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SUMS OF FOUR POLYGONAL NUMBERS WITH COEFFICIENTS

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Abstract. Let $m \geqslant 3$ be an integer. The polygonal numbers of order $m+2$ are given by $p_{m+2}(n) = m\binom{n}{2} + n$ $(n = 0, 1, 2, \ldots)$. A famous claim of Fermat proved by Cauchy asserts that each nonnegative integer is the sum of $m+2$ polygonal numbers of order $m+2$. For $(a, b) = (1, 1), (2, 2), (1, 3), (2, 4)$, we study whether any sufficiently large integer can be expressed as

$$p_{m+2}(x_1) + p_{m+2}(x_2) + ap_{m+2}(x_3) + bp_{m+2}(x_4)$$

with x_1, x_2, x_3, x_4 nonnegative integers. We show that the answer is positive if $(a, b) \in \{(1, 3), (2, 4)\}$, or $(a, b) = (1, 1)$ & $4 \mid m$, or $(a, b) = (2, 2)$ & $m \not\equiv 2 \pmod 4$. In particular, we confirm a conjecture of Z.-W. Sun which states that any natural number can be written as $p_6(x_1) + p_6(x_2) + 2p_6(x_3) + 4p_6(x_4)$ with x_1, x_2, x_3, x_4 nonnegative integers.

1. Introduction

Let $m \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. The *polygonal numbers of order* $m+2$ (or $(m+2)$-*gonal numbers*), which are constructed geometrically from the regular polygons with $m+2$ sides, are given by

$$p_{m+2}(n) := m\binom{n}{2} + n = \frac{mn^2 - (m-2)n}{2} \text{ for } n \in \mathbb{N} = \{0, 1, 2, \ldots\}. \quad (1.1)$$

Clearly,

$$p_{m+2}(0) = 0,\ p_{m+2}(1) = 1,\ p_{m+2}(2) = m+2,\ p_{m+2}(3) = 3m+3,$$

and $p_{m+2}(x)$ with $x \in \mathbb{Z}$ are called *generalized* $(m+2)$-*gonal numbers*. It is easy to see that generalized hexagonal numbers coincide with triangular numbers (i.e., those $p_3(n) = n(n+1)/2$ with $n \in \mathbb{N}$). Note that

$$p_4(n) = n^2,\ p_5(n) = \frac{n(3n-1)}{2},\ p_6(n) = n(2n-1) = p_3(2n-1).$$

Fermat's claim that each $n \in \mathbb{N}$ can be written as the sum of $m+2$ polygonal numbers of order $m+2$ was proved by Lagrange in the case $m = 2$, by Gauss in the case $m = 1$, and by Cauchy in the case $m \geqslant 3$ (cf. [9, pp. 3-35] and [7,

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pp. 54-57]). In 1830 Legendre refined Cauchy's polygonal number theorem by showing that any integer $N \geqslant 28m^3$ with $m \geqslant 3$ can be written as

$$p_{m+2}(x_1)+p_{m+2}(x_2)+p_{m+2}(x_3)+p_{m+2}(x_4)+\delta_m(N)$$

where $x_1,x_2,x_3,x_4\in\mathbb{N}$, $\delta_m(N)=0$ if $2\nmid m$, and $\delta_m(N)\in\{0,1\}$ if $2\mid m$. Nathanson ([8] and [9, p. 33]) simplified the proofs of Cauchy's and Legendre's theorems.

In 1917 Ramanujan [10] listed 55 possible quadruples (a,b,c,d) of positive integers with $a\leqslant b\leqslant c\leqslant d$ such that any $n\in\mathbb{N}$ can be written as $ax^2+by^2+cz^2+dw^2$ with $x,y,z,w\in\mathbb{Z}$, and 54 of them were later confirmed by Dickson [2] while the remaining one on the list was actually wrong.

Recently, Sun [12] showed that any positive integer can be written as the sum of four generalized octagonal numbers one of which is odd. He also proved that for many triples (b,c,d) of positive integers (including (1,1,3), (1,2,2) and (1,2,4)) we have

$$\{p_8(x_1)+bp_8(x_2)+cp_8(x_3)+dp_8(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{Z}\}=\mathbb{N}.$$

In [12, Conjecture 5.3], Sun conjectured that any $n\in\mathbb{N}$ can be written as $p_6(x_1)+p_6(x_2)+2p_6(x_3)+4p_6(x_4)$ with $x_1,x_2,x_3,x_4\in\mathbb{N}$.

Motivated by the above work, for $(a,b)=(1,1),(2,2),(1,3),(2,4)$ and $m\in\{3,4,5,\ldots\}$, we study whether any sufficiently large integer can be written as

$$p_{m+2}(x_1)+p_{m+2}(x_2)+ap_{m+2}(x_3)+bp_{m+2}(x_4)\ \text{ with } x_1,x_2,x_3,x_4\in\mathbb{N}.$$

Now we state our main results.

Theorem 1.1. *Let $m\in\mathbb{Z}^+$ with $4\mid m$.*

(i) *Any integer $N\geqslant 28m^3$ can be expressed as*

$$p_{m+2}(x_1)+p_{m+2}(x_2)+p_{m+2}(x_3)+p_{m+2}(x_4)\ \ (x_1,x_2,x_3,x_4\in\mathbb{N}). \tag{1.2}$$

(ii) *There are infinitely many positive integers not of the form $p_{m+4}(x_1)+p_{m+4}(x_2)+p_{m+4}(x_3)+p_{m+4}(x_4)$ with $x_1,x_2,x_3,x_4\in\mathbb{N}$.*

Remark 1.1. This can be viewed as a supplement to Legendre's theorem. By Theorem 1.1(ii), there are infinitely many positive integers none of which is the sum of four octagonal numbers; in contrast, Sun [12] showed that any $n\in\mathbb{N}$ is the sum of four generalized octagonal numbers.

Corollary 1.1. *We have*

$$\begin{aligned}&\{p_6(x_1)+p_6(x_2)+p_6(x_3)+p_6(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}\\ =&\mathbb{N}\setminus\{5,\ 10,\ 11,\ 20,\ 25,\ 26,\ 38,\ 39,\ 54,\ 65,\ 70,\ 114,\ 130\}\end{aligned}\tag{1.3}$$

and hence any $n\in\mathbb{N}$ can be written as the sum of a triangular number and three hexagonal numbers. Also, any integer $n>2146$ can be written as the sum of four decagonal numbers and thus

$$\{p_{10}(x_1)+p_{10}(x_2)+p_{10}(x_3)+p_{10}(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{Z}\}=\mathbb{N}\setminus\{5,\ 6,\ 26\}.\tag{1.4}$$

Proof. Via a computer, we can easily verify that

$$5,\ 10,\ 11,\ 20,\ 25,\ 26,\ 38,\ 39,\ 54,\ 65,\ 70,\ 114,\ 130$$

are the only natural numbers smaller than 28×4^3 which cannot be written as the sum of four hexagonal numbers, but all these numbers can be expressed as the sum of a triangular number and three hexagonal numbers. Also, every $n=2147,\ldots,28\times8^3-1$ is the sum of four decagonal numbers, and 5, 6 and 26 are the only natural numbers smaller than 2147 which cannot be written as the sum of four generalized decagonal numbers. Now it suffices to apply Theorem 1.1 with $m=4,8$. □

Remark 1.2. Sun [11, Conjecture 1.10] conjectured that any $n\in\mathbb{N}$ can be written as the sum of two triangular numbers and a hexagonal number. Krachun [6] proved that

$$\begin{aligned}&\{p_6(-w)+p_6(-x)+p_6(y)+p_6(z):\ w,x,y,z\in\mathbb{N}\}\\ =&\{p_6(-w)+2p_6(-x)+p_6(y)+2p_6(z):\ w,x,y,z\in\mathbb{N}\}=\mathbb{N},\end{aligned}$$

which was first conjectured by the second author [12].

