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REFINING LAGRANGE'S FOUR-SQUARE THEOREM

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Abstract. Lagrange's four-square theorem asserts that any $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ can be written as the sum of four squares. This can be further refined in various ways. We show that any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x+y+z$ (or $x+2y$, or $x+y+2z$) is a square (or a cube). We also prove that any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z)$ is a square, whenever $P(x, y, z)$ is among the polynomials

$$x,\ 2x,\ x-y,\ 2x-2y,\ a(x^2-y^2)\ (a=1,2,3),\ x^2-3y^2,\ 3x^2-2y^2,$$
$$x^2+ky^2\ (k=2,3,5,6,8,12),\ (x+4y+4z)^2+(9x+3y+3z)^2,$$
$$x^2y^2+y^2z^2+z^2x^2,\ x^4+8y^3z+8yz^3,\ x^4+16y^3z+64yz^3.$$

We also pose some conjectures for further research; for example, our 1-3-5-Conjecture states that any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x+3y+5z$ is a square.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of all natural numbers (nonnegative integers). Lagrange's four-square theorem (cf. [N96, pp. 5-7]) states that any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{N}$.

It is known that for any $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ the set

$$E(a, b, c) := \{n \in \mathbb{N} :\ n \neq ax^2+by^2+cz^2 \text{ for any } x, y, z \in \mathbb{Z}\} \tag{1.1}$$

is not only nonempty but also infinite. A classical theorem of Gauss and Legendre (cf. [N96, p. 23] or [MW, p. 42]) asserts that

$$E(1, 1, 1) = \{4^k(8l+7) :\ k, l \in \mathbb{N}\}. \tag{1.2}$$

In this paper we study various refinements of Lagrange's theorem.

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Theorem 1.1. *Let $a \in \{1,4\}$ and $m \in \{4,5,6\}$. Then, any $n \in \mathbb{N}$ can be written as $ax^m + y^2 + z^2 + w^2$ with $x,y,z,w \in \mathbb{N}$.*

Remark 1.1. See [S16, A270969, A273915 and A273429] for related data; for example,

$$71 = 1^4 + 3^2 + 5^2 + 6^2,\ 240 = 2^5 + 0^2 + 8^2 + 12^2 \text{ and } 624 = 2^6 + 4^2 + 12^2 + 20^2.$$

In addition, we conjecture that any $n \in \mathbb{N}$ can be written as $x^2 + y^3 + z^4 + 2w^4$ with $x,y,z,w \in \mathbb{N}$ (cf. [S16, A262827]) and that each $n \in \mathbb{N}$ can be written as $x^5 + y^4 + z^2 + 3w^2$ with $x,y,z,w \in \mathbb{N}$ (cf. [S16, A273917]).

For convenience we introduce the following definition.

Definition 1.1. A polynomial $P(x,y,z,w)$ with integer coefficients is called a *suitable* polynomial if any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ such that $P(x,y,z,w)$ is a square.

Theorem 1.1 with $a = 1,4$ and $m = 4$ indicates that both x and $2x$ are suitable. For any squarefree integer $a \geqslant 3$, the polynomial ax is not suitable since $7 \neq (ax^2)^2 + y^2 + z^2 + w^2$ for all $x,y,z,w \in \mathbb{N}$.

Theorem 1.2. (i) *The polynomials $x-y$ and $2(x-y)$ are both suitable.*

(ii) *Let $c \in \{1,2,4\}$. Then, any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{Z})$ with $x+y = ct^3$ for some $t \in \mathbb{Z}$.*

(iii) *Let $d \in \{1,2\}$ and $m \in \{2,3\}$. Then, any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{Z}$ such that $x+2y = dt^m$ for some $t \in \mathbb{Z}$.*

(iv) *Any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{Z}$ such that $xy+yz+zw+wx = (x+z)(y+w)$ is a square.*

(v) *Each $n \in \mathbb{Z}^+$ can be written as $4^k(1+4x^2+y^2)+z^2$ with $k,x,y,z \in \mathbb{N}$.*

Remark 1.2. We even conjecture that any $n \in \mathbb{N}$ not of the form $2^{6k+3} \times 7$ $(k \in \mathbb{N})$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{N})$ with $x-y$ a cube, and that any $n \in \mathbb{Z}^+$ can be written as $4^k(1+4x^2+y^2)+z^2$ $(k,x,y,z \in \mathbb{N})$ with $x \leqslant y$ (or $x \leqslant z$).

Theorem 1.3. (i) *Let $m \in \{2,3\}$ and $(c,d) \in S(m)$, where*

$$S(3) = \{(1,1),(1,2),(2,1),(2,2),(2,4)\} \ \text{ and } \ S(2) = S(3) \cup \{(2,3),(2,6)\}.$$

Then any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{Z}$ such that $x+y+cz = dt^m$ for some $t \in \mathbb{Z}$.

(ii) *If any odd integer $n > 2719$ can be represented by $x^2+y^2+10z^2$ with $x,y,z \in \mathbb{Z}$, then each $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{Z})$ with $x+3y$ a square.*

(iii) *If any integer $n \geqslant 1190$ not divisible by 16 can be written as $x^2+10y^2+(2z^2+5^3r^4)/7$ with $r \in \{0,1,2,3\}$ and $x,y,z \in \mathbb{Z}$, then any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{Z})$ with $x+3y+5z$ a square.*

Remark 1.3. Concerning the condition of Theorem 1.3(ii), in 1916 S. Ramanujan [R] conjectured that any odd integer $n > 2719$ can be represented by $x^2+y^2+10z^2$ with $x, y, z \in \mathbb{Z}$, and this was proved by K. Ono and K. Soundararajan [OS] in 1997 under the GRH (Generalized Riemann Hypothesis). As for part (iii) of Theorem 1.3, we guess that any integer $n \geqslant 1190$ not divisible by 16 can be written as $x^2 + 10y^2 + (2z^2 + 5^3r^4)/7$ with $r \in \{0, 1, 2, 3\}$ and $x, y, z \in \mathbb{Z}$.

Theorem 1.4. (i) *Any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z) = 0$, whenever $P(x, y, z)$ is among the polynomials*

$$\begin{gathered} x(x-y),\ x(x-2y),\ (x-y)(x-2y),\ (x-y)(x-3y), \\ x(x+y-z),\ (x-y)(x+y-z),\ (x-2y)(x+y-z). \end{gathered} \tag{1.3}$$

(ii) *Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $(2x - 3y)(x + y - z) = 0$, provided that*

$$\{x^2 + y^2 + 13z^2 :\ x, y, z \in \mathbb{N}\} \supseteq \{8q + 5 :\ q \in \mathbb{N}\}. \tag{1.4}$$

(iii) *Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $z > 0$ such that $(x - y)z$ is a square.*

(iv) *Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y > 0$ such that $x + 4y + 4z$ and $9x + 3y + 3z$ are the two legs of a right triangle with positive integer sides.*

(v) *Each positive integer can be written as $x^2+y^2+z^2+w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $x^2y^2+y^2z^2+z^2x^2$ is a square. Also, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{Z}$ and $w \in \mathbb{Z}^+$ such that $x^2y^2 + 4y^2z^2 + 4z^2x^2$ is a square.*

(vi) *Any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $x^4 + 8y^3z + 8yz^3$ is a fourth power. Also, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $x^4 + 16y^3z + 64yz^3$ is a fourth power.*

Remark 1.4. Theorem 1.4(i) implies that xy, $2xy$ and $(x^2 + y^2)(x^2 + z^2)$ are all suitable. It seems that (1.4) does hold. We conjecture that $P(x, y, z)$ in Theorem 1.4(i) may be replaced by any of the following polynomials

$$\begin{aligned} &(x-y)(x+y-3z),\ (x-y)(x+2y-z),\ (x-y)(x+2y-2z), \\ &(x-y)(x+2y-7z),\ (x-y)(x+3y-3z),\ (x-y)(x+4y-6z), \\ &(x-y)(x+5y-2z),\ (x-2y)(x+2y-z),\ (x-2y)(x+2y-2z), \\ &(x-2y)(x+3y-3z),\ (x+y-z)(x+2y-2z),\ (x-y)(x+y+3z-3w), \\ &(x-y)(x+3y-z-5w),\ (x-y)(3x+3y-3z-5w), \\ &(x-y)(3x+5y-3z-7w),\ (x-y)(3x+7y-3z-9w). \end{aligned} \tag{1.5}$$

In contrast with Theorem 1.4(v), we also conjecture that $x^2y^2 + 9y^2z^2 + 9z^2x^2$ is suitable (cf. [S16, A268507]). See [S16, A273110 and A272351] for some data related to parts (iv) and (vi) of Theorem 1.4.

Theorem 1.5. (i) *Any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ such that x^2-y^2 is an even square. Also, $2(x^2-y^2)$, $3(x^2-y^2)$, x^2-3y^2 and $3x^2-2y^2$ are all suitable.*

(ii) *All the polynomials*

$$x^2+2y^2,\ x^2+3y^2,\ x^2+5y^2,\ x^2+6y^2,\ x^2+8y^2,\ x^2+12y^2,\ 2x^2+7y^2,$$
$$3x^2+4y^2,\ 4x^2+5y^2,\ 4x^2+9y^2,\ 5x^2+11y^2,\ 6x^2+10y^2,\ 7x^2+9y^2$$

are suitable.

We will prove Theorems 1.1-1.3 and Theorems 1.4-1.5 in Sections 2 and 3 respectively. Section 4 contains some open conjectures for further research.

2. Proofs of Theorems 1.1-1.3

Proof of Theorem 1.1. For $n = 0,1,2,\ldots,4^{m/\gcd(2,m)}-1$, the desired result can be verified directly.

Now let $n \geqslant 4^{m/\gcd(2,m)}$ be an integer and assume that the desired result holds for smaller values of n.

Case 1. $4^{m/\gcd(2,m)} \mid n$.

