)}$ was investigated by E. Grosswald [G].

In 1958 C. B. Haselgrove [H] disproved Pólya's conjecture that $\sum_{n\leqslant x}\lambda(n) \leqslant 0$ for all $x \geqslant 2$; he also showed that Turán's conjecture $\sum_{n\leqslant x}\lambda(n)/n > 0$ for $x \geqslant 1$, is also false. It is known that the least integer $x > 1$ with $\sum_{n\leqslant x}\lambda(n) > 0$ is $906150257 < 10^9$ (cf. [L] and [BFM]). Along this line we propose the following new hypothesis.

Hypothesis 1.1. (i) *For any* $x \geqslant 5$, *we have*

$$S(x) := \sum_{n \leqslant x} (-1)^{n-\Omega(n)} > 0, \tag{1.21}$$

i.e.,

$$|\{n \leqslant x : \ \Omega(n) \equiv n \pmod 2\}| > |\{n \leqslant x : \ \Omega(n) \not\equiv n \pmod 2\}|.$$

Moreover,

$$S(x) > \sqrt{x} \ \ \textit{for all } x \geqslant 325, \ \ \textit{and} \ \ S(x) < 2.3\sqrt{x} \ \ \textit{for all } x \geqslant 1.$$

(ii) *For any* $x \geqslant 1$ *we have*

$$T(x) := \sum_{n \leqslant x} \frac{(-1)^{n-\Omega(n)}}{n} < 0. \tag{1.22}$$

Moreover,

$$T(x)\sqrt{x} < -1 \ \ \textit{for all } x \geqslant 2, \quad \textit{and} \ \ T(x)\sqrt{x} > -2.3 \ \ \textit{for all } x \geqslant 3.$$

Remark 1.5. We have verified parts (i) and (ii) of the hypothesis for x up to 10^{11} and 2×10^9 respectively. Below are values of $S(x)$ for some particular x:

$$S(10^2) = 14, \ S(10^3) = 54, \ S(10^4) = 186, \ S(10^5) = 464, \ S(10^6) = 1302,$$
$$S(10^7) = 5426, \ S(10^8) = 19100, \ S(10^9) = 62824, \ S(10^{10}) = 172250,$$
$$S(2 \cdot 10^{10}) = 252292, \ S(3 \cdot 10^{10}) = 292154, \ S(4 \cdot 10^{10}) = 263326,$$
$$S(5 \cdot 10^{10}) = 360470, \ S(6 \cdot 10^{10}) = 363152, \ S(7 \cdot 10^{10}) = 406260,$$
$$S(8 \cdot 10^{10}) = 559558, \ S(9 \cdot 10^{10}) = 491100, \ S(10^{11}) = 457588.$$

Example 1.1. For $x_1 = 17593752$ and $x_2 = 123579784$, we have $S(x_1) = 9574$ and $S(x_2) = 11630$. Via a computer we find that

$$\max_{1 \leqslant x \leqslant 10^{11}} \frac{S(x)}{\sqrt{x}} = \frac{S(x_1)}{\sqrt{x_1}} \approx 2.28252$$

and

$$\min_{324 < x \leqslant 10^{11}} \frac{S(x)}{\sqrt{x}} = \frac{S(x_2)}{\sqrt{x_2}} \approx 1.04618.$$

We are unable to prove or disprove Hypothesis 1.1, but we can show the following relatively easy result.

Theorem 1.5. (i) *We have*

$$S(x) = o(x) \quad and \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-\Omega(n)}}{n} = 0. \tag{1.23}$$

(ii) *If $S(x) > 0$ for all $x \geqslant 5$, or $T(x) < 0$ for all $x \geqslant 1$, then the Riemann Hypothesis holds.*

Note that

$$S(x) > 0 \iff |\{n \leqslant x:\ 2 \mid n - \Omega(n)\}| > \frac{x}{2}.$$

In view of Hypothesis 1.1, it is natural to ask whether

$$|\{n \leqslant x:\ m \mid n - \Omega(n)\}| > \frac{x}{m} \text{ for sufficiently large } x.$$

For $m = 3, 4, \ldots, 18, 20$ we have the following conjecture based on our computation.

Conjecture 1.2. *We have*

$$|\{n \leqslant x : 4 \mid n - \Omega(n)\}| < \frac{x}{4} \quad \textit{for any } x \geqslant s(4),$$

and for $m = 3, 5, 6, \cdots, 18, 20$ we have

$$|\{n \leqslant x : m \mid n - \Omega(n)\}| > \frac{x}{m} \quad \textit{for all } x \geqslant s(m),$$

where

$$\begin{aligned}
&s(3) = 62,\ s(4) = 1793193,\ s(5) = 187,\ s(6) = 14,\ s(7) = 6044,\ s(8) = 73,\\
&s(9) = 65,\ s(10) = 61,\ s(11) = 4040389,\ s(12) = 14,\ s(13) = 6943303,\\
&s(14) = 4174,\ s(15) = 77,\ s(16) = 99,\ s(17) = 50147927,\ s(18) = 73,\ s(20) = 61.
\end{aligned}$$

Remark 1.7. The case $m = 19$ seems much more sophisticated. Perhaps the sign of $|\{n \leqslant x : 19|(n - \Omega(n))\}| - x/19$ changes infinitely often.

As there is an extended Riemann Hypothesis for algebraic number fields, we propose the following extension of Hypothesis 1.1 based on our computation.

