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ON $x(ax+1)+y(by+1)+z(cz+1)$ AND $x(ax+b)+y(ay+c)+z(az+d)$

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Abstract. In this paper we first investigate for what positive integers a,b,c every nonnegative integer n can be written as $x(ax+1)+y(by+1)+z(cz+1)$ with x,y,z integers. We show that (a,b,c) can be either of the following seven triples

$$(1,2,3),\ (1,2,4),\ (1,2,5),\ (2,2,4),\ (2,2,5),\ (2,3,3),\ (2,3,4),$$

and conjecture that any triple (a,b,c) among

$$(2,2,6),\ (2,3,5),\ (2,3,7),\ (2,3,8),\ (2,3,9),\ (2,3,10)$$

also has the desired property. For integers $0\leqslant b\leqslant c\leqslant d\leqslant a$ with $a>2$, we prove that any nonnegative integer can be written as $x(ax+b)+y(ay+c)+z(az+d)$ with x,y,z integers, if and only if the quadruple (a,b,c,d) is among

$$(3,0,1,2),\ (3,1,1,2),\ (3,1,2,2),\ (3,1,2,3),\ (4,1,2,3).$$

1. Introduction

Let $\mathbb{N}=\{0,1,2,\ldots\}$. Throughout this paper, for $f(x,y,z)\in\mathbb{Z}[x,y,z]$ we set

$$E(f(x,y,z))=\{n\in\mathbb{N}:\ n\neq f(x,y,z)\text{ for any }x,y,z\in\mathbb{Z}\}.$$

If $E(f(x,y,z))=\emptyset$, then we call $f(x,y,z)$ *universal over* $\mathbb{Z}$. The classical Gauss-Legendre theorem (cf. [N96, pp. 3-35]) states that

$$E(x^2+y^2+z^2)=\{4^k(8l+7):\ k,l\in\mathbb{N}\}.$$

Recall that those $T_x=x(x+1)/2$ with $x\in\mathbb{Z}$ are called *triangular numbers*. As $T_{-x-1}=T_x$, $T_{2x}=x(2x+1)$ and $T_{2x-1}=x(2x-1)$, we see that

$$\{T_x:\ x\in\mathbb{Z}\}=\{T_x:\ x\in\mathbb{N}\}=\{x(2x+1):\ x\in\mathbb{Z}\}.\tag{1.1}$$

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By the Gauss-Legendre theorem, any $n \in \mathbb{N}$ can be written as the sum of three triangular numbers (equivalently, $8n+3$ is the sum of three odd squares). In view of (1.1), this says that

$$\{x(2x+1)+y(2y+1)+z(2z+1):\ x,y,z\in\mathbb{Z}\}=\mathbb{N}. \tag{1.2}$$

Motivated by this, we are interested in finding all those $a,b,c\in\mathbb{Z}^+=\{1,2,3,\dots\}$ satisfying

$$\{x(ax+1)+y(by+1)+z(cz+1):\ x,y,z\in\mathbb{Z}\}=\mathbb{N}. \tag{1.3}$$

In the following theorem we determine all possible candidates $a,b,c\in\mathbb{Z}^+$ with (1.3) valid.

Theorem 1.1. *Let $a,b,c\in\mathbb{Z}^+$ with $a\leqslant b\leqslant c$. If $x(ax+1)+y(by+1)+z(cz+1)$ is universal over $\mathbb{Z}$, then (a,b,c) is among the following* 17 *triples:*

$$\begin{aligned}&(1,1,2),\ (1,2,2),\ (1,2,3),\ (1,2,4),\ (1,2,5),\\&(2,2,2),\ (2,2,3),\ (2,2,4),\ (2,2,5),\ (2,2,6),\\&(2,3,3),\ (2,3,4),\ (2,3,5),\ (2,3,7),\ (2,3,8),\ (2,3,9),\ (2,3,10).\end{aligned} \tag{1.4}$$

Remark 1.1. As proved by Liouville (cf. [D99, p. 23]),

$$\{2T_x+2T_y+T_z:\ x,y,z\in\mathbb{N}\}=\{2T_x+T_y+T_z:\ x,y,z\in\mathbb{N}\}=\mathbb{N}.$$

By [S15, Theorem 1.14], $T_x+T_y+2p_5(z)$ with $p_5(z)=z(3z-1)/2$ is also universal over $\mathbb{Z}$. These, together with (1.1) and (1.2), indicate that (1.3) holds for $(a,b,c)=(1,1,2),(1,2,2),(2,2,2),(2,2,3)$.