Theorem 1.2. *Let $m\geqslant3$ be an integer.*

(i) *Suppose that $2\nmid m$ or $4\mid m$. Then any integer $N\geqslant1628m^3$ can be written as*

$$p_{m+2}(x_1)+p_{m+2}(x_2)+2p_{m+2}(x_3)+2p_{m+2}(x_4)\ \ (x_1,x_2,x_3,x_4\in\mathbb{N}).\tag{1.5}$$

(ii) *If $m\equiv2\pmod4$, then there are infinitely many positive integers not represented by $p_{m+2}(x_1)+p_{m+2}(x_2)+2p_{m+2}(x_3)+2p_{m+2}(x_4)$ with $x_1,x_2,x_3,x_4\in\mathbb{N}$.*

Remark 1.3. Actually our proof of Theorem 1.2(i) given in Section 3 allows us to replace $1628m^3$ by $418m^3$ in the case $m\equiv1\pmod2$. By Theorem

1.2(ii), there are infinitely many positive integers not represented by $p_8(x_1)+p_8(x_2)+2p_8(x_3)+2p_8(x_4)$ with $x_1,x_2,x_3,x_4\in\mathbb{N}$; in contrast, Sun [12] proved that any $n\in\mathbb{N}$ can be written as $p_8(x_1)+p_8(x_2)+2p_8(x_3)+2p_8(x_4)$ with $x_1,x_2,x_3,x_4\in\mathbb{Z}$.

Corollary 1.2. *We have*

$$\{p_5(x_1)+p_5(x_2)+2p_5(x_3)+2p_5(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}=\mathbb{N}, \tag{1.6}$$

$$\begin{aligned}&\{p_6(x_1)+p_6(x_2)+2p_6(x_3)+2p_6(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}\\&\quad=\mathbb{N}\setminus\{22,\ 82,\ 100\},\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}&\{p_7(x_1)+p_7(x_2)+2p_7(x_3)+2p_7(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}\\&\quad=\mathbb{N}\setminus\{13,\ 26,\ 31,\ 65,\ 67,\ 173,\ 175,\ 215,\ 247\}.\end{aligned} \tag{1.8}$$

Also, for each $k=9,10,11$, any integer $n>C_k$ can be written as $p_k(x_1)+p_k(x_2)+2p_k(x_3)+2p_k(x_4)$ with $x_1,x_2,x_3,x_4\in\mathbb{N}$, where $C_9=925$, $C_{10}=840$ and $C_{11}=1799$. Therefore,

$$\{p_3(w)+p_6(x)+2p_6(y)+2p_6(z):\ w,x,y,z\in\mathbb{N}\}=\mathbb{N}, \tag{1.9}$$

$$\{2p_3(w)+p_6(x)+p_6(y)+2p_6(z):\ w,x,y,z\in\mathbb{N}\}=\mathbb{N}, \tag{1.10}$$

$$\{p_7(w)+p_7(x)+2p_7(y)+2p_7(z):\ w\in\mathbb{Z}\ \&\ x,y,z\in\mathbb{N}\}=\mathbb{N}, \tag{1.11}$$

$$\{p_9(w)+2p_9(x)+p_9(y)+2p_9(z):\ w,x\in\mathbb{Z}\ \&\ y,z\in\mathbb{N}\}=\mathbb{N}, \tag{1.12}$$

$$\{p_{10}(w)+2p_{10}(x)+p_{10}(y)+2p_{10}(z):\ w,x\in\mathbb{Z}\ \&\ y,z\in\mathbb{N}\}=\mathbb{N}, \tag{1.13}$$

and

$$\{p_{11}(w)+2p_{11}(x)+p_{11}(y)+2p_{11}(z):\ w,x\in\mathbb{Z}\ \&\ y,z\in\mathbb{N}\}=\mathbb{N}\setminus\{7\}. \tag{1.14}$$

Proof. Note that $\{p_6(w):\ w\in\mathbb{Z}\}=\{p_3(w):\ w\in\mathbb{N}\}$. It suffices to apply Theorem 1.2(i) with $m\in\{3,4,5,7,8,9\}$ and check those $n\in\mathbb{N}$ with $n<1628m^3$ via a computer. □

Remark 1.4. (1.6) appeared as part of [12, Conjecture 5.2(ii)], and it indicates that the set $\{p_5(x)+2p_5(y):\ x,y\in\mathbb{N}\}$ is an additive base of order 2. For positive integers a,b,c with $\{ap_5(x)+bp_5(y)+cp_5(z):\ x,y,z\in\mathbb{Z}\}=\mathbb{N}$, see [11] and [4].

Theorem 1.3. *Let $m\geqslant 3$ be an integer. Then each integer $N\geqslant 924m^3$ can be expressed as*

$$p_{m+2}(x_1)+p_{m+2}(x_2)+p_{m+2}(x_3)+3p_{m+2}(x_4)\quad(x_1,x_2,x_3,x_4\in\mathbb{N}). \tag{1.15}$$

Corollary 1.3. *We have*

$$\{p_5(x_1)+p_5(x_2)+p_5(x_3)+3p_5(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}=\mathbb{N}\setminus\{19\}, \tag{1.16}$$

$$\begin{aligned}&\{p_6(x_1)+p_6(x_2)+p_6(x_3)+3p_6(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}\\&\quad=\mathbb{N}\setminus\{14,\ 23,\ 41,\ 42,\ 83\}\end{aligned} \tag{1.17}$$

and

$$\begin{aligned}&\{p_7(x_1)+p_7(x_2)+p_7(x_3)+3p_7(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}\\=&\mathbb{N}\setminus\{13,\ 16,\ 27,\ 31,\ 33,\ 49,\ 50,\ 67,\ 87,\ 178,\ 181,\ 259\}.\end{aligned} \tag{1.18}$$

Also, for each $k=8,9,10$, any integer $n>M_k$ can be written as $p_k(x_1)+p_k(x_2)+p_k(x_3)+3p_k(x_4)$ with $x_1,x_2,x_3,x_4\in\mathbb{N}$, where $M_8=435$, $M_9=695$ and $M_{10}=916$. Therefore

$$\{p_7(w)+p_7(x)+p_7(y)+3p_7(z):\ w\in\mathbb{Z}\ \&\ x,y,z\in\mathbb{N}\}=\mathbb{N}, \tag{1.19}$$

$$\{p_9(x_1)+p_9(x_2)+p_9(x_3)+3p_9(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{Z}\}=\mathbb{N}\setminus\{17\}, \tag{1.20}$$

$$\{p_{10}(x_1)+p_{10}(x_2)+p_{10}(x_3)+3p_{10}(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{Z}\}=\mathbb{N}\setminus\{16,\ 19\}. \tag{1.21}$$

Proof. It suffices to apply Theorem 1.3 with $m\in\{3,4,5,6,7,8\}$ and check those $n\in\mathbb{N}$ with $n<924m^3$ via a computer. □

Remark 1.5. Guy [5] thought that 10, 16 and 76 might be the only natural numbers which cannot be written as the sum of three generalized heptagonal numbers. The second author [12, Remark 5.2 and Conjecture 1.2] conjectured that $\{p_7(x)+p_7(y)+p_7(z):\ x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{10,16,76,307\}$ and

$$\{p_8(x)+p_8(y)+3p_8(z):\ x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{7,14,18,91\}.$$

Theorem 1.4. *Let $m\geqslant 3$ be an integer. Then any integer $N\geqslant 1056m^3$ can be written as*

$$p_{m+2}(x_1)+p_{m+2}(x_2)+2p_{m+2}(x_3)+4p_{m+2}(x_4)\quad(x_1,x_2,x_3,x_4\in\mathbb{N}). \tag{1.22}$$