Hypothesis 1.2 (Extended Hypothesis). *Let K be any algebraic number field. Then we have*

$$S_K(x) := \sum_{N(A) \leqslant x} (-1)^{N(A)-\Omega(A)} > 0 \quad \textit{for all sufficiently large } x,$$

where A runs over all nonzero integral ideals in K whose norm (with respect to the field extension $K/\mathbb{Q}$) are not greater than x, and $\Omega(A)$ denotes the total number of prime ideals in the factorization of A as a product of prime ideals (counted with multiplicity). In particular, for $K = \mathbb{Q}(i)$ we have $S_K(x) > 0$ for all $x \geqslant 9$, and for $K = \mathbb{Q}(\sqrt{-2})$ we have $S_K(x) > 0$ for all $x \geqslant 132$.

Now we give one more conjecture based on our computation.

Conjecture 1.3. *For an integer* $d \equiv 0, 1 \pmod 4$ *define*

$$S_d(x) = \sum_{n \leqslant x} (-1)^{n-\Omega(n)} \left(\frac{d}{n}\right),$$

where $(\frac{d}{n})$ *denotes the Kronecker symbol. Then*

$$S_{-4}(x) < 0,\ S_{-7}(x) < 0,\ S_{-8}(x) < 0$$

for all $x \geqslant 1$*, and*

$$S_5(x) > 0 \text{ for } x \geqslant 11,\ \ S_{-3}(x) > 0 \text{ for } x \geqslant 406759,\ S_{-11}(x) > 0 \text{ for } x \geqslant 771862,$$

and

$$S_{24}(x) < 0 \text{ for } x \geqslant 90601, \quad \text{and}\ \ S_{28}(x) < 0 \text{ for } x \geqslant 629819.$$

We will show Theorems 1.1 and 1.2 in the next section, and prove Theorems 1.3-1.5 in Sections 3-5 respectively.

2. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. (i) Clearly $\mu_m^*(1)\nu_m^*(1) = 1 \cdot 1 = 1$. Let N be any integer greater than one, and let n be the product of all distinct prime factors of N. Then

$$\begin{aligned}\sum_{d \mid N} \mu_m^*(d)\nu_m^*\left(\frac{N}{d}\right) =& \sum_{d \mid n} e^{2\pi i \Omega(d)/m}(-e^{2\pi i/m})^{\Omega(n/d)+\Omega(N/n)} \\ =& (-1)^{\Omega(N/n)} e^{2\pi i \Omega(N)/m} \sum_{d \mid n} \mu\left(\frac{n}{d}\right) = 0.\end{aligned}$$

Therefore μ_m^* is the inverse of ν_m^* with respect to the Dirichlet convolution $*$.

Let $s = \sigma + it$ be a complex number with $\Re(s) = \sigma > 1$. Since

$$\max\left\{\left|\frac{\mu_m^*(n)}{n^s}\right|, \left|\frac{\nu_m^*(n)}{n^s}\right|\right\} \leqslant \left|\frac{1}{n^{\sigma+it}}\right| = \left|\frac{e^{-it\log n}}{n^\sigma}\right| = \frac{1}{n^\sigma}$$

for any $n \in \mathbb{Z}^+$, both $\sum_{n=1}^\infty \mu_m^*(n)/n^s$ and $\sum_{n=1}^\infty \nu_m^*(n)/n^s$ converge absolutely. Therefore

$$\zeta_m^*(s) \sum_{n=1}^\infty \frac{\mu_m^*(n)}{n^s} = \sum_{n=1}^\infty \frac{\mu_m^*(n)}{n^s} \sum_{n=1}^\infty \frac{\nu_m^*(n)}{n^s} = \sum_{n=1}^\infty \frac{\mu_m^* * \nu_m^*(n)}{n^s} = 1.$$

Now we prove (1.6). Since $|p^s| = p^\sigma > p \geqslant |1+e^{2\pi i/m}|$ for any prime p, we have

$$\prod_p \left(1 - \frac{1+e^{2\pi i/m}}{p^s}\right)^{-1} = \prod_p \sum_{k=0}^{\infty} \frac{(1+e^{2\pi i/m})^k}{p^{ks}} = \sum_{n=0}^{\infty} \frac{(1+e^{2\pi i/m})^{\Omega(n)}}{n^s}.$$

Note that

$$\begin{aligned}\zeta_m(s) =& \prod_p \frac{p^s - 1 - e^{2\pi i/m}}{p^s - 1} = \prod_p \frac{1-(1+e^{2\pi i/m})/p^s}{1-1/p^s} \\ =& \zeta(s) \prod_p \left(1 - \frac{1+e^{2\pi i/m}}{p^s}\right).\end{aligned}$$

So (1.6) does hold.

(ii) Now assume that $m > 4$. Then $2\pi/m < \pi/2$ and $0 < \cos(2\pi/m) < 1$. For any prime p we have

$$\left|1 + \frac{e^{2\pi i/m}}{p}\right| = \left|\left(1 + \frac{\cos(2\pi/m)}{p}\right) + i\frac{\sin(2\pi/m)}{p}\right| \geqslant 1 + \frac{\cos(2\pi/m)}{p}.$$

Therefore

$$\left|\prod_{p \leqslant x}\left(1 + \frac{e^{2\pi i/m}}{p}\right)\right| \geqslant \prod_{p\leqslant x}\left(1 + \frac{\cos(2\pi/m)}{p}\right) \geqslant 1 + \cos\frac{2\pi}{m}\sum_{p\leqslant x}\frac{1}{p},$$

and hence (1.7) holds since $\sum_p 1/p$ diverges (cf. [IR, p. 21]).