In Section 2 we will prove Theorem 1.1 as well as the following related result.

Theorem 1.2. (1.3) *holds if (a,b,c) is among the following* 7 *triples:*

$$(1,2,3),\ (1,2,4),\ (1,2,5),\ (2,2,4),\ (2,2,5),\ (2,3,3),\ (2,3,4).$$

In view of Theorems 1.1-1.2 and Remark 1.1, we have reduced the converse of Theorem 1.1 to our following conjecture.

Conjecture 1.1. (1.3) *holds if (a,b,c) is among the following six triples:*

$$(2,2,6),\ (2,3,5),\ (2,3,7),\ (2,3,8),\ (2,3,9),\ (2,3,10).$$

Remark 1.2. It is easy to show that (1.3) holds for $(a,b,c)=(2,3,7)$ if and only if for any $n\in\mathbb{N}$ we can write $168n+41$ as $21x^2+14y^2+6z^2$ with $x,y,z\in\mathbb{Z}$.

Inspired by (1.2), we want to know for what $a,b,c,d\in\mathbb{N}$ with $b\leqslant c\leqslant d\leqslant a$ we have

$$\{x(ax+b)+y(ay+c)+z(az+d):\ x,y,z\in\mathbb{Z}\}=\mathbb{N}. \tag{1.5}$$

We achieve this in the following theorem which will be proved in Section 3.

Theorem 1.3. *Let $a > 2$ be an integer and let $b, c, d \in \mathbb{N}$ with $b \leqslant c \leqslant d \leqslant a$. Then* (1.5) *holds if and only if (a, b, c, d) is among the following five quadruples:*

$$(3,0,1,2),\ (3,1,1,2),\ (3,1,2,2),\ (3,1,2,3),\ (4,1,2,3). \tag{1.6}$$

Remark 1.3. For $a \in \{1,2\}$ and $b, c, d \in \mathbb{N}$ with $b \leqslant c \leqslant d \leqslant a$, we can easily show that if (1.5) holds then (a, b, c, d) is among the following five quadruples:

$$(1,0,0,1),\ (1,0,1,1),\ (2,0,0,1),\ (2,0,1,1),\ (2,1,1,1).$$

The converse also holds since

$$x^2+y^2+2T_z,\ x^2+2T_y+2T_z,\ 2x^2+2y^2+T_z,\ 2x^2+T_y+T_z,\ T_x+T_y+T_z$$

are all universal over $\mathbb{Z}$ (cf. [S07]).

We also note some other universal sums. For example, we have

$$\{x^2+y(3y+1)+z(3z+2) : x, y, z \in \mathbb{Z}\} = \{x^2+y(4y+1)+z(4z+3) : x, y, z \in \mathbb{Z}\} = \mathbb{N}$$

which can be easily proved.

Based on our computation, we formulate the following conjecture for further research.