By the induction hypothesis, we can write

$$\frac{n}{4^{m/\gcd(2,m)}} = ax^m + y^2 + z^2 + w^2 \quad \text{with } x,y,z,w \in \mathbb{N}.$$

It follows that

$$n = a\left(2^{2/\gcd(2,m)}x\right)^m + \left(2^{m/\gcd(2,m)}y\right)^2 + \left(2^{m/\gcd(2,m)}z\right)^2 + \left(2^{m/\gcd(2,m)}w\right)^2.$$

Case 2. $4^{m/\gcd(2,m)} \nmid n$ and $n \neq 4^k(8l+7)$ for any $k,l \in \mathbb{N}$.

In this case, $n \notin E(1,1,1)$ and hence there are $y,z,w \in \mathbb{N}$ such that $n = a \times 0^m + y^2 + z^2 + w^2$.

Case 3. $n = 4^k(8l+7)$ with $k,l \in \mathbb{N}$ and $k < m/\gcd(2,m)$.

If $k \in \{0,1\}$, or $k=2$ and $m > 4 = a$, then $n-a \notin E(1,1,1)$ by (1.2), and hence $n = a\times 1^m + y^2+z^2+w^2$ for some $y,z,w \in \mathbb{N}$. When $a=1$, $k=2$ and $m \in \{5,6\}$, we have $n-2^m = 4^2(8l+7-2^{m-4}) \notin E(1,1,1)$, and hence $n = a2^m+y^2+z^2+w^2$ for some $y,z,w \in \mathbb{N}$. If $k \in \{3,4\}$ and $m=5$, then $n-a2^m \notin E(1,1,1)$ by (1.2), and hence $n = a2^m+y^2+z^2+w^2$ for some $y,z,w \in \mathbb{N}$.

In view of the above, we have completed our induction proof of Theorem 1.1. $\square$

Lemma 2.1. *We have*

$$E(1,1,2) = \{4^k(16l+14) : \ k,l \in \mathbb{N}\}. \tag{2.1}$$

Remark 2.1. (2.1) can be found in L. E. Dickson [D39, pp. 112-113].

Proof of Theorem 1.2. (i) Let $a \in \{1,2\}$. We claim that any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ such that $a(x-y)$ is a square, and want to prove this by induction.

For every $n = 0,1,\ldots,15$, we can verify the claim directly.

Now we fix an integer $n \geqslant 16$ and assume that the claim holds for smaller values of n.

Case 1. $16 \mid n$.

In this case, by the induction hypothesis, there are $x,y,z,w \in \mathbb{N}$ with $a(x-y)$ a square such that $n/16 = x^2+y^2+z^2+w^2$, and hence $n = (4x)^2+(4y)^2+(4z)^2+(4w)^2$ with $a(4x-4y)$ a square.

Case 2. $16 \nmid n$ and $n \notin E(1,1,2)$.

In this case, there are $x,y,z,w \in \mathbb{N}$ with $x=y$ and $n = x^2+y^2+z^2+w^2$, thus $a(x-y) = 0^2$ is a square.

Case 3. $16 \nmid n$ and $n \in E(1,1,2)$.

In this case, $n = 4^k(16l+14)$ for some $k \in \{0,1\}$ and $l \in \mathbb{N}$. Note that $n/2-(2/a)^2 \notin E(1,1,1)$ by (1.2). So, $n/2-(2/a)^2 = t^2+u^2+v^2$ for some $t,u,v \in \mathbb{N}$ with $t \geqslant u \geqslant v$. As $n/2-(2/a)^2 \geqslant 8-4 > 3$, we have $t \geqslant 2 \geqslant 2/a$. Thus

$$n = 2\left(\left(\frac{2}{a}\right)^2+t^2\right)+2(u^2+v^2) = \left(t+\frac{2}{a}\right)^2+\left(t-\frac{2}{a}\right)^2+(u+v)^2+(u-v)^2$$

with $a((t+2/a)-(t-2/a)) = 2^2$.

So far we have proved part (i) of Theorem 1.2.

(ii) We can easily verify the desired result in Theorem 1.2(ii) for all $n = 0,1,\ldots,63$.

Now let $n \geqslant 64$ and assume that any $r = 0,1,\ldots,n-1$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{Z})$ with $x+y \in \{ct^3 : \ t \in \mathbb{Z}\}$. If $64 \mid n$, then $n/64$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{Z})$ with $x+y = ct^3$ for some $t \in \mathbb{Z}$, hence $n = (8x)^2+(8y)^2+(8z)^2+(8w)^2$ with $8x+8y = c(2t)^3$.

Now we consider the case $64 \nmid n$. We claim that $\{2n, 2n-c^2, 2n-64c^2\} \not\subseteq E(1,1,1)$. If $\{2n, 2n-1\} \subseteq E(1,1,1)$, then (1.2) implies that $2n = 4^k(8l+7)$ for some $k \in \{2,3\}$ and $l \in \mathbb{N}$, hence $2n-64$ is $4^2(8l+3)$ or $4^3(8l+6)$, and thus $2n-64 \notin E(1,1,1)$. If $2n = 4^k(8l+7)$ for some $k \in \{1,2\}$ and $l \in \mathbb{N}$, then $2n-4$ is $4(8l+6)$ or $4(8(4l+3)+3)$, and hence $2n-2^2 \notin E(1,1,1)$; if $2n = 4^3(8l+7)$

with $l\in\mathbb{N}$, then $2n-64\times 2^2=4^3(8l+3)\not\in E(1,1,1)$. When $2n=4^k(8l+7)$ for some $k\in\{1,2,3\}$ and $l\in\mathbb{N}$, we have $2n-4^2\not\in E(1,1,1)$ since

$$2n-16=4^k(8l+7)-16=\begin{cases}4(8l+7)-16=4(8l+3)&\text{if } k=1,\\ 4^2(8l+7)-16=4^2(8l+6)&\text{if } k=2,\\ 4^3(8l+7)-16=4^2(8(4l+3)+3)&\text{if } k=3.\end{cases}$$

So the claim is true.

By the claim, for some $\delta\in\{0,1,8\}$, we can write $2n-\delta^2c^2$ as the sum of three squares two of which have the same parity. Hence we may write $2n-\delta^2c^2=(2x-\delta c)^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$ and $y\equiv z \pmod 2$. It follows that

$$\begin{aligned}n=&\frac{(2x-\delta c)^2+\delta^2c^2}{2}+\frac{y^2+z^2}{2}\\ =&x^2+(\delta c-x)^2+\left(\frac{y+z}{2}\right)^2+\left(\frac{y-z}{2}\right)^2\end{aligned}$$

with $x+(\delta c-x)=c\delta\in\{ct^3:\ t=0,1,2\}$. This concludes the induction step.

(iii) For $n=0,1,\ldots,4^m-1$, we can verify the desired result in Theorem 1.2(iii) directly via a computer.

Fix $n\in\mathbb{N}$ with $n\geqslant 4^m$, and assume the required result for smaller values of n.

If $4^m\mid n$, then by the induction hypothesis $n/4^m$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{Z}$ such that $x+2y=dt^m$ for some $t\in\mathbb{Z}$, and hence $n=(2^mx)^2+(2^my)^2+(2^mz)^2+(2^mw)^2$ with $2^mx+2(2^my)=2^m(x+2y)=d(2t)^m$.

Now we assume $4^m\nmid n$. In light of (1.2), $\{5n,5n-d^2,5n-4^md^2\}\not\subseteq E(1,1,1)$. In fact, for any $l\in\mathbb{N}$ neither $8l+7-d^2$ nor $4(8l+7)-d^2$ belongs to $E(1,1,1)$; if $5n=4^2(8l+7)$ with $l\in\mathbb{N}$ then $5n-2^2=4(8(4l+3)+3)\not\in E(1,1,1)$ and $5n-4^31^2=4^2(8l+3)\not\in E(1,1,1)$. So, for some $\delta\in\{0,1,2^m\}$ we can write $5n-\delta^2d^2$ as $x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$. Note that a square is congruent to one of $0,1,-1$ modulo 5. It is easy to see that one of x^2,y^2,z^2 is congruent to $-(\delta d)^2$ modulo 5. Without loss of generality, we may assume that $(\delta d)^2+x^2\equiv y^2+z^2\equiv 0 \pmod 5$. As $x^2\equiv(2\delta d)^2 \pmod 5$ and $y^2\equiv(2z)^2 \pmod 5$, without loss of generality we simply assume that $x\equiv 2\delta d \pmod 5$ (otherwise we use $-x$ instead of x) and $y\equiv 2z \pmod 5$. Thus $r=(2x+\delta d)/5$, $s=(2\delta d-x)/5$, $u=(2y+z)/5$ and $v=(2z-y)/5$ are all integers. Note that

$$r^2+s^2+u^2+v^2=\frac{(\delta d)^2+x^2}{5}+\frac{y^2+z^2}{5}=n \text{ with } r+2s=\delta d\in\{dt^m:\ t=0,1,2\}.$$

This concludes our proof of the third part of Theorem 1.2.

(iv) For each $n=0,1,2$, there are $x,z\in\{0,1\}$ such that $n=x^2+0^2+z^2+0^2$ with $(x+z)(0+0)=0^2$. Note also that

$$3=1^2+1^2+(-1)^2+0^2\quad\text{with } (1+(-1))(1+0)=0^2.$$

Now let $n \in \{4, 5, 6, \dots\}$ and assume that any $r = 0, \dots, n-1$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $(x+z)(y+w)$ is a square.

If $4 \mid n$, then $n/4$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $(x+z)(y+w)$ a square, and hence $n = (2x)^2 + (2y)^2 + (2z)^2 + (2w)^2$ with $(2x+2z)(2y+2w) = 2^2(x+z)(y+w)$ a square.

If n is odd, then by (2.1) there are $x, y, z, w \in \mathbb{Z}$ with $w = -y$ such that $n = x^2 + 2y^2 + z^2 = x^2 + y^2 + z^2 + w^2$ with $(x+z)(y+w) = 0^2$.