Finally we prove the first identity in (1.8). For any prime p, we have

$$\left|1 + \frac{e^{2\pi i/3}}{p}\right|^2 = 1 + 2\frac{\cos 2\pi/3}{p} + \frac{1}{p^2} = 1 - \frac{1}{p} + \frac{1}{p^2} = \frac{1+p^{-3}}{1+p^{-1}}.$$

Thus

$$\begin{aligned}\left|\prod_{p\leqslant x}\left(1 + \frac{e^{2\pi i/3}}{p}\right)\right|^2 =& \prod_{p\leqslant x}\left(1 + \frac{1}{p^3}\right)\cdot\prod_{p\leqslant x}\left(1+\frac{1}{p}\right)^{-1} \\ \leqslant& \prod_p\left(1+\frac{1}{p^3}\right)\cdot\left(1 + \sum_{p\leqslant x}\frac{1}{p}\right)^{-1}.\end{aligned}$$

Since $\sum_p 1/p$ diverges while $\sum_p 1/p^3$ converges, the first equality in (1.8) follows.

The second equality in (1.8) is easy. In fact, as $x \to \infty$,

$$\left|\prod_{p\leqslant x}\left(1+\frac{e^{2\pi i/4}}{p}\right)\right|^2=\prod_{p\leqslant x}\left|1+\frac{i}{p}\right|^2$$

has the limit

$$\prod_p\left(1+\frac{1}{p^2}\right)=\frac{\prod_p(1-1/p^2)^{-1}}{\prod_p(1-1/p^4)^{-1}}=\frac{\zeta(2)}{\zeta(4)}=\frac{\pi^2/6}{\pi^4/90}=\frac{15}{\pi^2}.$$

In view of the above, we have completed the proof of Theorem 1.1. □

To prove Theorem 1.2, we need two lemmas.

Lemma 2.1 (Selberg [S]). *Let z be a complex number. For $x \geqslant 2$ we have*

$$\sum_{n\leqslant x} z^{\omega(n)} = F(z)x(\log x)^{z-1} + O\left(x(\log x)^{\Re(z)-2}\right) \tag{2.1}$$

and

$$\sum_{\substack{n\leqslant x\\ n \text{ is squarefree}}} z^{\omega(n)} = G(z)x(\log x)^{z-1} + O\left(x(\log x)^{\Re(z)-2}\right), \tag{2.2}$$

where

$$F(z)=\frac{1}{\Gamma(z)}\prod_p\left(1+\frac{z}{p-1}\right)\left(1-\frac{1}{p}\right)^z$$

and

$$G(z)=\frac{1}{\Gamma(z)}\prod_p\left(1+\frac{z}{p}\right)\left(1-\frac{1}{p}\right)^z.$$

When $|z|<2$, for $x\geqslant 2$ we also have

$$\sum_{n\leqslant x} z^{\Omega(n)} = H(z)x(\log x)^{z-1} + O\left(x(\log x)^{\Re(z)-2}\right), \tag{2.3}$$

where

$$H(z)=\frac{1}{\Gamma(z)}\prod_p\left(1-\frac{z}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^z.$$

Lemma 2.2. *Let $a(1), a(2), \ldots$ be a sequence of complex numbers. Suppose that*

$$\sum_{n\leqslant x} a(n) = cx(\log x)^{z-1} + O(x(\log x)^{\Re(z)-2}) \quad (x \geqslant 2), \tag{2.4}$$

where c and z are (absolute) complex numbers with $z \neq 0$ and $\Re(z) \neq 1$. Then, for $x, y \geqslant 2$ we have

$$\begin{aligned}&\sum_{n\leqslant x} \frac{a(n)}{n} - \frac{c}{z}(\log x)^z - \Big(\sum_{n\leqslant y} \frac{a(n)}{n} - \frac{c}{z}(\log y)^z\Big)\\ &\quad= O((\log x)^{z-1}) + O((\log y)^{z-1}).\end{aligned} \tag{2.5}$$

Thus, if $\Re(z) < 1$ then

$$\sum_{n\leqslant x} \frac{a(n)}{n} = \frac{c}{z}(\log x)^z + c_z + O((\log x)^{\Re(z)-1}) \quad (x \geqslant 2), \tag{2.6}$$

where c_z is a suitable constant.

Proof. Let $A(t) = \sum_{n\leqslant t} a(n)$ for $t \geqslant 2$. By the partial summation formula,

$$\begin{aligned}\sum_{n\leqslant x} \frac{a(n)}{n} - \sum_{n\leqslant y} \frac{a(n)}{n} =&\frac{A(x)}{x} - \frac{A(y)}{y} - \int_y^x A(t)(t^{-1})' dt\\ =&\frac{A(x)}{x} - \frac{A(y)}{y} + \int_y^x \frac{A(t)}{t^2} dt.\end{aligned}$$

Note that

$$\frac{A(t)}{t} = c(\log t)^{z-1} + O((\log t)^{\Re(z)-2}) \quad \text{for } t \geqslant 2.$$

Clearly

$$\int_y^x \frac{(\log t)^{z-1}}{t} dt = \frac{(\log t)^z}{z}\bigg|_{t=y}^x = \frac{(\log x)^z - (\log y)^z}{z}$$

and

$$\int_y^x \frac{(\log t)^{\Re(z)-2}}{t} dt = \frac{(\log t)^{\Re(z)-1}}{\Re(z)-1}\bigg|_{t=y}^x = \frac{(\log x)^{\Re(z)-1} - (\log y)^{\Re(z)-1}}{\Re(z)-1}.$$

So the desired (2.5) follows from the above.