Conjecture 1.2. (i) *Any positive integer $n \neq 225$ can be written as $p(p-1)/2 + q(q-1)/2 + r(r-1)/2$ with p prime and $q, r \in \mathbb{Z}^+$.*

(ii) *Each $n \in \mathbb{N}$ can be written as $x^2 + y(3y+1)/2 + z(2z-1)$ with $x, y, z \in \mathbb{N}$. Also, any $n \in \mathbb{N}$ can be written as $x^2 + y(3y+1)/2 + z(5z+3)/2$ with $x, y, z \in \mathbb{N}$.*

(iii) *Every $n \in \mathbb{Z}^+$ can be written as $x^3 + y^2 + T_z$ with $x, y \in \mathbb{N}$ and $z \in \mathbb{Z}^+$. We also have $\{x^2 + y(y+1) + z(z^2+1) :\ x, y, z \in \mathbb{N}\} = \mathbb{N}$.*

(iv) *Any $n \in \mathbb{N}$ can be written as $x^4 + y(3y+1)/2 + z(7z+1)/2$ with $x, y, z \in \mathbb{Z}$.*

2. Proofs of Theorems 1.1-1.2

Proof of Theorem 1.1. For $x \in \mathbb{Z} \setminus \{0\}$, clearly $ax^2 + x \geqslant |x|(a|x|-1) \geqslant a-1$. As $1 = x(ax+1) + y(by+1) + z(cz+1)$ for some $x, y, z \in \mathbb{Z}$, we must have $a \leqslant 2$.

Case 1. $a = b = 1$.

As $1 \notin \{x(x+1) + y(y+1) :\ x, y \in \mathbb{Z}\}$, we must have $1 \in \{z(cz+1) :\ z \in \mathbb{Z}\}$ and hence $c = 2$. (Note that if $c > 2$ then $cz^2 + z \geqslant c-1 > 1$ for all $z \in \mathbb{Z} \setminus \{0\}$.)

Case 2. $a = 1 < b$.

If $b > 2$, then $y(by+1) \geqslant b-1 > 1$ and $z(cz+1) \geqslant c-1 > 1$ for all $y, z \in \mathbb{Z}\setminus\{0\}$. As $1 = x(x+1) + y(by+1) + z(cz+1)$ for some $x, y, z \in \mathbb{Z}$, we must have $b = 2$. It is easy to see that $4 \notin \{x(x+1) + y(2y+1) :\ x, y \in \mathbb{Z}\}$. If $c > 5$, then $z(cz+1) \geqslant c-1 > 4$ for all $z \in \mathbb{Z} \setminus \{0\}$. As $4 = x(x+1) + y(2y+1) + z(cz+1)$ for some $x, y, z \in \mathbb{Z}$, we must have $c \in \{2, 3, 4, 5\}$.

Case 3. $a=b=2$.

In view of (1.1),

$$5\not\in\{T_x+T_y:\ x,y\in\mathbb{N}\}=\{x(2x+1)+y(2y+1):\ x,y\in\mathbb{Z}\}.$$

If $c>6$, then $z(cz+1)\geqslant c-1>5$ for all $z\in\mathbb{Z}\setminus\{0\}$. As $5=x(2x+1)+y(2y+1)+z(cz+1)$ for some $x,y,z\in\mathbb{Z}$, we must have $c\in\{2,3,4,5,6\}$.

Case 4. $a=2<b$.

Clearly, $2\not\in\{x(2x+1):\ x\in\mathbb{Z}\}$. If $b>3$, then $y(by+1)\geqslant b-1>2$ and $z(cz+1)\geqslant c-1>2$ for all $y,z\in\mathbb{Z}\setminus\{0\}$. As $2=x(2x+1)+y(by+1)+z(cz+1)$ for some $x,y,z\in\mathbb{Z}$, we must have $b=3$. Note that $x(2x+1)+y(3y+1)\neq 9$ for all $x,y\in\mathbb{Z}$. If $c>10$, then $z(cz+1)\geqslant c-1>9$ for all $z\in\mathbb{Z}\setminus\{0\}$. Since $9=x(2x+1)+y(3y+1)+z(cz+1)$ for some $x,y,z\in\mathbb{Z}$, we must have $c\leqslant 10$. Note that $48\neq x(2x+1)+y(3y+1)+z(6z+1)$ for all $x,y,z\in\mathbb{Z}$. So $c\in\{3,4,5,7,8,9,10\}$.