Corollary 1.4. *We have*

$$\{p_5(x_1)+p_5(x_2)+2p_5(x_3)+4p_5(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}=\mathbb{N}, \tag{1.23}$$

$$\{p_6(x_1)+p_6(x_2)+2p_6(x_3)+4p_6(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}=\mathbb{N}, \tag{1.24}$$

$$\{p_7(x_1)+p_7(x_2)+2p_7(x_3)+4p_7(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}=\mathbb{N}\setminus\{17,51\}, \tag{1.25}$$

$$\begin{aligned}&\{p_8(x_1)+p_8(x_2)+2p_8(x_3)+4p_8(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}\\&\quad=\mathbb{N}\setminus\{19,\ 30,\ 39,\ 59,\ 78,\ 91\},\end{aligned} \tag{1.26}$$

and

$$\begin{aligned}&\{p_9(x_1)+p_9(x_2)+2p_9(x_3)+4p_9(x_4):\ x_1,x_2,x_3,x_4\in\mathbb{N}\}\\ =&\mathbb{N}\setminus\{17,\ 21,\ 34,\ 41,\ 44,\ 67,\ 89,\ 104,\ 119,\ 170,\ 237,\ 245,\ 290\}.\end{aligned}\tag{1.27}$$

Also, for each $k=10,11,12$ *any integer* $n>N_k$ *can be written as* $p_k(x_1)+p_k(x_2)+2p_k(x_3)+4p_k(x_4)$ *with* $x_1,x_2,x_3,x_4\in\mathbb{N}$*, where* $N_{10}=333$*,* $N_{11}=734$ *and* $N_{12}=1334$*. Therefore,*

$$\{p_k(w)+p_k(x)+2p_k(y)+4p_k(z):\ w\in\mathbb{Z}\ \&\ x,y,z\in\mathbb{N}\}=\mathbb{N}\tag{1.28}$$

for $k=7,9$*, and*

$$\{p_k(w)+p_k(x)+2p_k(y)+4p_k(z):\ w,x,y,z\in\mathbb{Z}\}=\mathbb{N}\tag{1.29}$$

for $k=8,10,11,12$.

Proof. It suffices to apply Theorem 1.4 with $m\in\{3,\ldots,10\}$ and check those $n\in\mathbb{N}$ with $n<1056m^3$ via a computer. □

Remark 1.6. (1.23) and (1.24) were first conjectured by the second author [12, Conjecture 5.2(ii) and Conjecture 5.3]. Sun [12, Remark 5.2] also conjectured that

$$\{p_7(x)+2p_7(y)+4p_7(z):\ x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{131,146\}.$$

We will show Theorems 1.1-1.4 in Sections 2-5 respectively.

Throughout this paper, for a prime p and $a,n\in\mathbb{N}$, by $p^a\|n$ we mean $p^a\mid n$ and $p^{a+1}\nmid n$. For example, $4\|n$ if and only if $n\equiv 4\pmod 8$.

2. Proof of Theorem 1.1

We first give a lemma which is a slight variant of [9, Lemma 1.10].

Lemma 2.1. *Let* $l,m,N\in\mathbb{Z}^+$ *with* $N\geqslant 7l^2m^3$*. Then the length of the interval*

$$I_1=\left[\frac{1}{2}+\sqrt{\frac{6N}{m}-3},\ \frac{2}{3}+\sqrt{\frac{8N}{m}}\right]\tag{2.1}$$

is greater than lm*.*

Proof. Let L_1 denote the length of the interval I_1. Then $L_1=\sqrt{8x}-\sqrt{6x-3}+1/6$, where $x=N/m\geqslant 7l^2m^2$. Let $l_0=lm-1/6$. Then

$$\begin{aligned}L_1>lm&\iff\sqrt{8x}>\sqrt{6x-3}+l_0\\&\iff 2x+3-l_0^2>2l_0\sqrt{6x-3}\\&\iff 4x(x+3-7l_0^2)+(l_0^2-3)^2+12l_0^2>0\end{aligned}$$

As $x \geqslant 7l_0^2$, by the above we have $L_1 > lm$. This completes the proof. □

The following lemma is a slight modification of [9, Lemma 1.11].

Lemma 2.2. *Let $a, b, m, N \in \mathbb{Z}^+$ with $m \geqslant 3$ and*

$$N = \frac{m}{2}(a-b) + b \geqslant \frac{2}{3}m.$$

Suppose that b belongs to the interval I_1 given by (2.1). Then

$$b^2 < 4a \quad \text{and} \quad 3a < b^2 + 2b + 4. \tag{2.2}$$

Proof. Observe that

$$a = \left(1 - \frac{2}{m}\right) b + \frac{2N}{m}$$

and

$$\begin{aligned} b^2 - 4a =& b^2 - 4\left(1 - \frac{2}{m}\right) b - \frac{8N}{m} \\ =& \left(b - 2\left(1 - \frac{2}{m}\right)\right)^2 - 4\left(\left(1 - \frac{2}{m}\right)^2 + \frac{2N}{m}\right). \end{aligned}$$

As $m \geqslant 3$ and $b \in I_1$, we have

$$b \leqslant \frac{2}{3} + \sqrt{\frac{8N}{m}} < 2\left(1 - \frac{2}{m}\right) + 2\sqrt{\left(1 - \frac{2}{m}\right)^2 + \frac{2N}{m}}$$

and hence $b^2 - 4a < 0$. On the other hand, since $(1/2 - 3/m)^2 < 1$ and $b \in I_1$ we have

$$b \geqslant \frac{1}{2} + \sqrt{\frac{6N}{m} - 3} > \frac{1}{2} - \frac{3}{m} + \sqrt{\left(\frac{1}{2} - \frac{3}{m}\right)^2 - 4 + \frac{6N}{m}}$$

and hence

$$\begin{aligned} b^2 + 2b + 4 - 3a =& b^2 - \left(1 - \frac{6}{m}\right) b + \left(4 - \frac{6N}{m}\right) \\ =& \left(b - \left(\frac{1}{2} - \frac{3}{m}\right)\right)^2 - \left(\left(\frac{1}{2} - \frac{3}{m}\right)^2 - 4 + \frac{6N}{m}\right) \\ >& 0. \end{aligned}$$

This proves (2.2). □

Lemma 2.3. *Let a, b, c be positive integers and let x, y, z be real numbers. Then we have the inequality*

$$(ax + by + cz)^2 \leqslant (a + b + c)(ax^2 + by^2 + cz^2). \tag{2.3}$$

Proof. By the Cauchy-Schwarz inequality (cf. [9, p. 178]),

$$\begin{aligned}&\left(\sqrt{a}(\sqrt{a}x)+\sqrt{b}(\sqrt{b}y)+\sqrt{c}(\sqrt{c}z)\right)^2\\ \leqslant&\left((\sqrt{a})^2+(\sqrt{b})^2+(\sqrt{c})^2\right)\left((\sqrt{a}x)^2+(\sqrt{b}y)^2+(\sqrt{c}z)^2\right).\end{aligned}$$

This yields the desired (2.3). □

The following lemma with $2\nmid ab$ is usually called Cauchy's Lemma (cf. [9, pp. 31–34]).

Lemma 2.4. *Let a and b be positive integers satisfying* (2.2)*. Suppose that $2\nmid ab$, or $2\|a$ and $2\mid b$. Then there exist $s,t,u,v\in\mathbb{N}$ such that*

$$a=s^2+t^2+u^2+v^2\quad\text{and}\quad b=s+t+u+v. \tag{2.4}$$

Proof. By the Gauss-Legendre theorem (cf. [9, Section 1.5]), we have

$$\{x^2+y^2+z^2:\ x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{4^k(8l+7):\ k,l\in\mathbb{N}\}. \tag{2.5}$$

We claim that there are $x,y,z\in\mathbb{Z}$ with $4a-b^2=x^2+y^2+z^2$ such that all the numbers

$$\begin{aligned}s&=\frac{b+x+y+z}{4},\quad t=\frac{b+x-y-z}{4},\\ u&=\frac{b-x+y-z}{4},\quad v=\frac{b-x-y+z}{4}\end{aligned} \tag{2.6}$$

are integers.

Case 1. $2\nmid ab$.

In this case, $4a-b^2\equiv 3\pmod 8$ and hence $4a-b^2=x^2+y^2+z^2$ for some $x,y,z\in\mathbb{Z}$ with $2\nmid xyz$. Without loss of generality, we may assume that $x\equiv y\equiv z\equiv b\pmod 4$. (If $x\equiv -b\pmod 4$ then $-x\equiv b\pmod 4$.) Thus the numbers in (2.6) are all integers.

Case 2. $2\|a$ and $2\mid b$.