Now assume that $\Re(z) < 1$. For any $\varepsilon > 0$ we can find a positive integer N such that for $x, y \geqslant N$ the absolute value of the right-hand side of (2.5) is smaller than ε. Therefore, in view of (2.5) and Cauchy's convergence criterion, $\sum_{n\leqslant x} a(n)/n - c(\log x)^z/z$ has a finite limit c_z as $x \to \infty$. Letting $y \to \infty$ in (2.5) we immediately obtain (2.6). This ends the proof. □

Proof of Theorem 1.2. When $z=0$, (1.9)-(1.11) obviously hold with $c(0)=c_*(0)=C(0)=0$.

Now assume $z\neq 0$. As $\Gamma(1+z)=z\Gamma(z)$, we see that

$$\mathcal{F}(z)=\frac{F(z)}{z},\ \mathcal{G}(z)=\frac{G(z)}{z},\ \text{and}\ \mathcal{H}(z)=\frac{H(z)}{z},$$

where the functions F, G and H are given in Lemma 2.1. Combining Lemmas 2.1 and 2.2 we immediately get the desired (1.9)-(1.11). □

3. Proof of Theorem 1.3

We first present two lemmas.

Lemma 3.1. *Let $m\in\mathbb{Z}^+$ and $x\geqslant 1$. Then we have*

$$\sum_{n\leqslant x}\mu_m(n)\left\lfloor\frac{x}{n}\right\rfloor=\sum_{n\leqslant x}(1-e^{2\pi i/m})^{\omega(n)}. \tag{3.1}$$

Proof. We first claim that

$$\sum_{d\mid n}\mu_m(d)=(1-e^{2\pi i/m})^{\omega(n)} \tag{3.2}$$

for any $n\in\mathbb{Z}^+$. Clearly (3.2) holds for $n=1$. If $n=p_1^{a_1}\cdots p_k^{a_k}$ with $p_1,\ldots,p_k$ distinct primes and $a_1,\ldots,a_k\in\mathbb{Z}^+$, then

$$\sum_{d\mid n}\mu_m(d)=\sum_{I\subseteq\{1,\ldots,k\}}\mu_m\Big(\prod_{i\in I}p_i\Big)=\sum_{r=0}^{k}\binom{k}{r}(-e^{2\pi i/m})^r=(1-e^{2\pi i/m})^{\omega(n)}.$$

Observe that

$$\sum_{d\leqslant x}\mu_m(d)\left\lfloor\frac{x}{d}\right\rfloor=\sum_{d\leqslant x}\mu_m(d)\sum_{q\leqslant x/d}1=\sum_{dq\leqslant x}\mu_m(d)=\sum_{n\leqslant x}\sum_{d\mid n}\mu_m(d).$$

Combining this with (3.2) we immediately obtain (3.1). □

Lemma 3.2. *Let $m\in\mathbb{Z}^+$, $m\neq 2$, and $x\geqslant 2$. Then we have*

$$\sum_{n\leqslant x}\mu_m(n)\left\{\frac{x}{n}\right\}=o(x),\ \sum_{n\leqslant x}\nu_m(n)\left\{\frac{x}{n}\right\}=o(x),\ \sum_{n\leqslant x}\nu_m^*(n)\left\{\frac{x}{n}\right\}=o(x), \tag{3.3}$$

where $\{\alpha\}$ denotes the fractional part of a real number α.

Proof. As $m\neq 2$, $\Re(e^{2\pi i/m})=\cos\frac{2\pi}{m}\neq -1$. Applying (2.1)-(2.3) we obtain

$$\begin{aligned}\sum_{n\leqslant x}\mu_m(x)=&xG(-e^{2\pi i/m})(\log x)^{-e^{2\pi i/m}-1}+O\left(x(\log x)^{-\cos(2\pi/m)-2}\right)=o(x),\\ \sum_{n\leqslant x}\nu_m(x)=&xF(-e^{2\pi i/m})(\log x)^{-e^{2\pi i/m}-1}+O\left(x(\log x)^{-\cos(2\pi/m)-2}\right)=o(x),\\ \sum_{n\leqslant x}\nu^*_m(x)=&xH(-e^{2\pi i/m})(\log x)^{-e^{2\pi i/m}-1}+O\left(x(\log x)^{-\cos(2\pi/m)-2}\right)=o(x).\end{aligned}$$

(Note that $F(-1)=G(-1)=H(-1)=0$.)

Let w be any of the three functions μ_m, ν_m and ν^*_m. By the above, $W(x)=\sum_{n\leqslant x}w(n)=o(x)$. We want to show that

$$\Delta(x):=\sum_{n\leqslant x}w(n)\left\{\frac{x}{n}\right\}=o(x).$$

Clearly

$$r(u):=\sup_{t\geqslant u}\frac{|W(t)|}{t}\leqslant 1\quad\text{for } u\geqslant 1.$$

Also, $r(u)\to 0$ as $u\to\infty$.

Let $0<\varepsilon<1$. Then

$$\begin{aligned}|\Delta(x)|\leqslant&\Bigg|\sum_{n\leqslant\varepsilon x}w(n)\left\{\frac{x}{n}\right\}\Bigg|+\Bigg|\sum_{\varepsilon x<n\leqslant x}w(n)\left\{\frac{x}{n}\right\}\Bigg|\\ \leqslant&\varepsilon x+\Bigg|\sum_{\varepsilon x<n\leqslant x}(W(n)-W(n-1))\left\{\frac{x}{n}\right\}\Bigg|\\ \leqslant&\varepsilon x+\Bigg|\sum_{\varepsilon x<n<\lfloor x\rfloor}W(n)\left(\left\{\frac{x}{n}\right\}-\left\{\frac{x}{n+1}\right\}\right)\Bigg|\\ &+\left|W(\lfloor x\rfloor)\left\{\frac{x}{\lfloor x\rfloor}\right\}-W(\lfloor\varepsilon x\rfloor)\left\{\frac{x}{\lfloor\varepsilon x\rfloor+1}\right\}\right|.\end{aligned}$$