Now we consider the case $n = 2m$ with m odd. By (2.1), we can write m as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $w = y$. Therefore,

$$n = 2m = (x+y)^2 + (x-y)^2 + (z+w)^2 + (z-w)^2 = (x+y)^2 + (z+w)^2 + (y-x)^2 + (w-z)^2$$

with

$$((x+y) + (y-x))((z+w) + (w-z)) = 2y(2w) = (2y)^2.$$

This ends our induction proof of Theorem 1.2(iv).

(v) Clearly,

$$1 = 4^0(1 + 4 \times 0^2 + 0^2) + 0^2, \ 2 = 4^0(1 + 4 \times 0^2 + 1^2) + 0^2, \ 3 = 4^0(1 + 4 \times 0^2 + 1^2) + 1^2.$$

Now let $n \in \mathbb{Z}^+$ with $n \geqslant 4$. If $4 \mid n$ and $n/4 = 4^k(1 + 4x^2 + y^2) + z^2$ for some $k, x, y, z \in \mathbb{N}$, then $n = 4^{k+1}(1 + 4x^2 + y^2) + (2z)^2$. If $n \equiv 2, 3 \pmod 4$, then $n - 1 \equiv 1, 2 \pmod 4$ and hence for some $x, y, z \in \mathbb{Z}$ we have $n - 1 = (2x)^2 + y^2 + z^2$ and thus $n = 4^0(1 + 4x^2 + y^2) + z^2$. By Dickson [D39, pp. 112–113], if $q \in \mathbb{N}$ is congruent to 1 modulo 4 then $q \notin E(1, 4, 16)$. Thus, when $n \equiv 1 \pmod 4$, we have $n - 4 \notin E(1, 4, 16)$, hence there are $x, y, z \in \mathbb{Z}$ such that $n - 4 = 16x^2 + 4y^2 + z^2$, i.e., $n = 4(1 + 4x^2 + y^2) + z^2$. This proves Theorem 1.2(v) by induction.

In view of the above, we have completed the proof of Theorem 1.2. □

Remark 2.2. In contrast with (2.1), we conjecture that any odd integer $n > 1248$ can be written as $x^2 + y^2 + 2z^2$ with $x, y, z \in \mathbb{N}$ and $x \geqslant y \geqslant z$. Modifying the proof of Theorem 1.2(iii) slightly, we see that under this conjecture the polynomial $(x-z)(y-w)$ is suitable and also any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $w \in \mathbb{Z}$, $x, y, z \in \mathbb{N}$ and $x \leqslant y \geqslant z \geqslant |w|$ such that $(x+y)(z+w)$ is a square.

Lemma 2.2. *We have*

$$E(1,2,6) = \{4^k(8l+5) : \ k, l \in \mathbb{N}\}, \tag{2.2}$$

$$E(2,3,6) = \{3q+1 : \ q \in \mathbb{N}\} \cup \{4^k(8l+7) : \ k, l \in \mathbb{N}\}, \tag{2.3}$$

$$E(1,2,3) = \{4^k(16l+10) : \ k, l \in \mathbb{N}\}, \tag{2.4}$$

$$E(1,3,6) = \{3q+2 : \ q \in \mathbb{N}\} \cup \{4^k(16l+14) : \ k, l \in \mathbb{N}\}, \tag{2.5}$$

$$E(1,5,5) = \{n \in \mathbb{N} : \ n \equiv 2, 3 \pmod 5\} \cup \{4^k(8l+7) : \ k, l \in \mathbb{N}\}. \tag{2.6}$$

Remark 2.3. (2.2)-(2.6) are known results, see, e.g., Dickson [D39, pp. 112-113].

Proof of Theorem 1.3. (i) For $n = 0, \ldots, 4^m - 1$ we can easily verify the desired result in Theorem 1.3(i) directly.

Now let $n \in \mathbb{N}$ with $n \geqslant 4^m$. Assume that any $r \in \{0, \ldots, n-1\}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + y + cz \in \{dt^m : t \in \mathbb{Z}\}$. If $4^m \mid n$, then there are $x, y, z, w \in \mathbb{Z}$ with $x^2 + y^2 + z^2 + w^2 = n/4^m$ such that $x + y + cz = dt^m$ for some $t \in \mathbb{Z}$, and hence

$$n = (2^m x)^2 + (2^m y)^2 + (2^m z)^2 + (2^m w)^2$$

with $2^m x + 2^m y + c(2^m z) = 2^m(x + y + cz) = d(2t)^m$. Below we suppose that $4^m \nmid n$.

Case 1. $c = 1$.

In this case, it suffices to show that there are $x, y, z \in \mathbb{Z}$ and $\delta \in \{0, 1, 2^m\}$ such that

$$n = x^2 + (y+z)^2 + (z-y)^2 + (\delta d - 2z)^2 = x^2 + 2y^2 + 6z^2 - 4\delta dz + \delta^2 d^2. \quad (2.7)$$

(Note that $(y+z) + (z-y) + (\delta d - 2z) = \delta d \in \{dt^m : t \in \mathbb{Z}\}$.) Suppose that this fails for $\delta = 0$. By (2.2), $n = 4^k(8l + 5)$ for some $k, l \in \mathbb{N}$ with $k < m$.

We first handle the subcase $d = 1$. Clearly,

$$3n - 1 = \begin{cases} 3(8l+5) - 1 = 2(12l+7) & \text{if } k = 0, \\ 3 \times 4(8l+5) - 1 = 8(12l+7) + 3 & \text{if } k = 1. \end{cases}$$

Thus, if $k \in \{0, 1\}$, then $3n - 1 \notin E(2, 3, 6)$ by (2.3), hence for some $x, y, z \in \mathbb{Z}$ we have

$$3n - 1 = 3x^2 + 6y^2 + 2(3z-1)^2 = 3(x^2 + 2y^2 + 2(3z^2 - 2z)) + 2$$

and thus

$$n = x^2 + 2y^2 + 6z^2 - 4z + 1 = x^2 + (y+z)^2 + (z-y)^2 + (1-2z)^2$$

which gives (2.7) with $\delta = 1$. When $k = 2$ and $m = 3$, we have

$$3n - 64 = 3 \times 4^2(8l+5) - 64 = 4^2(8(3l+1) + 3) \notin E(2, 3, 6)$$

in view of (2.3), hence for some $x, y, z \in \mathbb{Z}$ we have

$$3n - 4^3 = 3x^2 + 6y^2 + 2(3z-8)^2 = 3(x^2 + 2y^2 + 2(3z^2 - 16z)) + 2 \times 4^3$$

and thus

$$n = x^2 + 2y^2 + 6z^2 - 32z + 64 = x^2 + (y+z)^2 + (z-y)^2 + (2^3 - 2z)^2$$

which gives (2.7) with $\delta = 2^m$.

Now we handle the subcase $d = 2$. Clearly,

$$3n - 4 = 3 \times 4^k(8l + 5) - 4 = \begin{cases} 8(3l+1)+3 & \text{if } k = 0, \\ 4(8(3l+1)+6) & \text{if } k = 1, \\ 4(8(12l+7)+3) & \text{if } k = 2. \end{cases}$$

It follows that $3n - 4 \notin E(2, 3, 6)$ by (2.3). So, for some $x, y, z \in \mathbb{Z}$ we have

$$3n - 4 = 3x^2 + 6y^2 + 2(3z - 2)^2 = 3x^2 + 6y^2 + 18z^2 - 24z + 8$$

and hence

$$n = x^2 + 2y^2 + 6z^2 - 8z + 4 = x^2 + (y + z)^2 + (z - y)^2 + (2 - 2z)^2$$

with $(y + z) + (z - y) + (2 - 2z) = 2 \times 1^m$.

Case 2. $c = 2$.

If $n \notin E(1, 2, 3)$, then for some $x, y, z \in \mathbb{Z}$ we have

$$n = x^2 + 2y^2 + 3z^2 = x^2 + (y + z)^2 + (z - y)^2 + (-z)^2$$

with $(y + z) + (z - y) + 2(-z) = d \times 0^m$. Now let $n \in E(1, 2, 3)$. By (2.4), $n = 4^k(16l + 10)$ for some $k, l \in \mathbb{N}$ with $k < m$.

Subcase 2.1. $m = 2$ and $d \in \{3, 6\}$.

For $n = 16, \ldots, 23$ we can verify the desired result directly. Now let $n \geqslant 24 = 216/9$. No matter $k = 0$ or $k = 1$, we have

$$n - \frac{216}{d^2} = 4^k(16l + 10) - \frac{216}{d^2} \notin E(1, 2, 3)$$

by (2.4). So, there are $x, y, z \in \mathbb{Z}$ such that

$$n - \frac{216}{d^2} = x^2 + 2y^2 + 3\left(\frac{6}{d} - z\right)^2$$

and hence

$$n = x^2 + 2y^2 + 3z^2 - \frac{36}{d}z + \left(\frac{18}{d}\right)^2 = x^2 + (y + z)^2 + (z - y)^2 + \left(\frac{18}{d} - z\right)^2$$

with $(y + z) + (z - y) + 2(18/d - z) = d(6/d)^2$.

Subcase 2.2. $d = 1$.