Write $a=2a_0$ and $b=2b_0$ with $a_0,b_0\in\mathbb{Z}$ and $2\nmid a_0$. Since $2a_0-b_0^2\equiv 1,2\pmod 4$, we have $2a_0-b_0^2=x_0^2+y_0^2+z_0^2$ for some $x_0,y_0,z_0\in\mathbb{Z}$. Without loss of generality, we may assume that $x_0\equiv b_0\pmod 2$ and $y_0\equiv z_0\pmod 2$. Set $x=2x_0$, $y=2y_0$ and $z=2z_0$. Then $4a-b^2=4(2a_0-b_0^2)=x^2+y^2+z^2$, and the numbers in (2.6) are all integers.

In either case, there are $x,y,z\in\mathbb{Z}$ for which $4a-b^2=x^2+y^2+z^2$ and $s,t,u,v\in\mathbb{Z}$, where s,t,u,v are as in (2.6). Obviously, $s+t+u+v=b$ and

$$\begin{aligned}&s^2+t^2+u^2+v^2\\ =&2\left(\frac{s+t}{2}\right)^2+2\left(\frac{s-t}{2}\right)^2+2\left(\frac{u+v}{2}\right)^2+2\left(\frac{u-v}{2}\right)^2\\ =&2\left(\frac{b+x}{4}\right)^2+2\left(\frac{y+z}{4}\right)^2+2\left(\frac{b-x}{4}\right)^2+2\left(\frac{y-z}{4}\right)^2\\ =&\frac{b^2+x^2+y^2+z^2}{4}=a.\end{aligned}$$

In view of Lemma 2.3 and the second inequality in (2.2), we have

$$(|x|+|y|+|z|)^2\leqslant 3(x^2+y^2+z^2)=3(4a-b^2)<(b+4)^2.$$

Therefore

$$s,t,u,v\geqslant\frac{b-|x|-|y|-|z|}{4}>-1$$

and hence $s,t,u,v\in\mathbb{N}$. □

Now we need one more lemma which is well known (cf. [1, p. 59]).

Lemma 2.5. *For any $n\in\mathbb{Z}^+$, we have*

$$r_4(n)=8\sum_{\substack{d\mid n\\ 4\nmid d}}d \tag{2.7}$$

where

$$r_4(n):=|\{(w,x,y,z)\in\mathbb{Z}^4:\ w^2+x^2+y^2+z^2=n\}|.$$

Proof of Theorem 1.1. (i) Let $I_1=[\alpha,\beta]$ be the interval given by (2.1). As $N\geqslant 7\times 2^2m^3$, by Lemma 2.1 the length of the interval I_1 is greater than $2m$. Choose $b_0\in\{\lceil\alpha\rceil+r:\ r=0,\ldots,m-1\}$ with $b_0\equiv N\pmod m$. Then $b_1=b_0+m\leqslant\lceil\alpha\rceil+2m-1<\alpha+2m<\beta$. Thus both b_0 and b_1 lie in I_1. Note that

$$\frac{2}{m}(N-b_1)+b_1-\left(\frac{2}{m}(N-b_0)+b_0\right)=m-2\equiv 2\pmod 4.$$

So, for some $b\in\{b_0,b_1\}$ and $a=\frac{2}{m}(N-b)+b$ we have $2\nmid ab$, or $2\|a$ and $2\mid b$. Obviously,

$$b\geqslant\min I_1>0,\ a=\frac{2}{m}N+\left(1-\frac{2}{m}\right)b>0,\ \text{and } N=\frac{m}{2}(a-b)+b.$$

Applying Lemmas 2.2 and 2.4, we see that there are $s,t,u,v\in\mathbb{N}$ satisfying (2.4). Therefore,

$$\begin{aligned}N=&\frac{m}{2}(a-b)+b\\=&\frac{m}{2}(s^2-s+t^2-t+u^2-u+v^2-v)+s+t+u+v\\=&p_{m+2}(s)+p_{m+2}(t)+p_{m+2}(u)+p_{m+2}(v).\end{aligned}$$

(ii) Write $m=4l$ with $l\in\mathbb{Z}^+$. Let φ denote Euler's totient function. We want to show that none of the positive integers

$$4l^2\times\frac{4^{k\varphi(2l+1)}-1}{2l+1}\ \ (k=1,2,\ldots)$$

can be written as $\sum_{i=1}^4 p_{m+4}(x_i)$ with $x_1,x_2,x_3,x_4\in\mathbb{N}$.

Suppose that for some $n\in\mathbb{Z}^+$ divisible by $\varphi(2l+1)$ there are $w,x,y,z\in\mathbb{N}$ such that

$$\begin{aligned}4l^2\times\frac{4^n-1}{2l+1}=&p_{m+4}(w)+p_{m+4}(x)+p_{m+4}(y)+p_{m+4}(z)\\=&\frac{4l+2}{2}(w^2+x^2+y^2+z^2-w-x-y-z)+w+x+y+z.\end{aligned}$$

Then

$$4^{n+1}l^2=((2l+1)w-l)^2+((2l+1)x-l)^2+((2l+1)y-l)^2+((2l+1)z-l)^2.$$

As $r_4(4^{n+1}l^2)=r_4(4l^2)$ by Lemma 2.5, there are $w_0,x_0,y_0,z_0\in\mathbb{Z}$ with $w_0^2+x_0^2+y_0^2+z_0^2=4l^2$ such that

$$\begin{aligned}&(2l+1)w-l=2^nw_0,\ (2l+1)x-l=2^nx_0,\\&(2l+1)y-l=2^ny_0,\ (2l+1)z-l=2^nz_0.\end{aligned}$$

Since $2^n\equiv 1\pmod{2l+1}$ by Euler's theorem, we have

$$w_0\equiv x_0\equiv y_0\equiv z_0\equiv -l\pmod{2l+1}.$$

As $w_0^2+x_0^2+y_0^2+z_0^2=4l^2$, we must have $w_0=x_0=y_0=z_0=-l$. Thus $(2l+1)w=2^nw_0+l=l(1-2^n)<0$, which contradicts $w\in\mathbb{N}$. □

3. Proof of Theorem 1.2

Lemma 3.1. *Let $l,m,N\in\mathbb{Z}^+$ with $N\geqslant 11lm^2(lm+1)$. Then the length of the interval*

$$I_2=\left[\frac{3}{2}+\sqrt{\frac{10N}{m}-3},\ 1+\sqrt{\frac{12N}{m}}\right]\tag{3.1}$$

is greater than lm.

Proof. The length L_2 of the interval I_2 is $\sqrt{12x}-\sqrt{10x-3}-1/2$, where $x=N/m\geqslant 11lm(lm+1)$. Let $l_0=lm+1/2$. Then

$$\begin{aligned} L_2>lm &\iff \sqrt{12x}>\sqrt{10x-3}+l_0\\ &\iff 2x+3-l_0^2+3>2l_0\sqrt{10x-3}\\ &\iff 4x(x-11l_0^2+3)+(l_0^2-3)^2+12l_0^2>0. \end{aligned}$$

As

$$x\geqslant 11lm(lm+1)>11l^2m^2+11lm+\frac{11}{4}-3=11l_0^2-3,$$

we have $L_2>lm$ as desired. □

Lemma 3.2. *Let $a,b,m,N\in\mathbb{Z}^+$ with $m\geqslant 3$ and*

$$N=\frac{m}{2}(a-b)+b\geqslant\frac{3}{5}m.$$

Suppose that b belongs to the interval I_2 given by (3.1)*. Then*

$$b^2<6a \quad \text{and} \quad 5a<b^2+2b+6. \tag{3.2}$$

Proof. Note that

$$a=\left(1-\frac{2}{m}\right)b+\frac{2N}{m}.$$

Thus

$$\begin{aligned} b^2-6a=&b^2-6\left(1-\frac{2}{m}\right)b-\frac{12N}{m}\\ =&\left(b-3\left(1-\frac{2}{m}\right)\right)^2-9\left(1-\frac{2}{m}\right)^2-\frac{12N}{m}. \end{aligned}$$