Note that

$$\left|W(\lfloor x\rfloor)\left\{\frac{x}{\lfloor x\rfloor}\right\}\right|=|W(\lfloor x\rfloor)|\frac{\{x\}}{\lfloor x\rfloor}\leqslant 1$$

and

$$\left|W(\lfloor\varepsilon x\rfloor)\left\{\frac{x}{\lfloor\varepsilon x\rfloor+1}\right\}\right|\leqslant|W(\lfloor\varepsilon x\rfloor)|\leqslant\lfloor\varepsilon x\rfloor\leqslant\varepsilon x.$$

Therefore

$$\begin{aligned}|\Delta(x)| \leqslant& 1+2\varepsilon x+\sum_{\varepsilon x<n<\lfloor x\rfloor}\frac{|W(n)|}{n}x\left|\left\{\frac{x}{n}\right\}-\left\{\frac{x}{n+1}\right\}\right|\\ \leqslant& 1+2\varepsilon x+xr(\varepsilon x)\sum_{\varepsilon x<n<\lfloor x\rfloor}\left|\frac{x}{n}-\frac{x}{n+1}-\left(\left\lfloor\frac{x}{n}\right\rfloor-\left\lfloor\frac{x}{n+1}\right\rfloor\right)\right|\\ \leqslant& 1+2\varepsilon x+xr(\varepsilon x)\sum_{\varepsilon x<n<\lfloor x\rfloor}\left(\left(\frac{x}{n}-\frac{x}{n+1}\right)+\left(\left\lfloor\frac{x}{n}\right\rfloor-\left\lfloor\frac{x}{n+1}\right\rfloor\right)\right)\\ \leqslant& 1+2\varepsilon x+xr(\varepsilon x)\left(2\frac{x}{\lfloor\varepsilon x\rfloor+1}-\frac{x}{\lfloor x\rfloor}-\left\lfloor\frac{x}{\lfloor x\rfloor}\right\rfloor\right)\end{aligned}$$

and hence

$$\frac{|\Delta(x)|}{x}\leqslant\frac{1}{x}+2\varepsilon+\frac{2}{\varepsilon}r(\varepsilon x).$$

It follows that

$$\limsup_{x\to\infty}\frac{|\Delta(x)|}{x}\leqslant 2\varepsilon. \tag{3.4}$$

As (3.4) holds for any given $\varepsilon\in(0,1)$, we must have $\Delta(x)=o(x)$ as desired. $\square$

Proof of Theorem 1.3. For $z=-e^{2\pi i/m}$ we have $\Re(z)=-\cos(2\pi/m)<1$ as $m\neq 2$. Combining (3.1) with (2.1), we obtain

$$\sum_{n\leqslant x}\mu_m(n)\left\lfloor\frac{x}{n}\right\rfloor=F(1+z)x(\log x)^z+O\left(x(\log x)^{-1-\cos(2\pi/m)}\right).$$

By Lemma 3.2,

$$\sum_{n\leqslant x}\mu_m(x)\left\{\frac{x}{n}\right\}=o(x).$$

Therefore

$$x\sum_{n\leqslant x}\frac{\mu_m(n)}{n}=\sum_{n\leqslant x}\mu_m(n)\left(\left\lfloor\frac{x}{n}\right\rfloor+\left\{\frac{x}{n}\right\}\right)=F(1+z)x(\log x)^z+o(x)$$

and hence

$$\sum_{n\leqslant x}\frac{\mu_m(n)}{n}=\mathcal{G}(z)(\log x)^z+o(1) \tag{3.5}$$

since $F(1+z)=G(z)/z=\mathcal{G}(z)$. Combining (3.5) with (1.10) and noting that $(\log x)^{z-1}\to 0$ as $x\to\infty$, we get $c_*(z)=0$. So (1.10) reduces to (1.15).

For $m=5,6,\dots$ we clearly have $\cos(2\pi/m)>0$ and hence (1.15) implies that $\sum_{n=1}^{\infty}\mu_m(n)/n=0$. This concludes the proof. $\square$

Remark 3.1. The way we prove (1.14) can be modified to show the equality

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}=0. \tag{3.6}$$

Since $\lambda=\nu_1^*$, we have $\sum_{n\leqslant x}\lambda(n)\{x/n\}=o(x)$ by Lemma 3.2. So it suffices to prove $\sum_{n\leqslant x}\lambda(n)\lfloor x/n\rfloor=o(x)$. In fact,

$$\begin{aligned}\sum_{d\leqslant x}\lambda(d)\left\lfloor\frac{x}{d}\right\rfloor=&\sum_{d\leqslant x}\lambda(d)\sum_{q\leqslant x/d}1=\sum_{dq\leqslant x}\lambda(d)=\sum_{n\leqslant x}\sum_{d\mid n}\lambda(d)\\ =&|\{1\leqslant n\leqslant x:\ n\text{ is a square}\}|=\lfloor\sqrt{x}\rfloor=o(x).\end{aligned}$$

4. Proof of Theorem 1.4

Lemma 4.1. *Let* $m\in\{1,5,6,\ldots\}$*. Then the series*

$$\zeta_m(1):=\sum_{n=1}^{\infty}\frac{\nu_m(n)}{n}\quad\textit{and}\quad\zeta_m^*(1):=\sum_{n=1}^{\infty}\frac{\nu_m^*(n)}{n}$$

converge. Moreover, we have

$$\lim_{s\to1+}\zeta_m(s)=\zeta_m(1)\quad\textit{and}\quad\lim_{s\to1+}\zeta_m^*(s)=\zeta_m^*(1). \tag{4.1}$$