If $k=0$, then $6n-1=6(16l+10)-1\equiv 3 \pmod 8$ and hence $6n-1\notin E(2,3,6)$ by (2.3). When $k=1$, we have

$$6n-4^m=6\times 4(16l+10)-4^m=4^2(8(3l+1)+7-4^{m-2})\notin E(2,3,6)$$

by (2.3). If $k=2$ and $m=3$, then

$$6n-4^m=6\times 4^2(16l+10)-4^3=4^3(8(3l+1)+6)\notin E(2,3,6)$$

by (2.3). So, for some $\delta\in\{1,2^m\}$ we have $6n-\delta^2\notin E(2,3,6)$. Hence there are $x,u,v\in\mathbb{Z}$ such that $6n-\delta^2=6x^2+3u^2+2v^2$. As $u\equiv\delta \pmod 2$, we can write $u=2y+\delta$ with $y\in\mathbb{Z}$. Since $v^2\equiv\delta^2 \pmod 3$, without loss of generality we may assume that $v=3z+\delta$ with $z\in\mathbb{Z}$. Therefore,

$$6n=6x^2+3(2y+\delta)^2+2(3z+\delta)^2+\delta^2$$

and hence

$$n=x^2+2y^2+3z^2+2\delta y+2\delta z+\delta^2=x^2+(y+z+\delta)^2+(z-y)^2+(-z)^2$$

with $(y+z+\delta)+(z-y)+2(-z)=\delta\in\{t^m:\ t=1,2\}$.

Subcase 2.3. $d=2$.

Observe that

$$3n-2=3\times 4^k(16l+10)-2=\begin{cases}4(12l+7) & \text{if } k=0,\\ 16(12l+7)+6 & \text{if } k=1.\end{cases}$$

If $k=2$ and $m=3$, then

$$3n-2^7=3\times 4^2(16l+10)-2^7=4^2(16(3l+1)+6).$$

Combining this with (2.5), we find that for some $\delta\in\{1,2^m\}$ we have $3n-2\delta^2\notin E(1,3,6)$. Thus, there are $x,y,z\in\mathbb{Z}$ for which

$$3n-2\delta^2=3x^2+6y^2+(3z-\delta)^2=3x^2+6y^2+9z^2-6\delta z+\delta^2$$

and hence

$$n=x^2+2y^2+3z^2-2\delta z+\delta^2=x^2+(y+z)^2+(z-y)^2+(\delta-z)^2$$

with $(y+z)+(z-y)+2(\delta-z)=2\delta\in\{2t^m:\ t=1,2\}$.

Subcase 2.4. $d=4$.

Observe that

$$3n-8 = 3\times 4^k(16l+10)-8 = \begin{cases} 16(3l+1)+6 & \text{if } k=0,\\ 4^2(12l+7) & \text{if } k=1,\\ 4(16(12l+7)+6) & \text{if } k=2.\end{cases}$$

Clearly, $3n-8 \notin E(1,3,6)$ by (2.5). Thus, for some $x,y,z\in\mathbb{Z}$ we have

$$3n-8 = 3x^2+6y^2+(3z-2)^2 = 3x^2+6y^2+9z^2-12z+4$$

and hence

$$n = x^2+2y^2+3z^2-4z+4 = x^2+(y+z)^2+(z-y)^2+(2-z)^2$$

with $(y+z)+(z-y)+2(2-z) = 4\times 1^m$.

Combining the above, we have proved part (i) of Theorem 1.3.

(ii) Suppose that any odd integer $n>2719$ can be represented by $x^2+y^2+10z^2$ with $x,y,z\in\mathbb{Z}$. We want to prove by induction the claim that each $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{Z})$ with $x+3y$ a square.

For $n=0,1,\ldots,15$, the claim can be verified via a computer.

Now fix an integer $n\geqslant 16$ and assume that the claim holds for smaller values of n.

If $16\mid n$, then by the induction hypothesis there are $x,y,z,w\in\mathbb{Z}$ with $n/16 = x^2+y^2+z^2+w^2$ such that $x+3y$ is a square, and hence $n=(4x)^2+(4y)^2+(4z)^2+(4w)^2$ with $4x+3(4y)=4(x+3y)$ a square.

Now we let $16\nmid n$. If $2\nmid n$ and $n\leqslant 2719$, then we can easily verify that $5n$ or $5n-8$ can be written as $2x^2+5y^2+5z^2$ with $x,y,z\in\mathbb{Z}$. If $2\nmid n$ and $n>2719$, then there are $x,y,z\in\mathbb{Z}$ such that $n=10x^2+y^2+z^2$ and hence $5n=2(5x)^2+5y^2+5z^2$. If n is even and n is not of the form $4^k(16l+6)$ $(k,l\in\mathbb{N})$, then by Dickson [D27] there are $x,y,z\in\mathbb{Z}$ such that $n=10x^2+y^2+z^2$ and hence $5n=2(5x)^2+5y^2+5z^2$. When $n=4^k(16l+6)$ for some $k\in\{0,1\}$ and $l\in\mathbb{N}$, clearly

$$\frac{5n-8}{2} = 5\times 4^k(8l+3)-4 \notin E(1,5,5)$$

by (2.6), thus there are $x,y,z\in\mathbb{Z}$ such that $(5n-8)/2 = x^2+5y^2+5z^2$ and hence $5n-8=2x^2+5(y+z)^2+5(y-z)^2$.

Since $5n$ or $5n-8$ can be written as $2x^2+5y^2+5z^2$ with $x,y,z\in\mathbb{Z}$, for some $\delta\in\{0,2\}$ and $x,y,z\in\mathbb{Z}$ we have

$$10n-\delta^4 = 2(2x^2+5y^2+5z^2) = (2x)^2+10y^2+10z^2.$$

As $(2x)^2\equiv -\delta^4\equiv (3\delta^2)^2 \pmod{10}$, without loss of generality we may assume that $2x=10w+3\delta^2$ with $w\in\mathbb{Z}$. Then

$$10n = \delta^4+(10w+3\delta^2)^2+10y^2+10z^2$$

and hence

$$n = 10w^2 + y^2 + z^2 + 6\delta^2 w + \delta^4 = (3w+\delta^2)^2 + (-w)^2 + y^2 + z^2$$

with $(3w+\delta^2) + 3(-w) = \delta^2$ a square. This concludes the induction step.

(iii) Suppose that any integer $n \geqslant 1190$ not divisible by 16 can be written as $x^2 + 10y^2 + (2z^2 + 5^3r^4)/7$ with $r \in \{0,1,2,3\}$ and $x,y,z \in \mathbb{Z}$. We want to prove by induction the claim that each $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{Z}$ such that $x+3y+5z$ is a square.

For $n = 0,1,\ldots,1189$, the claim can be verified via a computer.

Now fix an integer $n \geqslant 1190$ and assume that the claim holds for smaller values of n.

If $16 \mid n$, then by the induction hypothesis there are $x,y,z,w \in \mathbb{Z}$ with $n/16 = x^2+y^2+z^2+w^2$ such that $x+3y+5z$ is a square, and hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $4x + 3(4y) + 5(4z) = 4(x+3y+5z)$ a square.

Below we let $16 \nmid n$. Then, for some $r \in \{0,1,2,3\}$ and $x,y,z \in \mathbb{Z}$ we have

$$n = x^2 + 10y^2 + \frac{2z^2 + 5^3r^4}{7}, \text{ i.e., } 7n - 5(5r^2)^2 = 7x^2 + 70y^2 + 2z^2.$$

Thus,

$$7(n - (5r^2)^2) = 7x^2 + 70y^2 + 2\left(z^2 - (5r^2)^2\right).$$

As $5r^2$ is congruent to z or $-z$ modulo 7, without loss of generality we may assume that $z = 7t - 5r^2$ for some $t \in \mathbb{Z}$. It follows that

$$n - (5r^2)^2 = x^2 + 10y^2 + \frac{2}{7}\left((7t-5r^2)^2 - (5r^2)^2\right) = x^2 + 10y^2 + 14t^2 - 4(5r^2)t$$

and hence

$$n = x^2 + 10y^2 + 10t^2 + (2t-5r^2)^2 = x^2 + (3y+t)^2 + (3t-y)^2 + (5r^2-2t)^2$$

with

$$(3y+t) + 3(3t-y) + 5(5r^2-2t) = (5r)^2.$$

This concludes the induction step.

The proof of Theorem 1.3 is now complete. □

Remark 2.4. Those $3z^2 - 2z$ with $z \in \mathbb{Z}$ appeared in the proof of Theorem 1.3(i) are called generalized octagonal numbers, one may consult [S15] and [S16a] for related results.

3. Proofs of Theorems 1.4-1.5

Lemma 3.1. *We have*

$$E(1,1,5)=\{4^k(8l+3):\ k,l\in\mathbb{N}\},\tag{3.1}$$

and

$$E(1,1,10)\cap 2\mathbb{Z}=\{4^k(16l+6):\ k,l\in\mathbb{N}\}.\tag{3.2}$$

Remark 3.1. (3.1) can be found in Dickson [D39, pp. 112-113], and (3.2) was a conjecture of Ramanujan [R] proved by Dickson [D27].

Combining (1.2), (2.1), (2.2), (2.4) and (3.1)-(3.2), we immediately get the following lemma.

Lemma 3.2. *The six sets*

$$E(1,1,1),\ E(1,1,2),\ E(1,2,3),\ E(1,2,6),\ E(1,1,5)\ \text{ and }\ E(1,1,10)\cap 2\mathbb{Z}\tag{3.3}$$

are pairwise disjoint.

Lemma 3.3. *Let $n\in\mathbb{N}$. Then $n\notin E(1,2,6)$ if and only if $n=x^2+y^2+z^2+w^2$ for some $x,y,z,w\in\mathbb{N}$ with $x+y=z$. Also, $n\notin E(1,2,3)$ if and only if $n=x^2+y^2+z^2+w^2$ for some $x,y,z,w\in\mathbb{Z}$ with $x+y=2z$.*

Proof. (i) Assume that $n\notin E(1,2,6)$. Then, there are $x,y,z\in\mathbb{N}$ for which

$$n=x^2+2y^2+6z^2=x^2+(y+z)^2+|y-z|^2+(2z)^2.$$

Clearly $(y+z)+|y-z|=2z$ if $y\leqslant z$, and $|y-z|+2z=y+z$ if $y>z$. Therefore $n=x^2+u^2+v^2+w^2$ for some $u,v,w\in\mathbb{N}$ with $u+v=w$.