In view of the above, we have completed the proof of Theorem 1.1. $\square$

Lemma 2.1. *Let u and v be integers with u^2+v^2 a positive multiple of* 5. *Then $u^2+v^2=x^2+y^2$ for some $x,y\in\mathbb{Z}$ with $5\nmid xy$.*

Proof. Let a be the 5-adic order of $\gcd(u,v)$, and write $u=5^au_0$ and $v=5^av_0$ with $u_0,v_0\in\mathbb{Z}$ not all divisible by 5. Choose $\delta,\varepsilon\in\{\pm1\}$ such that $u_0'\not\equiv 2v_0'\pmod 5$, where $u_0'=\delta u_0$ and $v_0'=\varepsilon v_0$. Clearly, $5^2(u_0^2+v_0^2)=u_1^2+v_1^2$, where $u_1=3u_0'+4v_0'$ and $v_1=4u_0'-3v_0'$. Note that u_1 and v_1 are not all divisible by 5 since $u_1\not\equiv v_1\pmod 5$. Continue this process, we finally write $u^2+v^2=5^{2a}(u_0^2+v_0^2)$ in the form x^2+y^2 with $x,y\in\mathbb{Z}$ not all divisible by 5. As $x^2+y^2=u^2+v^2\equiv 0\pmod 5$, we must have $5\nmid xy$. This concludes the proof. $\square$

With the help of Lemma 2.1, we are able to deduce the following result.

Lemma 2.2. *For any $n\in\mathbb{N}$ and $r\in\{6,14\}$, we can write $20n+r$ as $5x^2+5y^2+z^2$ with $x,y,z\in\mathbb{Z}$ and $2\nmid z$.*

Proof. As $20n+r\equiv r\equiv 2\pmod 4$, by the Gauss-Legendre theorem we can write $20n+r$ as $(2w)^2+u^2+v^2$ with $u,v,w\in\mathbb{Z}$ and $2\nmid uv$. If $(2w)^2\equiv -r\pmod 5$, then $u^2+v^2\equiv 2r\pmod 5$ and hence $u^2\equiv v^2\equiv r\pmod 5$. If $(2w)^2\equiv r\pmod 5$, then $u^2+v^2\equiv 2\pmod 4$ is a positive multiple of 5 and hence by Lemma 2.1 we can write it as s^2+t^2, where s and t are odd integers with $s^2\equiv -r\pmod 5$ and $t^2\equiv r\pmod 5$. If $5\mid w$, then one of u^2 and v^2 is divisible by 5 and the other is congruent to r modulo 5.

By the above, we can always write $20n+r=x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$, $2\nmid z$ and $z^2\equiv r\pmod 5$. Note that $x^2\equiv -y^2=(\pm 2y)^2\pmod 5$. Without loss of generality, we assume that $x\equiv 2y\pmod 5$ and hence $2x\equiv -y\pmod 5$. Set $\bar{x}=(x-2y)/5$ and $\bar{y}=(2x+y)/5$. Then

$$20n+r=x^2+y^2+z^2=5\bar{x}^2+5\bar{y}^2+z^2.$$

This concludes the proof. $\Box$

Remark 2.1. Let $n\in\mathbb{N}$ and $r\in\{6,14\}$. In contrast with Lemma 2.2, we conjecture that $20n+r$ can be written as $5x^2+5y^2+(2z)^2$ with $x,y,z\in\mathbb{Z}$ unless $r=6$ and $n\in\{0,11\}$, or $r=14$ and $n\in\{1,10\}$.

Lemma 2.3. (i) *For any positive integer $w=x^2+2y^2$ with $x,y\in\mathbb{Z}$, we can write w in the form u^2+2v^2 with $u,v\in\mathbb{Z}$ such that u or v is not divisible by* 3.