Proof. Let $a(n)=\nu_m(n)$ for all $n\in\mathbb{Z}^+$, or $a(n)=\nu_m^*(n)$ for all $n\in\mathbb{Z}^+$. Set $A(x):=\sum_{n\leqslant x}a(n)$ for $x\geqslant1$, and $f_s(t)=t^{-s}$ for $s\geqslant1$ and $t\geqslant2$. By the partial summation formula, for $x\geqslant x_0\geqslant2$ we have

$$\sum_{x_0<n\leqslant x}a(n)f_s(n)=A(x)f_s(x)-A(x_0)f_s(x_0)-\int_{x_0}^{x}A(t)f_s'(t)dt$$

and hence

$$\sum_{x_0<n\leqslant x}\frac{a(n)}{n^s}=\frac{A(x)}{x^s}-\frac{A(x_0)}{x_0^s}+s\int_{x_0}^{x}\frac{A(t)}{t^{s+1}}dt. \tag{4.2}$$

In view of (2.1) or (2.3) with $z=-e^{2\pi i/m}$, there is a constant $c>0$ depending on m such that

$$|A(t)|\leqslant\frac{ct}{(\log t)^{\cos(2\pi/m)+1}}\quad\text{for all } t\geqslant2. \tag{4.3}$$

For any $s\geqslant1$, we have

$$\left|\frac{A(x)}{x^s}\right|\leqslant\frac{|A(x)|}{x}\leqslant\frac{c}{(\log x)^{\cos(2\pi/m)+1}},\quad\left|\frac{A(x_0)}{x_0^s}\right|\leqslant\frac{|A(x_0)|}{x_0}\leqslant\frac{c}{(\log x_0)^{\cos(2\pi/m)+1}}$$

and

$$\left|\int_{x_0}^{x}\frac{A(t)}{t^{s+1}}dt\right| \leqslant \int_{x_0}^{x}\frac{ct}{t^2}(\log t)^{-\cos(2\pi/m)-1}dt = \frac{c(\log t)^{-\cos(2\pi/m)}}{-\cos(2\pi/m)}\Bigg|_{t=x_0}^{x}$$
$$=\frac{c}{\cos(2\pi/m)}\left(\frac{1}{(\log x_0)^{\cos(2\pi/m)}}-\frac{1}{(\log x)^{\cos(2\pi/m)}}\right)$$

with the help of (4.3).

Let $\varepsilon > 0$. Since $\cos(2\pi/m) > 0$, by the above, there is an integer $N(\varepsilon) \geqslant 2$ such that if $x > x_0 \geqslant N(\varepsilon)$ then for any $s \geqslant 1$ we have

$$\left|\sum_{x_0<n\leqslant x}\frac{a(n)}{n^s}\right| \leqslant \left|\frac{A(x)}{x^s}\right| + \left|\frac{A(x_0)}{x_0^s}\right| + s\left|\int_{x_0}^{x}\frac{A(t)}{t^{s+1}}dt\right| \leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2}+s\varepsilon = (1+s)\varepsilon.$$

Therefore the series $\sum_{n=1}^{\infty} a(n)/n^s$ converges for any $s \geqslant 1$, in particular $\sum_{n=1}^{\infty} a(n)/n$ converges!

In view of the general properties of Dirichelt's series (cf. [T, p. 291]), we immediately have

$$\lim_{s\to 1+}\sum_{n=1}^{\infty}\frac{a(n)}{n^s} = \sum_{n=1}^{\infty}\frac{a(n)}{n}.$$

This concludes the proof. □

Lemma 4.2. *Let $m > 4$ be an integer. For $\Re(s) > 1$, we have*

$$\frac{d}{ds}\log\zeta_m(s) + e^{2\pi i/m}\frac{d}{ds}\log\zeta(s) = v(s) \tag{4.4}$$

and

$$\frac{d}{ds}\log\zeta_m^*(s) + e^{2\pi i/m}\frac{d}{ds}\log\zeta(s) = v^*(s), \tag{4.5}$$

where $v(s)$ is a suitable holomorphic function in the region $\Re(s) > \log_2(2\cos\frac{\pi}{m})$ and $v^*(s)$ is a suitable holomorphic functions in the half plane $\Re(s) > 1/2$.

Proof. (i) Equation (4.5) can be proved in a way similar to the proof of [KY, Theorem 1]. Let $z = -e^{2\pi i/m}$ and

$$v^*(s) = \sum_p(\log p)\sum_{k=2}^{\infty}\frac{z-z^k}{p^{ks}} \quad \text{for } \Re(s) > \frac{1}{2}.$$

If $\sigma = \Re(s) > 1/2$, then $|p^s| = p^\sigma \geqslant \sqrt{2}$ for any prime p, and hence

$$\sum_p(\log p)\left|\sum_{k=2}^{\infty}\frac{z-z^k}{p^{ks}}\right| \leqslant \sum_p(\log p)\sum_{k=2}^{\infty}\frac{2}{p^{k\sigma}} = \sum_p\frac{2\log p}{p^{2\sigma}(1-p^{-\sigma})}$$
$$\leqslant \frac{2}{1-1/\sqrt{2}}\sum_p\frac{\log p}{p^{2\sigma}} \leqslant \frac{2\sqrt{2}}{\sqrt{2}-1}\sum_{n=1}^{\infty}\frac{\log n}{n^{2\sigma}} < \infty.$$

So $v^*(s)$ is a holomorphic function in the region $\Re(s) > 1/2$.