As $b\in I_2$, we have

$$b\leqslant 1+\sqrt{\frac{12N}{m}}<3\left(1-\frac{2}{m}\right)+\sqrt{9\left(1-\frac{2}{m}\right)^2+\frac{12N}{m}}$$

and hence $b^2-6a<0$. On the other hand,

$$\begin{aligned} b^2+2b+6-5a=&b^2-\left(3-\frac{10}{m}\right)b+\left(6-\frac{10N}{m}\right)\\ =&\left(b-\left(\frac{3}{2}-\frac{5}{m}\right)\right)^2-\left(\frac{3}{2}-\frac{5}{m}\right)^2+\left(6-\frac{10N}{m}\right)\\ >&0 \end{aligned}$$

since $(3/2-5/m)^2\leqslant 9/4<3$ and

$$b\geqslant\frac{3}{2}+\sqrt{\frac{10N}{m}-3}>\frac{3}{2}-\frac{5}{m}+\sqrt{\left(\frac{3}{2}-\frac{5}{m}\right)^2-6+\frac{10N}{m}}.$$

This proves (3.2). □

Lemma 3.3. *Let a and b be positive integers with $a \equiv b \pmod 2$ satisfying (3.2). Suppose that $2 \nmid a$ or $4 \| a$, and that $3 \mid a$ or $3 \nmid b$. Then there exist $s, t, u, v \in \mathbb{N}$ such that*

$$a = s^2 + t^2 + 2u^2 + 2v^2 \text{ and } b = s + t + 2u + 2v. \tag{3.3}$$

Proof. If $n \in \mathbb{N}$ is not of the form $4^k(8l+7)$ with $k, l \in \mathbb{N}$, then by (2.5) there are $x, u, v \in \mathbb{Z}$ with $u \equiv v \pmod 2$ such that

$$n = x^2 + u^2 + v^2 = x^2 + 2\left(\frac{u-v}{2}\right)^2 + 2\left(\frac{u+v}{2}\right)^2.$$

We claim that there are $x, y, z \in \mathbb{Z}$ with $6a - b^2 = x^2 + 2y^2 + 2z^2$ such that all the numbers

$$\begin{aligned} s &= \frac{b + x + 2y + 2z}{6}, \quad t = \frac{b - x - 2y + 2z}{6}, \\ u &= \frac{b - x + y - z}{6}, \quad v = \frac{b + x - y - z}{6} \end{aligned} \tag{3.4}$$

are integral.

Case 1. $3 \nmid b$.

If $a \equiv b \equiv 1 \pmod 2$, then $6a - b^2 \equiv 1 \pmod 4$. When $4 \| a$ and $2 \mid b$, we have $6a - b^2 \equiv 4, 8 \pmod{16}$. Thus $6a - b^2 = x^2 + 2y^2 + 2z^2$ for some $x, y, z \in \mathbb{Z}$. Clearly, $x \equiv b \pmod 2$ and $y \equiv z \pmod 2$. Note that

$$x^2 + 2y^2 + 2z^2 \equiv 6a - b^2 \equiv 2 \pmod 3.$$

As $x^2 \not\equiv 2 \pmod 3$, we have $3 \nmid y$ or $3 \nmid z$. Without loss of generality, we assume that $3 \nmid z$ and moreover $z \equiv b \pmod 3$. (If $z \equiv -b \pmod 3$ then $-z \equiv b \pmod 3$.) As $x^2 - y^2 \equiv x^2 + 2y^2 \equiv 0 \pmod 3$, without loss of generality we may assume that $x \equiv y \pmod 3$. Now it is easy to see that all the numbers in (3.4) are integers.

Case 2. $3 \mid a$ and $3 \mid b$.

In this case, $a = 3a_0$ and $b = 3b_0$ for some $a_0, b_0 \in \mathbb{Z}$. If $a \equiv b \equiv 1 \pmod 2$, then $2a_0 - b_0^2 \equiv 1 \pmod 4$. When $4 \| a$ and $2 \mid b$, we have $2a_0 - b_0^2 \equiv 4, 8 \pmod{16}$. Thus $2a_0 - b_0^2 = x_0^2 + 2y_0^2 + 2z_0^2$ for some $x_0, y_0, z_0 \in \mathbb{Z}$. It follows that $x_0 \equiv b_0 \pmod 2$ and $y_0 \equiv z_0 \pmod 2$ since $a_0 \equiv b_0 \pmod 2$. Set $x = 3x_0$, $y = 3y_0$ and $z = 3z_0$. Then $6a - b^2 = 9(2a_0 - b_0^2) = x^2 + 2y^2 + 2z^2$ and all the numbers in (3.4) are integers.

By the above, in either case, there are $x, y, z \in \mathbb{Z}$ for which $x^2 + 2y^2 + 2z^2 = 6a - b^2$ and $s, t, u, v \in \mathbb{Z}$, where s, t, u, v are as in (3.4). Observe that

$$s + t + 2(u + v) = \frac{b + 2z}{3} + 2 \times \frac{b - z}{3} = b$$

and also

$$\begin{aligned}&s^2+t^2+2u^2+2v^2\\ =&2\left(\frac{s+t}{2}\right)^2+2\left(\frac{s-t}{2}\right)^2+(u+v)^2+(u-v)^2\\ =&2\left(\frac{b+2z}{6}\right)^2+2\left(\frac{x+2y}{6}\right)^2+\left(\frac{b-z}{3}\right)^2+\left(\frac{x-y}{3}\right)^2\\ =&\frac{b^2+x^2+2y^2+2z^2}{6}=a.\end{aligned}$$

In view of Lemma 2.3 and (3.2),

$$(|x|+2|y|+2|z|)^2\leqslant 5(x^2+2y^2+2z^2)=5(6a-b^2)<(b+6)^2$$

and hence

$$b-|x|-2|y|-2|z|>-6.$$

So we have

$$s,t,u,v\geqslant\frac{b-|x|-2|y|-2|z|}{6}>-1,$$

and hence $s,t,u,v\in\mathbb{N}$.

In view of the above, we have completed the proof of Lemma 3.3. □

Remark 3.1. For s,t,u,v given in (3.4), the identity

$$\begin{aligned}6(s^2+t^2+2u^2+2v^2)=&b^2+x^2+2y^2+2z^2\\ =&(s+t+2u+2v)^2+(s-t-2u+2v)^2\\ &+2(s-t+u-v)^2+2(s+t-u-v)^2\end{aligned}$$

is a special case of our following general identity

$$\begin{aligned}&(a+b)(c+d)(w^2+abx^2+cdy^2+abcdz^2)\\ =&ac(w+bx+dy+bdz)^2+ad(w+bx-cy-bcz)^2\\ &+bc(w-ax+dy-adz)^2+bd(w-ax-cy+acz)^2.\end{aligned}\tag{3.5}$$

We have also found another similar identity:

$$\begin{aligned}&(3b+4)(w^2+2x^2+(b+1)y^2+2bz^2)\\ =&(w+2x+(b+1)y+2bz)^2+2(w-(b+1)y+bz)^2\\ &+(b+1)(w-2x+y)^2+2b(w+x-2z)^2.\end{aligned}\tag{3.6}$$

Proof of Theorem 1.2. (i) Let $I_2=[\alpha,\beta]$ be the interval given by (3.1). As $N\geqslant 1628m^3=(12+1/3)m\times 132m^2\geqslant 11m^2\times 12(12m+1)$, by Lemma 3.1 the length of the interval I_2 is greater than $12m$. We distinguish two cases to construct integers $b\in I_2$ and $a\equiv b\pmod 2$ for which $N=\frac{m}{2}(a-b)+b$, and $2\nmid a$ or $4\|a$, and $3\mid a$ or $3\nmid b$.

Case 1. $3\nmid m$ or $3\nmid N$.

Choose $b_0\equiv N\pmod m$ with $b_0\in\{\lceil\alpha\rceil+r:\ r=0,\ldots,m-1\}$, and let $b_j=b_0+jm$ for $j=1,\ldots,7$. Since $\lceil\alpha\rceil+8m-1<\alpha+8m<\beta$, we have $b_i\in I_2$ for all $i=0,\ldots,7$.