Now suppose that $n=x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ and $x+y=z$. If $x\equiv y\pmod 2$, then

$$n=2\left(\frac{x+y}{2}\right)^2+2\left(\frac{x-y}{2}\right)^2+(x+y)^2+w^2=w^2+2\left(\frac{x-y}{2}\right)^2+6\left(\frac{x+y}{2}\right)^2$$

and hence $n\notin E(1,2,6)$. When $x\not\equiv y\pmod 2$, without loss of generality we may assume that $y\equiv z\pmod 2$, hence

$$n=(z-y)^2+2\left(\frac{y+z}{2}\right)^2+2\left(\frac{y-z}{2}\right)^2+w^2=w^2+2\left(\frac{y+z}{2}\right)^2+6\left(\frac{y-z}{2}\right)^2$$

and thus $n\notin E(1,2,6)$.

(ii) If $n\notin E(1,2,3)$, then there are $x,y,z\in\mathbb{Z}$ for which

$$n=x^2+2y^2+3z^2=x^2+(y+z)^2+(z-y)^2+z^2$$

with $(y+z)+(z-y)=2z$.

If $n=x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{Z}$ and $x+y=2z$, then

$$n=2\left(\frac{x+y}{2}\right)^2+2\left(\frac{x-y}{2}\right)^2+z^2+w^2=w^2+2\left(\frac{x-y}{2}\right)^2+3z^2$$

and hence $n\not\in E(1,2,3)$.

In view of the above, we have completed the proof of Lemma 3.3. $\Box$

Proof of Theorem 1.4. (i) Clearly, $5y^2=(2y)^2+y^2$ and $10y^2=(3y)^2+y^2$. In view of Lemmas 3.2 and 3.3, each $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ such that $P(x,y,z)=0$, where $P(x,y,z)$ may be any of the polynomials listed in (1.3). For example, for $P(x,y,z)=(x-2y)(x+y-z)$ we use the fact that $E(1,1,5)\cap E(1,2,6)=\emptyset$.

(ii) Suppose that (1.4) holds. If $n=4^k(8l+5)$ for some $k,l\in\mathbb{N}$, then there are $x,y,z\in\mathbb{N}$ such that $8l+5=x^2+y^2+13z^2$ and hence

$$n=(2^kx)^2+(2^ky)^2+13(2^kz)^2=(2^kx)^2+(2^ky)^2+(2^k\times 3z)^2+(2^{k+1}z)^2$$

with $2(2^k\times 3z)=3(2^{k+1}z)$. If $n\in\mathbb{N}\setminus\{4^k(8l+5):\ k,l\in\mathbb{N}\}$, then $n\not\in E(1,2,6)$ by (2.2), and hence by Lemma 3.3 there are $x,y,z,w\in\mathbb{N}$ such that $n=x^2+y^2+z^2+w^2$ with $x+y=z$. Therefore, any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ with $(2x-3y)(x+y-z)=0$.

(iii) If $n=2x^2$ for some $x\in\mathbb{Z}^+$, then $n=x^2+0^2+x^2+0^2$ with $x\in\mathbb{Z}^+$ and $(x-0)x=x^2$. If $n\not\in E(1,1,2)$ and $n\neq 2x^2$ for all $x\in\mathbb{N}$, then there are $x,z,w\in\mathbb{N}$ with $z>0$ such that $n=x^2+x^2+z^2+w^2$ with $(x-x)z=0^2$.

Now suppose that $n\in E(1,1,2)$. Then $n\not\in E(1,2,6)$ by Lemma 3.2, and hence by Lemma 3.3 there are $x,y,z,w\in\mathbb{N}$ with $x=y+z$ such that $x^2+y^2+z^2+w^2=n$. If y and z are both zero, then $n=w^2\not\in E(1,1,2)$ which leads a contradiction. Without loss of generality we assume that $z>0$. Note that $(x-y)z=z^2$ is a square. This concludes the proof of Theorem 1.4(iii).

(iv) If $n\not\in E(1,1,1)$, then there are $y,z,w\in\mathbb{N}$ with $y>0$ such that $n=0^2+y^2+z^2+w^2$. As

$$(0+4y+4z)^2+(9\times 0+3y+3z)^2=(5y+5z)^2,$$

by Pythagoras' theorem there is a right triangle with positive integer sides $4y+4z$, $3y+3z$ and $5y+5z$.

Now let $n\in E(1,1,1)$. Then $n\not\in E(1,2,6)$ by Lemma 3.2, and hence by Lemma 3.3 there are $x,y,z,w\in\mathbb{N}$ with $x=y+z$ such that $n=x^2+y^2+z^2+w^2$. Clearly $y>0$ since $n\in E(1,1,1)$. Observe that

$$(x+4y+4z)^2+(9x+3y+3z)^2=(5x)^2+(12x)^2=(13x)^2.$$

So $x+4y+4z$ and $9x+3y+3z$ are the two legs of a right triangle with positive integer sides.

(v) Fix $n \in \mathbb{Z}^+$. If $n \notin E(1,1,1)$, then there are $x, y \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $n = x^2+y^2+0^2+w^2$ and hence $x^2y^2+y^20^2+0^2x^2 = x^2y^2+4y^20^2+4\times 0^2x^2 = (xy)^2$ is a square.

Now suppose that $n \in E(1,1,1)$. Then $n \notin E(1,2,6)$ by Lemma 3.2, and hence by Lemma 3.3 there are $x, y, z, w \in \mathbb{N}$ with $x+y = z$ such that $n = x^2+y^2+z^2+w^2$. Clearly $w > 0$ since $n \in E(1,1,1)$. Observe that

$$\begin{aligned} x^2y^2 + y^2z^2 + z^2x^2 =&(xy)^2 + (x^2+y^2)(x+y)^2 \\ =&(xy)^2 + (x^2+xy+y^2-xy)(x^2+xy+y^2+xy) \\ =&(x^2+xy+y^2)^2. \end{aligned}$$

As $E(1,2,3) \cap E(1,1,1) = \emptyset$ by Lemma 3.2, we have $n \notin E(1,2,3)$, and hence by Lemma 3.3 we can write n as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{Z})$ with $x+y = 2z$. Clearly $w \neq 0$ since $n \in E(1,1,1)$. Note that

$$\begin{aligned} x^2y^2 + 4y^2z^2 + 4z^2x^2 =&(xy)^2 + (x^2+y^2)(2z)^2 \\ =&(xy)^2 + (x^2+y^2)(x+y)^2 = (x^2+xy+y^2)^2. \end{aligned}$$

This concludes the proof of Theorem 1.4(v).

(vi) If $n \notin E(1,1,1)$, then there are $x, z, w \in \mathbb{N}$ with $w > 0$ such that $n = x^2+0^2+z^2+w^2$ and hence $x^4 + 8\times 0^3z + 8\times 0z^3 = x^4 + 16\times 0^3z + 16\times 0z^3 = x^4$ is a fourth power. Below we assume that $n \in E(1,1,1)$.

As $E(1,1,1) \cap E(1,2,6) = \emptyset$ (by Lemma 3.2), we have $n \notin E(1,2,6)$. In view of Lemma 3.3, there are $w, x, y, z \in \mathbb{N}$ with $x+z = y$ such that $n = x^2+y^2+z^2+w^2$. As $n \in E(1,1,1)$, we have $wx \neq 0$. Observe that

$$\begin{aligned} x^4 + 8yz(y^2+z^2) =&x^4 + ((y+z)^2-(y-z)^2)((y+z)^2+(y-z)^2) \\ =&x^4 + (y+z)^4 - (y-z)^4 = (y+z)^4. \end{aligned}$$

As $E(1,1,1) \cap E(1,2,3) = \emptyset$ by Lemma 3.2, we have $n \notin E(1,2,3)$ and hence there are $v, w, z \in \mathbb{N}$ such that $n = w^2+2v^2+3z^2 = w^2+(v+z)^2+(v-z)^2+z^2$. Clearly $w > 0$ since $n \notin E(1,1,1)$. Let $y = v+z$ and $x = |v-z| = |y-2z|$. Then $n = w^2+x^2+y^2+z^2$ and

$$\begin{aligned} x^4 + 16y^3z + 64yz^3 =&x^4 + 4y(2z)\left(2y^2+2(2z)^2\right) \\ =&x^4 + \left((y+2z)^2-(y-2z)^2\right)\left((y+2z)^2+(y-2z)^2\right) \\ =&x^4 + (y+2z)^4 - (y-2z)^4 = (y+2z)^4. \end{aligned}$$

The proof of Theorem 1.4 is now complete. $\square$

Proof of Theorem 1.5. (i) If $n \notin E(1,1,2)$, then for some $x, y, z, w \in \mathbb{N}$ with $x = y$ we have
$$n = x^2+y^2+z^2+w^2$$

with $a(x^2-y^2)=0^2$ for any $a=1,2,3$, and $3x^2-2y^2=x^2$. If $n\not\in E(1,1,1)$, then $n=x^2+0^2+y^2+z^2$ for some $x,y,z\in\mathbb{Z}$ with $x^2-3\times 0^2=x^2$. If $n\not\in E(1,1,5)$, then there are $x,y,z\in\mathbb{N}$ satisfying $n=5x^2+y^2+z^2=(2x)^2+x^2+y^2+z^2$ with $3((2x)^2-x^2)=(3x)^2$ and $(2x)^2-3x^2=x^2$. Since

$$E(1,1,2)\cap E(1,1,5)=E(1,1,1)\cap E(1,1,5)=\emptyset$$

by Lemma 3.2, we conclude that both $3(x^2-y^2)$ and x^2-3y^2 are suitable polynomials.