When $\Re(s) > 1$, in view of Euler's product $\prod_p(1-p^{-s})^{-1} = \zeta(s)$ and the formula (1.4), we have

$$\begin{aligned}&\frac{d}{ds}\log\zeta_m^*(s) + e^{2\pi i/m}\frac{d}{ds}\log\zeta(s)\\ =&-\sum_p \frac{d}{ds}\log(1+e^{2\pi i/m}p^{-s}) - e^{2\pi i/m}\sum_p\frac{d}{ds}\log(1-p^{-s})\\ =&-\sum_p\frac{e^{2\pi i/m}(-\log p)p^{-s}}{1+e^{2\pi i/m}p^s} - e^{2\pi i/m}\sum_p\frac{-(-\log p)p^{-s}}{1-p^{-s}}\\ =&-\sum_p(\log p)\sum_{k=1}^{\infty}\left(\frac{-e^{2\pi i/m}}{p^s}\right)^k - e^{2\pi i/m}\sum_p(\log p)\sum_{k=1}^{\infty}\frac{1}{p^{sk}}\\ =&\sum_p(\log p)\sum_{k=2}^{\infty}\frac{z-z^k}{p^{sk}} = v^*(s).\end{aligned}$$

This proves (4.5).

(ii) To prove (4.4), we set $z = -e^{2\pi i/m}$ and

$$v(s) = (z^2-z)\sum_p\frac{\log p}{(p^s-1)(p^s-1+z)} \quad \text{for } \Re(s) > \log_2\left(2\cos\frac{\pi}{m}\right).$$

Note that

$$|1-z| = \sqrt{\left(1+\cos\frac{2\pi}{m}\right)^2 + \left(\sin\frac{2\pi}{m}\right)^2} = 2\cos\frac{\pi}{m} > 2\cos\frac{\pi}{4} = \sqrt{2}.$$

If $\sigma = \Re(s) > \log_2(2\cos\frac{\pi}{m})$, then for any prime p we have $|p^s| = p^\sigma \geqslant 2^\sigma > 2\cos\frac{\pi}{m} = |1-z|$ and hence $p^s - 1 + z \neq 0$. For each prime $p > 3$ and $\sigma = \Re(s) > \log_2(2\cos\frac{\pi}{m}) > \frac{1}{2}$, as $p^\sigma > (2^\sigma)^2 > 2$ we have

$$|p^s - 1| \geqslant p^\sigma - 1 > \frac{p^\sigma}{2};$$

also,

$$|p^s - 1 + z| \geqslant p^\sigma - |1-z| > \left(1-\frac{1}{\sqrt{2}}\right)p^\sigma$$

since $p^\sigma > (2^\sigma)^2 > |1-z|^2 > |1-z|\sqrt{2}$. As $\sum_p(\log p)/p^{2\sigma}$ converges for any $\sigma > 1/2$, we see that

$$\sum_p\frac{\log p}{(p^s-1)(p^s-1+z)}$$

converges absolutely in the half plane $\Re(s) > \log_2(2\cos\frac{\pi}{m})$. Therefore $v(s)$ is indeed a holomorphic function in the region $\Re(s) > \log_2(2\cos\frac{\pi}{m})$.

When $\Re(s) > 1$, using Euler's product $\prod_p (1-p^{-s})^{-1} = \zeta(s)$ and the formula (1.3) we get

$$\begin{aligned}
&\frac{d}{ds}\log\zeta_m(s) + e^{2\pi i/m}\frac{d}{ds}\log\zeta(s) \\
=&\sum_p \frac{d}{ds}\log\left(1-\frac{e^{2\pi i/m}}{p^s-1}\right) - e^{2\pi i/m}\sum_p \frac{d}{ds}\log(1-p^{-s}) \\
=&\sum_p \frac{(-z/(p^s-1)^2)p^s\log p}{1+z/(p^s-1)} + z\sum_p \frac{-(-\log p)p^{-s}}{1-p^{-s}} \\
=&\sum_p (\log p)\bigg(-\frac{zp^s}{(p^s-1)(p^s-1+z)} + \frac{z}{p^s-1}\bigg) \\
=&\sum_p \frac{(z^2-z)\log p}{(p^s-1)(p^s-1+z))} = v(s).
\end{aligned}$$

This proves (4.4).

In view of the above, we have completed the proof of Lemma 4.2. $\square$

Proof of Theorem 1.4. Let $m \in \{1, 3, 4, \dots\}$ and $z = -e^{2\pi i/m}$. When $m = 3$, (1.9) and (1.11) yield (1.16) with $c_3 = c(z)$ and $C_3 = C(z)$. In the case $m = 4$, (1.9) and (1.11) give (1.17) with $c_4 = c(-i)$ and $C_4 = C(-i)$.

Now we assume that $m = 1$ or $m > 4$. Note that $\Re(z) = -\cos(2\pi/m) < 0$. By (1.9) and (1.11), we have

$$V_m(x) = \mathcal{F}(z)(\log x)^z + c_m + O((\log x)^{z-1})$$

and

$$V_m^*(x) = \mathcal{H}(z)(\log x)^z + C_m + O((\log x)^{z-1}),$$

where $c_m = c(z)$ and $C_m = C(z)$. It follows that

$$\lim_{x\to\infty} V_m(x) = c_m \quad \text{and} \quad \lim_{x\to\infty} V_m^*(x) = C_m.$$

Also, (1.19) and (1.20) hold if $c_m = C_m = 0$. So it suffices to show $V_m(x) = o(1)$ and $V_m^*(x) = o(1)$. This holds for $m = 1$ since $\zeta_1^*(1) = \sum_{n=1}^\infty \lambda(n)/n = 0$ by (3.6) and $\zeta_1(1) = \sum_{n=1}^\infty (-1)^{\omega(n)}/n = 0$ by [LD].