If $2\nmid m$, then we choose $i\in\{0,1\}$ with b_i odd. When $4\mid m$, we may choose $i\in\{0,1,2,3\}$ with

$$a_i:=\frac{2}{m}(N-b_i)+b_i\equiv 4\pmod 8$$

since $a_i-a_0=-2i+im=2i(m/2-1)$ with $m/2-1$ odd. If $3\mid m$ and $3\nmid N$, then $b=b_i\equiv N\not\equiv 0\pmod 3$. When $3\nmid m$, we choose $j\in\{i,i+4\}$ with $b=b_j\not\equiv 0\pmod 3$, and note that

$$a_{i+4}-a_i=4m-8\equiv\begin{cases}0\pmod 2 & \text{if } 2\nmid m,\\ 0\pmod 8 & \text{if } 4\mid m.\end{cases}$$

As $N\equiv b\pmod m$, we see that $a=2(N-b)/m+b$ is an integer with $a\equiv b\pmod 2$. By our choice, $2\nmid b$ if $2\nmid m$, and $a\equiv 4\pmod 8$ if $4\mid m$. Note also that $3\nmid b$.

Case 2. $m\equiv N\equiv 0\pmod 3$.

Choose $b_0\in\{\lceil\alpha\rceil+r:\ r=0,1,\ldots,3m-1\}$ with $b_0\equiv N\pmod{3m}$. If $2\nmid m$, then we choose $b\in\{b_0,b_0+3m\}$ with b odd and hence $a=\frac{2}{m}(N-b)+b\equiv b\equiv 1\pmod 2$. When $4\mid m$, we may choose $b\in\{b_0+3jm:\ j=0,1,2,3\}$ with $a=\frac{2}{m}(N-b)+b\equiv 4\pmod 8$, for

$$\frac{2}{m}(N-b_0-3jm)+b_0+3jm-\left(\frac{2}{m}(N-b_0)+b_0\right)=6j\left(\frac{m}{2}-1\right)$$

with $m/2-1$ odd. Note that

$$\alpha\leqslant b_0<b_0+9m\leqslant\lceil\alpha\rceil+3m-1+9m<\alpha+12m<\beta$$

and hence $b\in I_2$. Obviously, $a\equiv b\equiv N\equiv 0\pmod 3$.

Now we have constructed positive integers $b\in I_2$ and $a\equiv b\pmod 2$ with $N=\frac{m}{2}(a-b)+b$ such that $2\nmid a$ or $4\|a$, and $3\mid a$ or $3\nmid b$. So (3.2) holds by Lemma 3.2. In view of Lemma 3.3, (3.3) holds for some $s,t,u,v\in\mathbb{N}$. Therefore,

$$\begin{aligned}N=&\frac{m}{2}(s^2+t^2+2u^2+2v^2-s-t-2u-2v)+s+t+2u+2v\\ =&m\binom{s}{2}+m\binom{t}{2}+2m\binom{u}{2}+2m\binom{v}{2}+s+t+2u+2v\\ =&p_{m+2}(s)+p_{m+2}(t)+2p_{m+2}(u)+2p_{m+2}(v).\end{aligned}$$

This proves part (i) of Theorem 1.2.

(ii) Now assume that $m = 2l$ with $l \in \mathbb{Z}^+$ odd. We want to show that none of the positive integers

$$(l-1)^2 \times \frac{4^{k\varphi(l)}-1}{l} \quad (k = 1, 2, 3, \ldots)$$

can be written as $p_{m+2}(w)+p_{m+2}(x)+2p_{m+2}(y)+2p_{m+2}(z)$ with $w, x, y, z \in \mathbb{N}$.

Let $n \in \mathbb{Z}^+$ be a multiple of $\varphi(l)$. Then $2^n \equiv 1 \pmod{l}$ by Euler's theorem. Suppose that there are $w, x, y, z \in \mathbb{N}$ for which

$$\begin{aligned}&(l-1)^2 \times \frac{4^n-1}{l}\\ =&p_{m+2}(w) + p_{m+2}(x) + 2p_{m+2}(y) + 2p_{m+2}(z)\\ =&\frac{2l}{2}(w^2+x^2+2y^2+2z^2-w-x-2y-2z)+w+x+2y+2z.\end{aligned}$$

Then we have

$$\begin{aligned}&4^{n+1}(l-1)^2\\ =&(2lw-(l-1))^2+(2lx-(l-1))^2+2(2ly-(l-1))^2+2(2lz-(l-1))^2\\ =&(2lw-(l-1))^2+(2lx-(l-1))^2+(2l(y+z)-2(l-1))^2+(2l(y-z))^2\end{aligned}$$

and hence

$$4^n(l-1)^2 = \left(lw-\frac{l-1}{2}\right)^2+\left(lx-\frac{l-1}{2}\right)^2+(l(y+z-1)+1)^2+(l(y-z))^2.$$

As $4 \mid (l-1)^2$, by Lemma 2.5 we have $r_4(4^n(l-1)^2) = r_4((l-1)^2)$. So there are $w_0, x_0, y_0, z_0 \in \mathbb{Z}$ with

$$w_0^2+x_0^2+y_0^2+z_0^2 = (l-1)^2 \tag{3.7}$$

such that

$$lw-\frac{l-1}{2} = 2^n w_0,\ lx-\frac{l-1}{2} = 2^n x_0,\ l(y+z-1)+1 = 2^n y_0,\ l(y-z) = 2^n z_0.$$

As $2^n \equiv 1 \pmod{l}$, we see that

$$w_0 \equiv \frac{1-l}{2} \pmod{l},\ x_0 \equiv \frac{1-l}{2} \pmod{l},\ y_0 \equiv 1 \pmod{l},\ z_0 \equiv 0 \pmod{l}.$$

Observe that $l = m/2 \geqslant 2$ and hence $w_0 x_0 \neq 0$. Thus $y_0^2+z_0^2 \leqslant (l-1)^2-2$. Since $y_0 \equiv 1 \pmod{l}$ and $z_0 \equiv 0 \pmod{l}$, we must have $y_0 = 1$ and $z_0 = 0$. Now (3.7) yields $w_0^2+x_0^2 = (l-1)^2-1 = l^2-2l$. As $w_0 \equiv x_0 \equiv (1-l)/2 \pmod{l}$, we must have $\{w_0, x_0\} \subseteq \{(1-l)/2, (1+l)/2\}$. It is easy to verify directly that none of the numbers

$$\left(\frac{1-l}{2}\right)^2+\left(\frac{1+l}{2}\right)^2,\ 2\left(\frac{l-1}{2}\right)^2,\ 2\left(\frac{1+l}{2}\right)^2$$

is equal to $(l-1)^2-1=l^2-2l$. This contradiction concludes the proof of Theorem 1.2(ii). □

4. Proof of Theorem 1.3

Lemma 4.1. *Let a and b be positive integers satisfying* (3.2) *for which $a\equiv b \pmod 2$, and $a\equiv 3 \pmod 9$ or $3\nmid b$. Then there exist $s,t,u,v\in\mathbb{N}$ such that*

$$a=s^2+t^2+u^2+3v^2 \quad \text{and} \quad b=s+t+u+3v. \tag{4.1}$$

Proof. It is known that (cf. Dickson [3, pp.112-113])

$$\{x^2+y^2+3z^2:\ x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{9^k(9l+6):\ k,l\in\mathbb{N}\}. \tag{4.2}$$