(ii) *$w\in\mathbb{N}$ can be written as $3x^2+6y^2$ with $x,y\in\mathbb{Z}$, if and only if $3\mid w$ and $w=u^2+2v^2$ for some $u,v\in\mathbb{Z}$.*

(iii) *Let $n\in\mathbb{N}$ with $6n+1$ not a square. Then, for any $\delta\in\{0,1\}$ we can write $6n+1$ as $x^2+3y^2+6z^2$ with $x,y,z\in\mathbb{Z}$ and $x\equiv\delta\pmod 2$.*

Remark 2.2. Part (i) first appeared in the middle of a proof given on page 173 of [JP] (see also [S15, Lemma 2.1] for other similar results). Parts (ii) and (iii) are Lemmas 3.1 and 3.3 of the author [S15].

Proof of Theorem 1.2. Let us fix a nonnegative integer n.

(i) As $24n+11\equiv 3\pmod 8$, by the Gauss-Legendre theorem there are odd integers u,v,w such that $24n+11=u^2+v^2+w^2=w^2+2\bar{u}^2+2\bar{v}^2$, where $\bar{u}=(u+v)/2$ and $\bar{v}=(u-v)/2$. As $2(\bar{u}^2+\bar{v}^2)\equiv 11-w^2\equiv 10\equiv 2\pmod 8$, we have $\bar{u}\not\equiv\bar{v}\pmod 2$. Without loss of generality, we assume that $2\mid\bar{u}$ and $2\nmid\bar{v}$. If $3\nmid\bar{v}$, then $\gcd(6,\bar{v})=1$. When $3\mid\bar{v}$, we have $3\nmid\bar{u}$ (since $w^2\not\equiv 11\pmod 3$), and $w^2+2\bar{v}^2$ is a positive multiple of 3, thus by Lemma 2.3(i) there are $s,t\in\mathbb{Z}$ with $3\nmid st$ such that $s^2+2t^2=w^2+2\bar{v}^2\equiv 3\pmod 8$ and hence $2\nmid st$. Anyway, $24n+11$ can be written as $r^2+2s^2+2t^2$ with $r,s,t\in\mathbb{Z}$ and $\gcd(6,t)=1$. Since $r^2+2s^2\equiv 11-2t^2\equiv 0\pmod 3$, by Lemma 2.3(ii) we may write $r^2+2s^2=3r_0^2+6s_0^2$ with $r_0,s_0\in\mathbb{Z}$. Since $3r_0^2+6s_0^2=r^2+2s^2\equiv 11-2t^2\equiv 9\pmod 8$, we have $r_0^2+2s_0^2\equiv 3\pmod 8$ and hence $2\nmid r_0s_0$. Write $s_0=2x+1$, r_0 or $-r_0$ as $4y+1$, and t or $-t$ as $6z+1$, where $x,y,z\in\mathbb{Z}$. Then

$$24n+11=6(2x+1)^2+3(4y+1)^2+2(6z+1)^2$$

and hence $n=x(x+1)+y(2y+1)+z(3z+1)$. This proves (1.3) for $(a,b,c)=(1,2,3)$.

(ii) By the Gauss-Legendre theorem, there are $s,t,v\in\mathbb{Z}$ such that $32n+14=(2s+1)^2+(2t+1)^2+(2v)^2$ and hence $16n+7=(s+t+1)^2+(s-t)^2+2v^2$. As one of $s+t+1$ and $s-t$ is even, we have $16n+7=(2u)^2+w^2+2v^2$ for some $u,w\in\mathbb{Z}$. Clearly $2\nmid w$, $2v^2\equiv 7-w^2\equiv 2\pmod 4$, and $4u^2\equiv 7-2v^2-w^2\equiv 4\pmod 8$. So, u,v,w are all odd. Note that $w^2\equiv 7-4u^2-2v^2\equiv 7-4-2=1\pmod{16}$ and hence $w\equiv\pm 1\pmod 8$. Now we can write u as $2x+1$, v or $-v$ as $4y+1$, w or $-w$ as $8z+1$, where x,y,z are integers. Thus

$$16n+7=4(2x+1)^2+2(4y+1)^2+(8z+1)^2$$

and hence $n=x(x+1)+y(2y+1)+z(4z+1)$. This proves (1.3) for $(a,b,c)=(1,2,4)$.