Now suppose that $n\in E(1,1,2)$. By (2.1), we have $n=4^k(16l+14)$ for some $k,l\in\mathbb{N}$. As $n\not\in E(1,1,1)$ and $2\mid n$, there are $u,v,w\in\mathbb{N}$ satisfying $n=(2w)^2+0^2+u^2+v^2$ with $(2w)^2-0^2$ an even square. Since n is even and $n\neq 4^i(16j+6)$ for any $i,j\in\mathbb{N}$, we have $n\not\in E(1,1,10)$ by (3.2). Thus, there are $x,y,z\in\mathbb{N}$ satisfying $n=10x^2+y^2+z^2=(3x)^2+x^2+y^2+z^2$ with $2((3x)^2-x^2)=(4x)^2$ and $3(3x)^2-2x^2=(5x)^2$.

In view of the above, we have proved part (i) of Theorem 1.5.

(ii) We first prove that x^2+ky^2 $(k=2,3,5,6,8,12)$ and $7x^2+9y^2$ are all suitable.

Let $n\in\mathbb{N}$. If $n\not\in E(1,1,1)$, then there are $x,y,z\in\mathbb{N}$ such that $n=x^2+0^2+y^2+z^2$ with $x^2+k0^2=x^2$ for any $k\in\mathbb{Z}^+$, and $9x^2+7\times 0^2=(3x)^2$.

Now assume that $n\in E(1,1,1)$. As $n\not\in E(1,1,2)$ by Lemma 3.2, there are $x,y,z,w\in\mathbb{N}$ with $x=y$ and $n=x^2+y^2+z^2+w^2$, hence $x^2+3y^2=(2x)^2$, $x^2+8y^2=(3x)^2$ and $7x^2+9y^2=(4x)^2$. As $n\not\in E(1,1,5)$ by Lemma 3.2, there are $x,y,z,w\in\mathbb{N}$ with $2x=y$ and $n=x^2+y^2+z^2+w^2$, hence

$$x^2+2y^2=(3x)^2,\ y^2+5x^2=(3x)^2,\ x^2+6y^2=(5x)^2,\ x^2+12y^2=(7x)^2.$$

Next we show that $3x^2+4y^2,4x^2+5y^2$ and $4x^2+9y^2$ are all suitable.

Let $n\in\mathbb{N}$. If $n\not\in E(1,1,1)$, then for some $x,y,z\in\mathbb{N}$ we have $n=x^2+0^2+y^2+z^2$ with $4x^2+b\times 0^2=(2x)^2$ for $b=3,5,9$.

Now assume that $n\in E(1,1,1)$. As $n\not\in E(1,1,5)$ by Lemma 3.2, there are $x,y,z,w\in\mathbb{N}$ with $y=2x$ such that $n=x^2+y^2+z^2+w^2$ and hence $3y^2+4x^2=(4x)^2$ and $4y^2+9x^2=(5x)^2$. As $n\not\in E(1,1,2)$ by Lemma 3.2, there are $x,y,z,w\in\mathbb{N}$ with $y=x$ such that $n=x^2+y^2+z^2+w^2$ and hence $4x^2+5y^2=(3x)^2$.

Finally we prove that $2x^2+7y^2$, $5x^2+11y^2$ and $6x^2+10y^2$ are suitable.

Let $n\in\mathbb{N}$. If $n\not\in E(1,1,2)$, then there are $x,y,z,w\in\mathbb{N}$ with $y=x$ such that $n=x^2+y^2+z^2+w^2$ and hence

$$2x^2+7y^2=(3x)^2\ \text{ and }\ 5x^2+11y^2=(4x)^2=6x^2+10y^2.$$

Below we assume that $n\in E(1,1,2)$. Then $2\mid n$ by (2.1). As $n\not\in E(1,1,5)$ by Lemma 3.2, there are $x,y,z,w\in\mathbb{N}$ with $y=2x$ such that $n=x^2+y^2+z^2+w^2$ and hence $5x^2+11y^2=(7x)^2$. Since $2\mid n$ and $n\not\in E(1,1,10)\cap 2\mathbb{Z}$ by Lemma 3.2, there are $x,y,z,w\in\mathbb{N}$ with $y=3x$ such that $n=x^2+y^2+z^2+w^2$ and hence $2y^2+7x^2=(5x)^2$ and $6y^2+10x^2=(8x)^2$.

In view of the above, we have completed the proof of Theorem 1.5. □

4. Some open conjectures

Motivated by our results in Section 1, we pose the following conjectures for further research.

Conjecture 4.1. *Let $a, b \in \mathbb{Z}^+$ with $\gcd(a, b)$ squarefree. Then $ax+by$ is suitable if and only if $\{a, b\} = \{1, 2\}, \{1, 3\}, \{1, 24\}$. Also, $ax - by$ is suitable if and only if (a, b) is among the ordered pairs*

$$(1, 1),\ (2, 1),\ (2, 2),\ (4, 3),\ (6, 2).$$

Remark 4.1. By Theorem 1.2(i), both $x - y$ and $2x - 2y$ are suitable. Though we have Theorem 1.2(iii) and Theorem 1.3(ii), we are not able to show that $x + 2y$ or $2x - y$ or $x + 3y$ is suitable. See [S16, A273404] for the number of ordered ways to write $n = x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $z \leqslant w$ such that $x + 24y$ is a square. We also guess that any $n \in \mathbb{N}$ with $n \neq 47$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 7y$ a square, and that any integer $n > 3$ can be expressed as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $3x - y$ a square.

Conjecture 4.2. (i) *For each $k = 1, 2, 3$ and $n \in \mathbb{Z}^+$, there are $x, y, w \in \mathbb{N}$ and $z \in \mathbb{Z}^+$ with $n = x^2 + y^2 + z^2 + w^2$ such that $(x + ky)z$ is a square.*

(ii) *Any positive integer can be written $x^2 + y^2 + z^2 + w^2$ with $x, y, w \in \mathbb{N}$ and $z \in \mathbb{Z}^+$ such that $(ax - by)z$ is a square, where (a, b) is either of the ordered pairs*

$$(1, 2),\ (2, 2),\ (3, 2),\ (3, 3),\ (4, 2),\ (6, 6).$$

Remark 4.2. In view of Theorem 1.4(i), $a(x - y)z$ $(a = 2, 3, 6)$, $(x - 2y)z$ and $(4x - 2y)z$ are all suitable. We also conjecture that $(ax + by)z$ is suitable for any ordered pair (a, b) among

$$(2, 5),\ (3, 3),\ (3, 6),\ (3, 15),\ (5, 6),\ (5, 11),\ (5, 13),\ (5, 15),\ (6, 15),\ (8, 46),\ (9, 23).$$

Conjecture 4.3. (i) (1-3-5-Conjecture) *Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 3y + 5z$ is a square.*

(ii) *Any integer $n > 15$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $3x + 5y + 6z$ is twice a square.*

(iii) *Let $a, b, c \in \mathbb{Z}^+$ with $b \leqslant c$ and $\gcd(a, b, c)$ squarefree. Then $ax - by - cz$ is suitable if and only if (a, b, c) is among the five triples*

$$(1, 1, 1),\ (2, 1, 1),\ (2, 1, 2),\ (3, 1, 2),\ (4, 1, 2).$$

(iv) *Let $a, b, c \in \mathbb{Z}^+$ with $a \leqslant b$ and $\gcd(a, b, c)$ squarefree. Then $ax + by - cz$*

is suitable if and only if (a,b,c) *is among the following* 52 *triples:*

$$\begin{gathered}(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,3),\ (1,3,1),\\ (1,3,3),\ (1,4,4),\ (1,5,1),\ (1,6,6),\ (1,8,6),\ (1,12,4),\ (1,16,1),\\ (1,17,1),\ (1,18,1),\ (2,2,2),\ (2,2,4),\ (2,3,2),\ (2,3,3),\ (2,4,1),\\ (2,4,2),\ (2,6,1),\ (2,6,2),\ (2,6,6),\ (2,7,4),\ (2,7,7),\ (2,8,2),\\ (2,9,2),\ (2,32,2),\ (3,3,3),\ (3,4,2),\ (3,4,3),\ (3,8,3),\ (4,5,4),\\ (4,8,3),\ (4,9,4),\ (4,14,14),\ (5,8,5),\ (6,8,6),\ (6,10,8),\ (7,9,7),\\ (7,18,7),\ (7,18,12),\ (8,9,8),\ (8,14,14),\ (8,18,8),\ (14,32,14),\\ (16,18,16),\ (30,32,30),\ (31,32,31),\ (48,49,48),\ (48,121,48).\end{gathered}$$

Remark 4.3. We guess that if a,b,c are positive integers with $\gcd(a,b,c)$ squarefree such that any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ with $ax+by+cz$ a square then we must have $\{a,b,c\}=\{1,3,5\}$. Concerning the 1-3-5-Conjecture, see [S16, A271518, A273294, A273302, A278560] for related data; for example, $43=1^2+5^2+4^2+1^2$ with $1+3\times5+5\times4=6^2$. We have verified parts (i) and (ii) of Conjecture 4.3 for n up to 3×10^7. The author would like to offer 1350 US dollars as the prize for the first complete solution of the 1-3-5-Conjecture. We also conjecture that any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{Z}$ such that $x+3y+5z$ is a cube. For parts (iii) and (iv) of Conjecture 4.3, see the comments in [S16, A271775].

Conjecture 4.4. (i) *For each* $c=1,2,4$*, any* $n\in\mathbb{N}$ *can be written as* $w^2+x^2+y^2+z^2$ *with* $w,x,y,z\in\mathbb{N}$ *and* $y\leqslant z$ *such that* $2x+y-z=ct^3$ *for some* $t\in\mathbb{N}$.