If $3\nmid b$, then $6a-b^2\equiv 2 \pmod 3$. If $a\equiv 3 \pmod 9$ and $3\mid b$, then $6a-b^2\equiv \pm 9 \pmod{27}$. By (4.2), $6a-b^2=x^2+y^2+3z^2$ for some $x,y,z\in\mathbb{Z}$. Clearly, $x^2+y^2\equiv 2b^2 \pmod 3$. Without loss of generality, we may assume that $x\equiv y\equiv b \pmod 3$. (If $x\equiv -b \pmod 3$ then $-x\equiv b \pmod 3$.) If a and b are both odd, then $x^2+y^2+3z^2=6a-b^2\equiv 1 \pmod 4$, and hence one of x and y is odd. If a and b are both even, then $x^2+y^2+3z^2=6a-b^2\equiv 0 \pmod 4$, and hence one of x and y is even. Without loss of generality, we may assume that $x\equiv a\equiv b \pmod 2$ and $y\equiv z \pmod 2$. Thus all the numbers

$$s=\frac{b+x+y+3z}{6},\ t=\frac{b+x+y-3z}{6},\ u=\frac{b+x-2y}{6},\ v=\frac{b-x}{6} \tag{4.3}$$

are integers. Observe that

$$s+t+u+3v=\frac{b+x}{2}+3\times\frac{b-x}{6}=b$$

and

$$\begin{aligned}
&s^2+t^2+u^2+3v^2\\
=&2\left(\frac{s+t}{2}\right)^2+2\left(\frac{s-t}{2}\right)^2+u^2+3v^2\\
=&2\left(\frac{b+x+y}{6}\right)^2+2\left(\frac{z}{2}\right)^2+\left(\frac{b+x-2y}{6}\right)^2+3\left(\frac{b-x}{6}\right)^2\\
=&\frac{b^2+x^2+y^2+3z^2}{6}=a.
\end{aligned}$$

In view of Lemma 2.3 and (3.2),

$$(|x|+|y|+3|z|)^2\leqslant 5(x^2+y^2+3z^2)=5(6a-b^2)<(b+6)^2$$

and hence

$$b-|x|-|y|-3|z|>-6.$$

So we have

$$s,t,u,v\geqslant\frac{b-|x|-|y|-3|z|}{6}>-1$$

and hence $s,t,u,v\in\mathbb{N}$. This concludes the proof. □

Proof of Theorem 1.3. As

$$N\geqslant 924m^3=99m^3\left(9+\frac{1}{3}\right)\geqslant 99m^3\left(9+\frac{1}{m}\right)=99m^2(9m+1),$$

the length of the interval $I_2=[\alpha,\beta]$ defined in Lemma 3.1 is greater than $9m$.

Let $b_0\in\{\lceil\alpha\rceil+r:\ r=0,1,\ldots,m-1\}$ with $b_0\equiv N\pmod m$. If $3\nmid m$ or $3\nmid N$, then we may choose $b\in\{b_0,b_0+m\}$ with $b\not\equiv 0\pmod 3$. When $3\mid m$ and $3\mid N$, we let $c_0\in\{\lceil\alpha\rceil+r:\ r=0,\ldots,3m-1\}$ with $c_0\equiv N\pmod{3m}$, and set $b=c_0+j3m$ with $j\in\{0,1,2\}$ such that

$$\frac{2}{m}(N-c_0)+c_0-6j\equiv 3\pmod 9.$$

Note that $b\in I_2$ since

$$\alpha\leqslant b\leqslant\lceil\alpha\rceil+3m-1+6m<\alpha+9m<\beta.$$

Let

$$a=\frac{2}{m}(N-b)+b,\quad\text{i.e., }N=\frac{m}{2}(a-b)+b.$$

Then

$$a=\frac{2}{m}N+\left(1-\frac{2}{m}\right)b>0\quad\text{and}\quad a\equiv b\pmod 2.$$

If $3\mid b$, then $3\mid m$ and

$$a=\frac{2}{m}(N-b)+b=\frac{2}{m}(N-c_0-3jm)+c_0+3jm\equiv\frac{2}{m}(N-c_0)-6j\equiv 3\pmod 9.$$

By Lemmas 3.2 and 4.1, there are $s,t,u,v\in\mathbb{N}$ satisfying (4.1). Therefore,

$$\begin{aligned}N=&\frac{m}{2}(s^2+t^2+u^2+3v^2-s-t-u-3v)+s+t+u+3v\\=&p_{m+2}(s)+p_{m+2}(t)+p_{m+2}(u)+3p_{m+2}(v).\end{aligned}$$

This completes the proof of Theorem 1.3. □

5. Proof of Theorem 1.4

Lemma 5.1. *Let $l,m,N\in\mathbb{Z}^+$ with $lm\geqslant 20$ and $N\geqslant 3lm^2(5lm+12)$. Then the length of the interval*

$$I_3=\left[\frac{5}{2}+\sqrt{\frac{14N}{m}-1},\ \frac{4}{3}+4\sqrt{\frac{N}{m}}\right] \tag{5.1}$$

is greater than lm.

Proof. The length of the interval I_3 is $4\sqrt{x}-\sqrt{14x-1}-7/6$, where $x=N/m$. Set $l_0=lm+7/6$. Then

$$\begin{aligned}&4\sqrt{x}-\sqrt{14x-1}-\frac{7}{6}>lm\\ \iff&4\sqrt{x}>\sqrt{14x-1}+l_0\\ \iff&2x+1-l_0^2>2l_0\sqrt{14x-1}\\ \iff&4x(x+1-15l_0^2)+(l_0^2-1)^2+4l_0^2>0.\end{aligned}$$

Note that

$$x\geqslant 15l^2m^2+36lm>15l^2m^2+35lm+\frac{245}{12}-1=15l_0^2-1.$$

So the desired result follows. □

Lemma 5.2. *Let a,b,m,N be positive integers with $m\geqslant 3$,*

$$N=\frac{m}{2}(a-b)+b\geqslant\frac{4}{7}m$$

and $b\in I_3$, where I_3 is the interval given by (5.1). Then

$$b^2<8a\quad\text{and}\quad 7a<b^2+2b+8. \tag{5.2}$$

Proof. Clearly,

$$a=\left(1-\frac{2}{m}\right)b+\frac{2N}{m}.$$

Thus

$$\begin{aligned}b^2-8a=&b^2-8\left(1-\frac{2}{m}\right)b-\frac{16N}{m}\\ =&\left(b-4\left(1-\frac{2}{m}\right)\right)^2-16\left(\left(1-\frac{2}{m}\right)^2+\frac{N}{m}\right).\end{aligned}$$

As $b\in I_3$, we have

$$b\leqslant\frac{4}{3}+4\sqrt{\frac{N}{m}}<4\left(1-\frac{2}{m}\right)+4\sqrt{\left(1-\frac{2}{m}\right)^2+\frac{N}{m}}$$

and hence $b^2-8a<0$. On the other hand,

$$\begin{aligned}&b^2+2b+8-7a\\ =&b^2-\left(5-\frac{14}{m}\right)b+\left(8-\frac{14N}{m}\right)\\ =&\left(b-\left(\frac{5}{2}-\frac{7}{m}\right)\right)^2-\left(\left(\frac{5}{2}-\frac{7}{m}\right)^2+\frac{14N}{m}-8\right)>0\end{aligned}$$

since $(5/2-7/m)^2\leqslant 25/4<7$ and

$$b\geqslant\frac{5}{2}+\sqrt{\frac{14N}{m}-1}>\frac{5}{2}-\frac{7}{m}+\sqrt{\left(\frac{5}{2}-\frac{7}{m}\right)^2+\frac{14N}{m}-8}.$$

Therefore (5.2) holds. □

Lemma 5.3. *Let a and b be positive integers satisfying* (5.2). *Then there are $s,t,u,v\in\mathbb{N}$ such that*

$$a=s^2+t^2+2u^2+4v^2\quad\text{and}\quad b=s+t+2u+4v,\tag{5.3}$$

under one of the following conditions (i)-(iii):

(i) $2\nmid ab$.

(ii) $2\mid a$ *and* $2\|b$.

(iii) $4\mid a$ *and* $4\|b$, *or* $a\equiv b+4\pmod{16}$ *and* $8\mid b$.

Proof. It is known (cf. [3, pp. 112-113]) that

$$\{x^2+2y^2+4z^2:\ x,y,z\in\mathbb{Z}\}=\mathbb{N}\setminus\{4^k(16l+14):\ k,l\in\mathbb{N}\}.\tag{5.4}$$

We claim that if one of (i)-(iii) holds then $8a-b^2=x^2+2y^2+4z^2$ for some $x,y,z\in\mathbb{Z}$ such that all the numbers

$$u=\frac{b+x-2y}{8},\ v=\frac{b-x}{8},\ s=u+\frac{y+z}{2},\ t=u+\frac{y-z}{2}\tag{5.5}$$

are integers.