(iii) By Dickson [D39, pp. 112-113] (or [JKS]),

$$E(10x^2+5y^2+2z^2)=\{8q+3:\ q\in\mathbb{N}\}\cup\bigcup_{k,l\in\mathbb{N}}\{25^k(5l+1),25^k(5l+4)\}.$$

So, there are $u,v,w\in\mathbb{Z}$ such that $40n+17=10u^2+5v^2+2w^2$. Clearly, $2\nmid v$, $2u^2+2w^2\equiv 17-5v^2\equiv 4\pmod 8$ and hence $2\nmid uw$. Note that $2w^2\equiv 17\equiv 2\pmod 5$ and hence $w\equiv\pm1\pmod 5$. Thus, we can write $u=2x+1$, v or $-v$ as $4y+1$, and w or $-w$ as $10z+1$, where x,y,z are integers. Now we have

$$40n+17=10(2x+1)^2+5(4y+1)^2+2(10z+1)^2$$

and hence $n=x(x+1)+y(2y+1)+z(5z+1)$. This proves (1.3) for $(a,b,c)=(1,2,5)$.

(iv) By the Gauss-Legendre theorem, there are $u,v,w\in\mathbb{Z}$ with $2\nmid w$ such that

$$16n+5=(2u)^2+(2v)^2+w^2=2(u+v)^2+2(u-v)^2+w^2.$$

As $w^2\equiv1\not\equiv5\pmod 8$, both $u+v$ and $u-v$ are odd. Since $w^2\equiv5-2-2=1\pmod{16}$, we have $w\equiv\pm1\pmod 8$. Now we can write $u+v$ or $-u-v$ as $4x+1$, $u-v$ or $v-u$ as $4y+1$, and w or $-w$ as $8z+1$, where $x,y,z\in\mathbb{Z}$. Thus

$$16n+5=2(4x+1)^2+2(4y+1)^2+(8z+1)^2$$

and hence $n=x(2x+1)+y(2y+1)+z(4z+1)$. This proves (1.3) for $(a,b,c)=(2,2,4)$.

(v) By Lemma 2.2, there are $u,v,w\in\mathbb{Z}$ with $2\nmid w$ such that $20n+6=5u^2+5v^2+w^2$. Clearly, $u\not\equiv v\pmod 2$, $w^2\equiv1\pmod 5$ and hence $w\equiv\pm1\pmod 5$. Thus w or $-w$ has the form $10z+1$ with $z\in\mathbb{Z}$. Observe that

$$40n+12=10u^2+10v^2+2w^2=5(u+v)^2+5(u-v)^2+2(10z+1)^2.$$

As $u+v$ and $u-v$ are both odd, we may write $u+v$ or $-u-v$ as $4x+1$, and $u-v$ or $v-u$ as $4y+1$, where x and y are integers. Then

$$40n+12=5(4x+1)^2+5(4y+1)^2+2(10z+1)^2$$

and hence $n=x(2x+1)+y(2y+1)+z(5z+1)$. This proves (1.3) for $(a,b,c)=(2,2,5)$.

(vi) By Dickson [D39, pp. 112-113],

$$E(x^2+y^2+3z^2)=\{9^k(9l+6):\ k,l\in\mathbb{N}\}.$$

So there are $u,v,w\in\mathbb{Z}$ such that $24n+7=u^2+v^2+3w^2$. As $u^2+v^2\not\equiv7\pmod 4$, we have $2\nmid w$ and hence $s=(u+v)/2\in\mathbb{Z}$ and $t=(u-v)/2\in\mathbb{Z}$.

Now $24n+7=2s^2+2t^2+3w^2$. As $2(s^2+t^2)\equiv 7-3w^2\equiv 4 \pmod 8$, we have $s^2+t^2\equiv 2 \pmod 4$ and hence $2\nmid st$. Note that $s^2+t^2\equiv (7-3)/2=2 \pmod 3$ and hence $3\nmid st$. Now we can write w or $-w$ as $4x+1$, s or $-s$ as $6y+1$, t or $-t$ as $6z+1$, where x,y,z are integers. Then

$$24n+7=3(4x+1)^2+2(6y+1)^2+2(6z+1)^2$$

and hence $n=x(2x+1)+y(3y+1)+z(3z+1)$. This proves (1.3) for $(a,b,c)=(2,3,3)$.