Below we fix $m \in \{5, 6, \dots\}$. By Lemma 4.2, we have

$$\frac{d}{ds}\log\left(\zeta_m(s)\zeta(s)^{e^{2\pi i/m}}\right) = v(s) \quad \text{for } s > 1,$$

where $v(s)$ is a holomorphic function in the half plane $\Re(s) > \log_2(2\cos\frac{\pi}{m})$. Choose a number $s_0 > 1$. Then

$$\zeta_m(s)\zeta(s)^{e^{2\pi i/m}} = \zeta_m(s_0)\zeta(s_0)^{e^{2\pi i/m}} e^{\int_{s_0}^{s} v(t)dt}$$

for all $s > 1$, and hence

$$\lim_{s\to 1+}\zeta_m(s) = \lim_{s\to 1+}\left(\frac{\zeta(s_0)}{\zeta(s)}\right)^{e^{2\pi i/m}} \zeta_m(s_0)e^{\int_{s_0}^{1} v(t)dt} = 0$$

since $1 > \log_2(2\cos\frac{\pi}{m})$, $\Re(e^{2\pi i/m}) = \cos\frac{2\pi}{m} > 0$ and $\lim_{s\to 1+}\zeta(s) = \infty$. Similarly, by applying (4.5) in Lemma 4.2 we get $\lim_{s\to 1+}\zeta_m^*(s) = 0$. Combining these with Lemma 4.1 we finally obtain

$$\zeta_m(1) = \lim_{s\to 1+}\zeta_m(s) = 0 \quad \text{and} \quad \zeta_m^*(1) = \lim_{s\to 1+}\zeta_m^*(s) = 0$$

as desired. This concludes the proof of Theorem 1.4. □

5. Proof of Theorem 1.5

Proof of Theorem 1.5. Let $L(x) = \sum_{n\leqslant x}(-1)^{\Omega(n)}$. Formula (2.3) with $z = -1$ yields that $L(x) = o(x)$. Observe that

$$S(x) + L(x) = \sum_{n\leqslant x}((-1)^n + 1)(-1)^{\Omega(n)} = 2\sum_{m\leqslant x/2}(-1)^{\Omega(2m)} = -2L\left(\frac{x}{2}\right).$$

Therefore

$$S(x) = -L(x) - 2L\left(\frac{x}{2}\right) = o(x).$$

For any complex number s, obviously

$$\sum_{n\leqslant x}\frac{(-1)^{n-\Omega(n)}}{n^s} + \sum_{n\leqslant x}\frac{\lambda(n)}{n^s} = 2\sum_{\substack{n\leqslant x\\ 2\mid n}}\frac{\lambda(n)}{n^s} = -2\sum_{m\leqslant x/2}\frac{\lambda(m)}{(2m)^s}$$

and hence

$$\sum_{n\leqslant x}\frac{(-1)^{n-\Omega(n)}}{n^s} = -2^{1-s}\sum_{n\leqslant x/2}\frac{\lambda(n)}{n^s} - \sum_{n\leqslant x}\frac{\lambda(n)}{n^s}.$$

Since $\sum_{n\leqslant x}\lambda(n)/n = o(1)$ by (3.6), we get $\sum_{n\leqslant x}(-1)^{n-\Omega(n)}/n = o(1)$ and hence $\sum_{n=1}^{\infty}(-1)^{n-\Omega(n)}/n = 0$.

Let $\Re(s) > 1$. Note that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-\Omega(n)}}{n^s} = -(1+2^{1-s})\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} = -(1+2^{1-s})\frac{\zeta(2s)}{\zeta(s)}.$$

On the other hand, by the partial summation method, we have

$$\sum_{n\leqslant x}\frac{(-1)^{n-\Omega(n)}}{n^s}=\frac{S(x)}{x^s}+s\int_1^x\frac{S(t)}{t^{s+1}}dt$$

and hence

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-\Omega(n)}}{n^s}=s\int_1^{\infty}\frac{S(t)}{t^{s+1}}dt.$$

Therefore

$$-(1+2^{1-s})\frac{\zeta(2s)}{\zeta(s)}=s\int_1^{\infty}\frac{S(t)}{t^{s+1}}dt. \tag{5.1}$$

Let σ_c be the least real number such that the integral in (5.1) converges whenever $\Re(s)>\sigma_c$. By the above, $\sigma_c\leqslant 1$.

Suppose that $S(x)>0$ for all $x\geqslant 5$. In view of (5.1), by applying Landau's theorem (cf. [MV, Lemma 15.1] or Ex. 16 of [Ap, p.248]) we obtain

$$\lim_{s\to\sigma_c+}-\frac{1+2^{1-s}}{s}\cdot\frac{\zeta(2s)}{\zeta(s)}=\infty$$

and hence $\sigma_c\leqslant 1/2$ since $\zeta(s)$ has no real zeroes with $s>1/2$. (Note that $(1-2^{1-s})\zeta(s)=\sum_{n=1}^{\infty}(-1)^{n-1}/n^s\neq 0$ for all $s>0$ with $s\neq 1$.) So the right-hand side of (5.1) converges for $\Re(s)>1/2$ and hence so is the left-hand side of (5.1). Therefore $\zeta(s)\neq 0$ for $\Re(s)>1/2$, i.e., the Riemann Hypothesis holds.

Similarly, if $T(x)<0$ for all $x\geqslant 1$, then we get the Riemann Hypothesis by applying Landau's theorem.

So far we have completed the proof of Theorem 1.5. □

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