Case 1. $2\nmid ab$.

Since $8a-b^2\equiv -1\pmod 8$, we have $8a-b^2=x^2+2y^2+4z^2$ for some $x,y,z\in\mathbb{Z}$. As $x^2+2y^2\equiv -1\pmod 4$, we have $2\nmid xy$. Since $4z^2\equiv -b^2-x^2-2y^2\equiv -1-1-2\pmod 8$, we also have $2\nmid z$. Note that

$$x^2=8a-b^2-2y^2-4z^2\equiv 8-b^2-2-4=2-b^2\equiv b^2\pmod{16}$$

and hence $x\equiv\pm b\pmod 8$. Without loss of generality, we may assume that $x\equiv b\pmod 8$ and $y\equiv b\pmod 4$. (If $y\equiv -b\pmod 4$ then $-y\equiv b\pmod 4$.) Thus all the four numbers in (5.5) are integers.

Case 2. $2 \mid a$ and $2\|b$.

Write $a = 2a_0$ and $b = 2b_0$ with $a_0, b_0 \in \mathbb{Z}$ and $2 \nmid b_0$. Since $4a_0 - b_0^2 \equiv 3 \pmod 4$, by (5.4) we have $4a_0 - b_0^2 = x_0^2 + 2y_0^2 + 4z_0^2$ for some $x_0, y_0, z_0 \in \mathbb{Z}$. As $x_0^2 + 2y_0^2 \equiv 3 \pmod 4$, both x_0 and y_0 are odd. Without loss of generality, we may assume that $x_0 \equiv y_0 \equiv b_0 \pmod 4$. Set $x = 2x_0$, $y = 2y_0$ and $z = 2z_0$. Then

$$8a - b^2 = 4(4a_0 - b_0^2) = x^2 + 2y^2 + 4z^2$$

and all the numbers in (5.5) are integers.

Case 3. $4 \mid a$ and $4\|b$, or $a \equiv b + 4 \pmod{16}$ and $8 \mid b$.

Write $a = 4a_0$ and $b = 4b_0$ with $a_0, b_0 \in \mathbb{Z}$. Then

$$2a_0 - b_0^2 \equiv \begin{cases} 1 \pmod 2 & \text{if } 4\|b \text{ (i.e., } 2 \nmid b_0), \\ 2(b_0+1) - b_0^2 \equiv 2 \pmod 8 & \text{if } a \equiv b+4 \pmod{16} \text{ and } 8 \mid b. \end{cases}$$

Thus, by (5.4) we have $2a_0 - b_0^2 = x_0^2 + 2y_0^2 + 4z_0^2$ for some $x_0, y_0, z_0 \in \mathbb{Z}$. Obviously $x_0 \equiv b_0 \pmod 2$. Set $x = 4x_0$, $y = 4y_0$ and $z = 4z_0$. Then all the numbers in (5.5) are integers.

Now assume that one of the conditions (i)-(iii) holds. By the claim we have proved, there are $x, y, z \in \mathbb{Z}$ such that $8a - b^2 = x^2 + 2y^2 + 4z^2$ and also $s, t, u, v \in \mathbb{Z}$, where s, t, u, v are given by (5.5). Clearly,

$$s + t + 2u + 4v = y + 2u + 2u + 4v = \frac{b+x}{2} + \frac{b-x}{2} = b$$

and

$$\begin{aligned} & s^2 + t^2 + 2u^2 + 4v^2 \\ =& 2\left(u + \frac{y}{2}\right)^2 + 2\left(\frac{z}{2}\right)^2 + 2\left(\frac{b+x-2y}{8}\right)^2 + 4\left(\frac{b-x}{8}\right)^2 \\ =& 2\left(\frac{b+x+2y}{8}\right)^2 + 2\left(\frac{b+x-2y}{8}\right)^2 + \frac{z^2}{2} + \left(\frac{b-x}{4}\right)^2 \\ =& \left(\frac{b+x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 + \frac{z^2}{2} + \left(\frac{b-x}{4}\right)^2 \\ =& \frac{b^2 + x^2 + 2y^2 + 4z^2}{8} = a. \end{aligned}$$

In view of Lemma 2.3 and (5.2),

$$(|x| + 2|y| + 4|z|)^2 \leqslant 7(x^2 + 2y^2 + 4z^2) = 7(8a - b^2) < (b+8)^2$$

and hence

$$b - |x| - 2|y| - 4|z| > -8.$$

So we have

$$u,v,s,t\geqslant\frac{b-|x|-2|y|-4|z|}{8}>-1$$

and hence $s,t,u,v\in\mathbb{N}$.

In view of the above, we have completed the proof of Lemma 5.3. □

Proof of Theorem 1.4. As

$$N\geqslant 96m^2\times 11m\geqslant 96m^2(10m+3)=24m^2(40m+12),$$

applying Lemma 5.1 with $l=8$ we find that the interval $I_3=[\alpha,\beta]$ given by (5.1) has length greater than $8m$.

Case 1. $4\nmid m$ or $8\nmid N$.

Choose $b_0\in\{\lceil\alpha\rceil+r:\ r=0,\ldots,m-1\}$ with $N\equiv b_0\pmod m$. Set $b_1=b_0+m$. Then

$$\alpha\leqslant b_0<b_1\leqslant\lceil\alpha\rceil+2m-1<\alpha+8m<\beta$$

and thus $b_j\in I_3$ for each $j=0,1$. Note that

$$a_j:=\frac{2}{m}(N-b_j)+b_j=\frac{2}{m}(N-b_0)+b_0+(m-2)j.$$

If $2\nmid m$, then $a_j\equiv b_j\equiv 1\pmod 2$ for some $j\in\{0,1\}$. If $2\mid m$ and $2\nmid N$, then $a_0\equiv b_0\equiv 1\pmod 2$.

If $2\|m$ and $2\mid N$, then for some $j\in\{0,1\}$ we have $b_j\equiv 2\pmod 4$ and $2\mid a_j$. When $4|m$ and $2\|N$, we have $b_0\equiv 2\pmod 4$ and $a_0\equiv b_0\equiv 0\pmod 2$. If $4\|m$ and $4\|N$, then $4\mid b_0$, and for some $j\in\{0,1\}$ we have $b_j\equiv 4\pmod 8$ and $a_j\equiv b_j\equiv 0\pmod 4$. When $8\mid m$ and $4\|N$, we have $b_0\equiv 4\pmod 8$ and $a_0\equiv b_0\pmod 2$, hence for some $j\in\{0,1\}$ we have $a_j\equiv 0\pmod 4$ and $b_j\equiv b_0\equiv 4\pmod 8$.

Case 2. $4\mid m$ and $8\mid N$.

Choose $b\in\{\lceil\alpha\rceil+r:\ r=0,\ldots,8m-1\}$ such that $b\equiv N-2m\pmod{8m}$. Since $\alpha\leqslant b\leqslant\lceil\alpha\rceil+8m-1<\alpha+8m<\beta$, we have $b\in I_3$. Clearly, $8\mid b$ since $8\mid N$ and $4\mid m$. Note that

$$\frac{2}{m}(N-b)+b\equiv 4+b\pmod 8.$$

By the above, in either case we can always find $b\in I_3$ and $a\in\mathbb{Z}$ for which one of (i)-(iii) in Lemma 5.3 holds and also

$$a=\frac{2}{m}(N-b)+b,\quad\text{i.e., }N=\frac{m}{2}(a-b)+b.$$

By Lemmas 5.2 and 5.3, there are $s,t,u,v\in\mathbb{N}$ satisfying (5.3). Therefore,

$$\begin{aligned}N=&\frac{m}{2}(s^2+t^2+2u^2+4v^2-s-t-2u-4v)+s+t+2u+4v\\=&p_{m+2}(s)+p_{m+2}(t)+2p_{m+2}(u)+4p_{m+2}(v).\end{aligned}$$

In view of the above, we have finished the proof of Theorem 1.4. □

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