(vii) Note that $48n+13\equiv 1 \pmod 6$ but $48n+13\not\equiv 1 \pmod 8$. By Lemma 2.3(iii), there are $u,v,w\in\mathbb{Z}$ such that $48n+13=6u^2+(2v)^2+3w^2$. Clearly, $2\nmid w$ and $3\nmid v$. As $6u^2\equiv 13-3\equiv 6 \pmod 4$, we must have $2\nmid u$. Since $4v^2\equiv 13-6u^2-3w^2\equiv 4 \pmod 8$, we have $2\nmid v$. Observe that

$$3w^2\equiv 13-6u^2-4v^2\equiv 13-6-4=3 \pmod{16}$$

and hence $w\equiv \pm 1 \pmod 8$. Now we can write u or $-u$ as $4x+1$, v or $-v$ as $6y+1$, and w or $-w$ as $8z+1$, where $x,y,z\in\mathbb{Z}$. Thus

$$48n+13=6(4x+1)^2+4(6y+1)^2+3(8z+1)^2$$

and hence $n=x(2x+1)+y(3y+1)+z(4z+1)$. This proves (1.3) for $(a,b,c)=(2,3,4)$.

So far we have completed the proof of Theorem 1.2. $\square$

3. Proof of Theorems 1.3

Lemma 3.1. *For any positive integer n, we can write $6n+1$ as $x^2+y^2+2z^2$ with $x,y,z\in\mathbb{Z}$ and $3\nmid xyz$.*

Remark 3.1. This is [S16, Lemma 4.3(ii)] proved by the author with the help of a result in [CL]. Combining it with Lemma 2.3(ii), for any $n\in\mathbb{Z}^+$ and $\delta\in\{0,1\}$ we can write $6n+1$ as $x^2+3y^2+6z^2$ with $x,y,z\in\mathbb{Z}$ and $x\equiv\delta \pmod 2$, which extends Lemma 2.3(iii) and confirms a conjecture in [S15, Remark 3.4].

Proof of Theorem 1.3. (i) If $|x|\geqslant 2$, then

$$x(ax+b)\geqslant |x|(a|x|-b)\geqslant 2(2a-b)\geqslant 2a,$$

and similarly $x(ax+c)\geqslant 2a$ and $x(ax+d)\geqslant 2a$. So, if (1.5) holds then we must have

$$\{0,1,\ldots,2a-1\}\subseteq\{x(ax+b)+y(ay+c)+z(az+d):\ x,y,z\in\{0,\pm1\}\}$$

and hence

$$2a\leqslant|\{x(ax+b)+y(ay+c)+z(az+d):\ x,y,z\in\{0,\pm1\}\}|\leqslant 3^3=27.$$

Note that $a \in \{3,4,\ldots,13\}$ and $0 \leqslant b \leqslant c \leqslant d \leqslant a$. Via a computer we find that if (a,b,c,d) is not among the five quadruples in (1.6) then one of $1,2,\ldots,17$ cannot be written as $x(ax+b)+y(ay+c)+z(az+d)$ with $x,y,z \in \mathbb{Z}$. For example, $x(4x+2)+y(4y+2)+z(4z+3) \neq 17$ for any $x,y,z \in \mathbb{Z}$. This proves the "only if" part of Theorem 1.3.

(ii) Now we turn to prove the "if" part of Theorem 1.3. Let us fix a nonnegative integer n.