(ii) *Any* $n\in\mathbb{N}$ *can be written as* $w^2+x^2+y^2+z^2$ *with* $w\in\mathbb{Z}$ *and* $x,y,z\in\mathbb{N}$ *such that* $w+x+2y-4z$ *is twice a nonnegative cube.*

(iii) *Any* $n\in\mathbb{N}$ *not of the form* $4^{2k+1}\times7$ $(k\in\mathbb{N})$ *can be written as* $w^2+x^2+y^2+z^2$ $(w,x,y,z\in\mathbb{N})$ *with* $w+2x+3y+5z$ *a square. Also, for any* $a,b,c,d\in\mathbb{Z}^+$*, there are infinitely many positive integers which cannot be written as* $w^2+x^2+y^2+z^2$ $(w,x,y,z\in\mathbb{N})$ *with* $aw+bx+cy+dz$ *a square.*

(iv) *Let* $a,b,c,d\in\mathbb{Z}^+$ *with* $a\leqslant b\leqslant c$ *and* $\gcd(a,b,c,d)$ *squarefree. Then* $ax+by+cz-dw$ *is suitable if and only if* (a,b,c,d) *is among the following quadruples*

$$\begin{gathered}(1,1,2,1),\ (1,2,3,1),\ (1,2,3,3),\ (1,2,4,2),\ (1,2,4,4),\ (1,2,5,5),\\ (1,2,6,2),\ (1,2,8,1),\ (2,2,4,4),\ (2,4,6,4),\ (2,4,6,6),\ (2,4,8,2).\end{gathered}$$

(v) *Any positive integer can be written as* $w^2+x^2+y^2+z^2$ *with* $w+x+y-z$ *a square, where* $w\in\mathbb{Z}$ *and* $x,y,z\in\mathbb{N}$ *with* $|w|\leqslant x\geqslant y\leqslant z<x+y$*. Also, any* $n\in\mathbb{N}$ *can be written as* $w^2+x^2+y^2+z^2$ *with* $w+x+y-z$ *a nonnegative cube, where* w,x,y,z *are integers with* $|x|\leqslant y\geqslant z\geqslant 0$.

(vi) *Any $n \in \mathbb{Z}^+$ can be written as $w^2+x^2+y^2+z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{Z}$ such that $aw + bx + cy + dz$ is a nonnegative cube, whenever (a, b, c, d) is among the quadruples*

$$(1, 1, 1, 3),\ (1, 1, 2, 2),\ (1, 1, 2, 4),\ (1, 1, 3, 3),\ (8, 2, 6, 8),\ (8, 4, 4, 8),\ (8, 4, 8, 12).$$

Remark 4.4. See [S16, A273432, A273568, A279522, A272620 and A273458] for related data.

Conjecture 4.5. *Let $a, b, c \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c$ and $\gcd(a, b, c)$ squarefree. Then the polynomial $w(ax + by + cz)$ is suitable if and only if (a, b, c) is among the five triples*

$$(1, 2, 3),\ (1, 3, 6),\ (1, 6, 9),\ (5, 6, 9),\ (18, 30, 114).$$

Remark 4.5. See [S16, A271724] for the number of ways to write $n = w^2 + x^2 + y^2 + z^2$ with $w, y, z \in \mathbb{N}$ and $x \in \mathbb{Z}^+$ such that $w(x + 2y + 3z)$ is a square.

Conjecture 4.6. (i) *Any $n \in \mathbb{Z}^+$ can be written as $w^2+x^2+y^2+z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $wx + 2xy + 2yz$ (or $2wx + xy + 4yz$) is a square. Also, any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$, $x, y, z \in \mathbb{N}$ and $x \leqslant y$ such that $2xy + yz - zw - wx$ is a square.*

(ii) *Any $n \in \mathbb{Z}^+$ can be written as $w^2+x^2+y^2+z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $w^2 + 4xy + 8yz + 32zx$ is a square.*

(iii) *For each $k = 1, 2, 8, 16, 48, 336$, any positive integer can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $w^2 + k(xy + yz)$ is a square.*

(iv) *Let $a, b, c \in \mathbb{Z}^+$ with $\gcd(a, b, c)$ squareferee. Then the polynomial $axy + byz + czx$ is suitable if and only if $\{a, b, c\}$ is among*

$$\{1, 2, 3\},\ \{1, 3, 8\},\ \{1, 8, 13\},\ \{2, 4, 45\},\ \{4, 5, 7\},\ \{4, 7, 23\},\ \{5, 8, 9\},\ \{11, 16, 31\}.$$

(v) *Any positive integer can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $wx + xy + 2yz + 3zx$ (or $wx + 3xy + 8yz + 5zx$) is twice a square. Also, each $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $6wx + 2xy + 3yz + 4zx = 3t^2$ for some $t \in \mathbb{N}$.*

Remark 4.6. See [S16, A271644, A273021 and A271665] for related data.

Conjecture 4.7. (i) *Any natural number can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x \geqslant y$ such that $ax^2 + by^2 + cz^2$ is a square, provided that the triple (a, b, c) is among*

$$(1, 8, 16),\ (4, 21, 24),\ (5, 40, 4),\ (9, 63, 7),\ (16, 80, 25),\ (36, 45, 40),\ (40, 72, 9).$$

(ii) *$ax^2+by^2+cz^2$ is suitable if (a,b,c) is among the triples*

$(1,3,12)$, $(1,3,18)$, $(1,3,21)$, $(1,3,60)$, $(1,5,15)$, $(1,8,24)$, $(1,12,15)$, $(1,24,56)$, $(3,4,9)$, $(3,9,13)$, $(4,5,12)$, $(4,5,60)$, $(4,9,60)$, $(4,12,21)$, $(4,12,45)$, $(5,36,40)$.

(iii) *If a,b,c are positive integers with $ax^2+by^2+cz^2$ suitable, then a,b,c cannot be pairwise coprime.*

Remark 4.7. See [S16, A271510 and A271513] for related data.

Conjecture 4.8. (i) *Any $n\in\mathbb{Z}^+$ can be written as $w^2+x^2+y^2+z^2$ with $w\in\mathbb{Z}^+$ and $x,y,z\in\mathbb{N}$ such that $(10w+5x)^2+(12y+36z)^2$ is a square.*

(ii) *Each positive integer can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ and $y>z$ such that $(x+y)^2+(4z)^2$ is a square.*

(iii) *Any integer $n>5$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$, $x+y>0$ and $z>0$ such that $(8x+12y)^2+(15z)^2$ is a square (i.e., $8x+12y$ and $15z$ are the two legs of a right triangle with positive integer sides).*

(iv) *Any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ and $x+y\geqslant z$ such that $(x+y+z)^2+(4(x+y-z))^2$ is a square.*

(v) *Any $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ and $y<z$ such that $x+8y+8z+15w$ and $6(x+y+z+w)$ are the two legs of a right triangle with positive integer sides. Also, all the polynomials*

$$(x+3y+6z+17w)^2+(20x+4y+8z+4w)^2,$$
$$(x+3y+9z+17w)^2+(20x+4y+12z+4w)^2,$$
$$(x+3y+11z+15w)^2+(12x+4y+4z+20w)^2,$$
$$(3(x+2y+3z+4w))^2+(4(x+4y+3z+2w))^2,$$
$$(3(x+2y+3z+4w))^2+(4(x+5y+3z+w))^2$$

are suitable.

Remark 4.8. This conjecture is particularly mysterious since it is related to Pythagorean triples. See [S16, A271714, A273108, A273107 and A273134] for related data.

Conjecture 4.9. (i) *Any $n\in\mathbb{Z}^+$ can be written as $w^2+x^2(1+y^2+z^2)$ with $w,x,y,z\in\mathbb{N}$, $x>0$ and $y\equiv z \pmod 2$. Moreover, any $n\in\mathbb{Z}^+$ with $n\neq 449$ can be written as $4^k(1+x^2+y^2)+z^2$ with $k,x,y,z\in\mathbb{N}$ and $x\equiv y \pmod 2$.*

(ii) *Each $n\in\mathbb{Z}^+$ can be written as $4^k(1+x^2+y^2)+z^2$ with $k,x,y,z\in\mathbb{N}$ and $x\leqslant y\leqslant z$.*

(iii) *Any $n\in\mathbb{Z}^+$ can be written as $4^k(1+5x^2+y^2)+z^2$ with $k,x,y,z\in\mathbb{N}$, and also each $n\in\mathbb{Z}^+$ can be written as $4^k(1+x^2+y^2)+5z^2$ with $k,x,y,z\in\mathbb{N}$.*

Remark 4.9. This conjecture was motivated by Theorem 1.2(v). We can show the first part provided that any integer $n>432$ can be written as $2x^2+2y^2+z(4z+1)$ with $x,y,z\in\mathbb{Z}$. Under the GRH we are able to prove Conjecture 4.9(iii) with the help of the work in [KS]. See [S16, A275738, A275656, A275675 and A275676] for related data.

Conjecture 4.10. (i) x^2+kyz $(k=12,24,32,48,84,120,252)$, $9x^2-4yz$, $9x^2+140yz$, $25x^2+24yz$ *and* $121x^2-20yz$ *are all suitable.*

(ii) *The polynomials* $w(x^2+8y^2-z^2)$, $(3x^2+13y^2)z$, $(5x^2+11y^2)z$, $(15x^2+57y^2)z$, $(15x^2+165y^2)z$ *and* $(138x^2+150y^2)z$ *are all suitable.*

(iii) *Any positive integer can be written as* $w^2+x^2+y^2+z^2$ *with* $w\in\mathbb{Z}^+$ *and* $x,y,z\in\mathbb{N}$ *such that* $36x^2y+12y^2z+z^2x$ (*or* $x^3+4yz(y-z)$, *or* $x^3+8yz(2y-z)$) *is a square.*

(iv) *Let* a *and* b *be nonzero integers with* $\gcd(a,b)$ *squarefree. Then the polynomial* ax^4+by^3z *is suitable if and only if* (a,b) *is among the ordered pairs*

$$(1,1),\ (1,15),\ (1,20),\ (1,36),\ (1,60),\ (1,1680),\ (9,260).$$

(v) *For each triple* $(a,b,c)=(1,20,60),\ (1,24,56),\ (9,20,60),\ (9,32,96)$, *any* $n\in\mathbb{Z}^+$ *can be written as* $x^2+y^2+z^2+w^2$ *with* $x,y,z\in\mathbb{N}$ *and* $w\in\mathbb{Z}^+$ *such that* $ax^4+by^3z+cyz^3$ *is a square.*

Remark 4.10. See [S16, A272888, A272332, A279056, A272336, A280831 and A272351] for related data and comments. Part (iii) of Conjecture 4.10 looks very curious. The author ever guessed that x^2-4yz, x^2+4yz and x^2+8yz are suitable, then his student You-Yin Deng made a clever observation:

$$x^2+2y^2+6z^2=x^2+(y+z)^2+(y-z)^2+(2z)^2 \text{ with } (2z)^2+4(y+z)(y-z)=(2y)^2.$$

Since $x^2\pm 4yz=x^2\pm(y+z)^2\mp(z-y)^2$, and $x(x+y-z)$ is suitable (cf. Theorem 1.4(i)), x^2+4yz and x^2-4yz are indeed suitable. The author's student Yu-Chen Sun has proved that x^2+8yz is also suitable.