(a) By [S15, Theorem 1.7(iv)], there are $u,v,x \in \mathbb{Z}$ such that $12n+5 = u^2+v^2+36x^2$. Clearly $u \not\equiv v \pmod 2$ and $3 \nmid uv$. Without loss of generality, we assume that $u \equiv \pm 1 \pmod 6$ and $v \equiv \pm 2 \pmod 6$. We may write u or $-u$ as $6y+1$, and v or $-v$ as $6z+2$, where y and z are integers. Thus

$$12n+5 = 36x^2+(6y+1)^2+(6z+2)^2$$

and hence $n = 3x^2+y(3y+1)+z(3z+2)$. This proves (1.5) for $(a,b,c,d) = (3,0,1,2)$.

(b) Let $\delta \in \{0,1\}$. By the Gauss-Legendre theorem, $12n+6+3\delta$ can be written as the sum of three squares. In view of [S16, Lemma 2.2], there are $u,v,w \in \mathbb{Z}$ with $3 \nmid uvw$ such that $12n+6+3\delta = u^2+v^2+w^2$. Clearly, u,v,w are neither all odd nor all even. Without loss of generality, we assume that $2 \nmid u$ and $2 \mid w$. Then $v \not\equiv \delta \pmod 2$. Obviously, $u \equiv \pm 1 \pmod 6$, $v \equiv \pm(1+\delta) \pmod 6$ and $w \equiv \pm 2 \pmod 6$. Thus we may write u or $-u$ as $6x+1$, v or $-v$ as $6y+1+\delta$, and w or $-w$ as $6z+2$, where $x,y,z \in \mathbb{Z}$. Therefore,

$$12n+6+3\delta = (6x+1)^2+(6y+1+\delta)^2+(6z+2)^2$$

and hence $n = x(3x+1)+y(3y+1+\delta)+z(3z+2)$. This proves (1.5) for $(a,b,c,d) = (3,1,1,2),(3,1,2,2)$.

(c) By Lemma 3.1, there are $u,v,w \in \mathbb{Z}$ with $3 \nmid uvw$ such that $6n+7 = u^2+v^2+2w^2$ and hence $12n+14 = (u+v)^2+(u-v)^2+(2w)^2$. As $(u+v)^2+(u-v)^2 \equiv 2 \pmod 4$, both $u+v$ and $u-v$ are odd. Since $(u+v)^2+(u-v)^2 \equiv 14-1 \equiv 1 \pmod 3$, without loss of generality we may assume that $u+v \equiv \pm 1 \pmod 6$ and $u-v \equiv 3 \pmod 6$. Now we may write $u+v$ or $-u-v$ as $6x+1$, w or $-w$ as $3y+1$, and $u-v$ as $6z+3$, where x,y,z are integers. Then

$$12n+14 = (6x+1)^2+(6y+2)^2+(6z+3)^2$$

and hence $n = x(3x+1)+y(3y+2)+z(3z+3)$. This proves (1.5) for $(a,b,c,d) = (3,1,2,3)$.

(d) As $16n+14 \equiv 2 \pmod 4$, by the Gauss-Legendre theorem $16n+14 = u^2+v^2+w^2$ for some $u,v,w \in \mathbb{Z}$ with $2 \nmid uv$ and $2 \mid w$. Since $w^2 \equiv 14-u^2-v^2 \equiv 12 \equiv 4 \pmod 8$, $w/2$ or $-w/2$ has the form $4y+1$ with $y \in \mathbb{Z}$. Thus $u^2+v^2 \equiv 14-4(w/2)^2 \equiv 10 \pmod{16}$. Without loss of generality, we assume that $u \equiv \pm 1 \pmod 8$ and $v \equiv \pm 3 \pmod 8$. Write u or $-u$ as $8x+1$, and v or $-v$ as $8z+3$, where $x, z \in \mathbb{Z}$. Then

$$16n + 14 = (8x+1)^2 + (8y+2)^2 + (8z+3)^2$$

and hence $n = x(4x+1)+y(4y+2)+z(4z+3)$. This proves (1.5) for $(a,b,c,d) = (4,1,2,3)$.

In view of the above, we have completed the proof of Theorem 1.3. □

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