Conjecture 4.11. (i) *Any positive integer can be written as* $x^2+y^2+z^2+w^2$ *with* $x,y,z,w\in\mathbb{N}$ *and* $z<w$ *such that* $4x^2+5y^2+20zw$ *is a square. Also, the polynomials* x^2+8y^2+8zw, $(3x+5y)^2-24zw$, $(x-2y)^2+24zw$ *and* $(x-3y)^2+16zw$ *are suitable.*

(ii) *The polynomials* $x^2+3y^2+4z^2+(x+y+z)^2$, $x^2+3y^2+5z^2-8w^2$, $(x-2y)^2+8z^2+16w^2$, $4(x-3y)^2+9z^2+12w^2$, $(x+2y)^2+8z^2+40w^2$ *and* $9(x+2y)^2+16z^2+24w^2$ *are all suitable.*

(iii) *The polynomial* $w^2x^2+3x^2y^2+2y^2z^2$ *is suitable.*

(iv) *Any positive integer can be written as* $w^2+x^2+y^2+z^2$ *with* $w\in\mathbb{Z}^+$ *and* $x,y,z\in\mathbb{N}$ *such that* $w^2x^2+5x^2y^2+80y^2z^2+20z^2w^2$ *is a square.*

Remark 4.11. See [S16, A272084, A271778, A271824, A273278, A269400 and A262357] for related data and more similar conjectures.

Conjecture 4.12. (i) *Let* $a,b,c,d\in\mathbb{Z}^+$ *with* $a\leqslant b$ *and* $c\leqslant d$, *and* $\gcd(a,b,c,d)$ *squarefree. Then* $ax+by-cz-dw$ *is suitable if and only if* (a,b,c,d) *is among the quadruples*

$$(1,2,1,1),\ (1,2,1,2),\ (1,3,1,2),\ (1,4,1,3),$$
$$(2,4,1,2),\ (2,4,2,4),\ (8,16,7,8),\ (9,11,2,9),\ (9,16,2,7).$$

(ii) *The polynomial* $ax^2 + by^2 - cz^2 - dw^2$ *is suitable if* (a, b, c, d) *is among the quadruples*

$$(3, 9, 3, 20),\ (5, 9, 5, 20),\ (5, 25, 4, 5),\ (9, 81, 9, 20),\ (12, 16, 3, 12),\ (16, 64, 15, 16),$$
$$(20, 25, 4, 20),\ (27, 81, 20, 27),\ (30, 64, 15, 30),\ (32, 64, 15, 32),\ (48, 64, 15, 48).$$

Remark 4.12. We also conjecture that $ax^2 - by^2 - cz^2$ is suitable if (a, b, c) is among the triples

$$(21, 5, 15),\ (36, 3, 8),\ (48, 8, 39),\ (64, 7, 8),\ (40, 15, 144),\ (45, 20, 144),\ (69, 20, 60).$$

Conjecture 4.13. (i) *The polynomial* $xyz(x + 9y + 11z + 10w)$ *is suitable.*

(ii) *Any* $n \in \mathbb{Z}^+$ *can be written as* $x^2 + y^2 + z^2 + w^2$ *with* $x \in \mathbb{Z}^+$, $y, z, w \in \mathbb{N}$ *and* $y \geqslant z$ *such that* $xyz(x + 3y + 13z)$ *is a square.*

(iii) *Any* $n \in \mathbb{Z}^+$ *can be written as* $x^2 + y^2 + z^2 + w^2$ *with* $xy(3x + 5y + 2z + 3w)$ (*or* $xy(x + 11y + z + 2w)$, *or* $2xy(x + 2y + z + 2w)$, *or* $2xy(x + 6y + z + 2w)$) *a square, where* x, y, z, w *are nonnegative integers with* $w > 0$ (*or* $z > 0$).

(iv) *The polynomial* $xy(ax^2 + by^2 + cz^2)$ *is suitable whenever* (a, b, c) *is among the triples*

$$(1, 8, 20),\ (3, 5, 15),\ (6, 14, 4),\ (7, 9, 5),\ (7, 29, 5),\ (18, 38, 18),\ (39, 81, 51).$$

(v) *If* (a, b, c) *is one of the six triples*

$$(1, 2, 4),\ (1, 2, 9),\ (1, 3, 4),\ (2, 3, 4),\ (2, 4, 6),\ (4, 8, 10),$$

then any $n \in \mathbb{Z}^+$ *can be written as* $w^2 + x^2 + y^2 + z^2$ *with* $w \in \mathbb{Z}^+$ *and* $x, y, z \in \mathbb{N}$ *such that* $w(25w + 24(ax + by + cz))$ *is a square.*

Remark 4.13. See [S16, A267121, A260625, A261876 and A268197] for related data.

Conjecture 4.14. (i) *Any* $n \in \mathbb{N}$ *can be written as* $x^2 + y^2 + z^2 + w^2$ *with* $x, y, z, w \in \mathbb{N}$ *such that* $xy + 2zw$ *or* $xy - 2zw$ *is a square. Also, each* $n \in \mathbb{N}$ *can be written as* $x^2 + y^2 + z^2 + w^2$ *with* $x, y, z, w \in \mathbb{N}$ *and* $\max\{x, y\} \geqslant \min\{z, w\}$ *such that* $xy + zw/2$ *or* $xy - zw/2$ *is a square.*

(ii) *Any* $n \in \mathbb{N}$ *can be written as* $x^2 + y^2 + z^2 + w^2$ *with* $x, y, z \in \mathbb{N}$, $w \in \mathbb{Z}$ *and* $x \geqslant z$ *such that* $3x^2y + z^2w$ *is a square. Also, for each ordered pair* $(a, b) = (7, 1),\ (8, 1),\ (9, 2)$, *any* $n \in \mathbb{N}$ *can be written as* $x^2 + y^2 + z^2 + w^2$ *with* $x, y, z \in \mathbb{N}$ *and* $w \in \mathbb{Z}$ *such that* $ax^2y + bz^2w$ *is a square.*

(iii) *Any* $n \in \mathbb{Z}^+$ *can be written as* $x^2 + y^2 + z^2 + w^2$ *with* $x \in \mathbb{Z}^+$, $y \in \mathbb{N}$ *and* $z, w \in \mathbb{Z}$ *such that* $xy + yz + zw$ *is a fourth power. Also, any* $n \in \mathbb{Z}^+$ *can be written as* $x^2 + y^2 + z^2 + w^2$ *with* $x, y, z \in \mathbb{Z}$ *and* $w \in \mathbb{Z}^+$ *such that* $xy + yz + 2zw + 2wx$ *is a fifth power.*

Remark 4.14. See [S16, A270073, A272977 and A273826] for related data. We have verified Conjecture 4.14(i) for all $n = 0, 1, \ldots, 2 \times 10^5$.

Conjecture 4.15. *Any $n \in \mathbb{N}$ can be written as $p_5(u) + p_5(v) + p_5(x) + p_5(y) + p_5(z)$ with $u, v, x, y, z \in \mathbb{N}$ such that $u + 2v + 4x + 5y + 6z$ is a pentagonal number, where $p_5(k)$ with $k \in \mathbb{N}$ denotes the pentagonal number $k(3k-1)/2$.*

Remark 4.15. As conjectured by Fermat and proved by Cauchy, each natural number can be written as the sum of five pentagonal numbers (cf. [N96, pp. 27-34] or [MW. pp. 54-57]). See [S16, A271608] for some data related to Conjecture 4.15.

Conjecture 4.16. (i) *Any $n \in \mathbb{N}$ can be written as $\sum_{i=1}^{9} x_i^3$ with $x_i \in \mathbb{N}$ such that*

$$x_1 + x_2 + x_3 + 2x_4 + 3x_5 + 4x_6 + 4x_7 + 9x_8 + 15x_9$$

is a cube.

(ii) *Any $n \in \mathbb{N}$ can be written as $\sum_{i=1}^{9} x_i^3$ with $x_i \in \mathbb{N}$ such that*

$$x_1^3 + x_2^3 + x_3^3 + 2x_4^3 + 3x_5^3 + 4x_6^3 + 5x_7^3 + 14x_8^3 + 19x_9^3$$

is a cube.

(iii) *Any $n \in \mathbb{N}$ can be written as $\sum_{i=1}^{9} x_i^3$ with $x_i \in \mathbb{N}$ such that $\sum_{i=1}^{9} ix_i^4$ (or $\sum_{i=1}^{9} ix_i^2$) is a square.*

Remark 4.16. It is well-known that any natural number is the sum of nine nonnegative cubes (cf. [N96, pp. 41-43]). We even conjecture further that any $n \in \mathbb{N}$ can be written as $u^3 + v^3 + 2x^3 + 2y^3 + 3z^3$ with $u, v, x, y, z \in \mathbb{N}$ (cf. [S16, A271099]).

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