

ON EXACTLY m TIMES COVERS

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ABSTRACT

In this paper it is shown that if every integer is covered by $a_1 + n_1\mathbb{Z}, \dots, a_k + n_k\mathbb{Z}$ exactly m times then for each $n = 1, \dots, m$ there exist at least $\binom{m}{n}$ subsets I of $\{1, \dots, k\}$ such that $\sum_{i \in I} 1/n_i$ equals n . The bound $\binom{m}{n}$ is best possible.

Let $a + n\mathbb{Z}$ denote the arithmetic sequence $\{x \in \mathbb{Z} : x \equiv a \pmod{n}\}$. We call

$$(1) \quad \{a_i + n_i\mathbb{Z}\}_{i=1}^k$$

an exactly m times cover if each integer belongs to exactly m of the sequences. An exactly one time cover is also said to be an exact cover.

It is well known [3] that $\sum_{i=1}^k 1/n_i$ equals m if (1) is an exactly m times cover, in particular $\sum_{i=1}^k 1/n_i$ equals 1 if (1) is an exact cover.

Porubský once asked whether each exactly m times cover is a union of m exact covers (cf. [2]). In 1976, Choi constructed an exactly 2 times cover which is not the union of two exact covers. (See Porubský [3].) For $m \geq 2$, the sequences in Choi's example, together with $m - 2$ sequences \mathbb{Z} , form an exactly m times cover which is not the union of m exact covers. Recently Ming-Zhi Zhang proved in [6] that for each $m = 2, 3, 4, \dots$ there exists an exactly m times cover no subcover of which is an exactly $n < m$ times cover.

Despite the negative answer to Porubský's question, we give here a result which has a positive aspect in some sense.

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THEOREM: Let (1) be an exactly m times cover. Then for each $n = 1, \dots, m$ there exist (at least) $\binom{m}{n}$ subsets I of $\{1, \dots, k\}$ such that $\sum_{i \in I} 1/n_i$ equals n , in particular there are (at least) m subsets $I \subseteq \{1, \dots, k\}$ with the property $\sum_{i \in I} 1/n_i = 1$.

To prove it we need a lemma.

LEMMA: Provided that $\{a_t + n_t \mathbb{Z}\}_{t=1}^k$ is an exactly m times cover, for any integer x we have the identity

$$\prod_{t=1}^k (1 - z^{N/n_t} e^{2\pi i(x+a_t)/n_t}) = (1 - z^N)^m$$

where $N = [n_1, \dots, n_k]$ is the least common multiple of n_1, \dots, n_k .

Proof: Notice that any zero θ of the left side satisfies $\theta^N = 1$. Furthermore, for each $u \in \{0, 1, \dots, N - 1\}$, $e^{2\pi i u/N}$ is a zero of multiplicity m of the left hand side because $-u - x$ is covered by $\{a_t + n_t \mathbb{Z}\}_{t=1}^k$ exactly m times. ■

We remark that both the above lemma and the main result of J. Beebee [1] are easy consequences of a more general theorem (cf. Theorem 3 of Sun [4] and Theorem 4 of Sun [5]).

Proof of Theorem: Suppose $r \geq 0$. Letting $z = r^{1/[n_1, \dots, n_k]}$ we obtain from the lemma

$$\prod_{t=1}^k (1 - r^{1/n_t} e^{2\pi i(n+a_t)/n_t}) = (1 - r)^m, \quad n = 1, 2, 3, \dots$$

Hence for all $n \in \mathbb{Z}^+$ we have

$$1 - \sum_{t=1}^k r^{1/n_t} e^{2\pi i(n+a_t)/n_t} + \sum_{1 \leq t_1 < t_2 \leq k} r^{1/n_{t_1} + 1/n_{t_2}} e^{2\pi i((n+a_{t_1})/n_{t_1} + (n+a_{t_2})/n_{t_2})} - \dots + (-1)^k r^{1/n_1 + \dots + 1/n_k} e^{2\pi i((n+a_1)/n_1 + \dots + (n+a_k)/n_k)} = (1 - r)^m.$$

For $s > 1$, $\sum_{n=1}^{\infty} e^{2\pi i cn} / n^s$ converges absolutely, and by the above

$$\begin{aligned} \sum_{\emptyset \subset I \subseteq \{1, \dots, k\}} (-1)^{|I|} r^{\sum_{i \in I} 1/n_i} e^{2\pi i \sum_{i \in I} a_i/n_i} \sum_{n=1}^{\infty} \frac{e^{2\pi i \sum_{i \in I} n/n_i}}{n^s} \\ = \sum_{n=1}^{\infty} \frac{1}{n^s} [(1 - r)^m - 1], \end{aligned}$$

i.e.

$$\left[\sum_{\substack{\emptyset \subset I \subseteq \{1, \dots, k\} \\ \sum_{t \in I} 1/n_t \in \mathbb{Z}}} (-1)^{|I|} r^{\sum_{t \in I} 1/n_t} e^{2\pi i \sum_{t \in I} a_t/n_t} + 1 - (1-r)^m \right] \zeta(s) + \sum_{\substack{\emptyset \subset I \subseteq \{1, \dots, k\} \\ \sum_{t \in I} 1/n_t \notin \mathbb{Z}}} (-1)^{|I|} r^{\sum_{t \in I} 1/n_t} e^{2\pi i \sum_{t \in I} a_t/n_t} \sum_{n=1}^{\infty} \frac{e^{2\pi i n \sum_{t \in I} 1/n_t}}{n^s} = 0.$$

Let $s \rightarrow 1$ from the right. Then $\zeta(s) \rightarrow \infty$, and if $\sum_{t \in I} 1/n_t \notin \mathbb{Z}$ then $\sum_{n=1}^{\infty} e^{2\pi i n \sum_{t \in I} 1/n_t} / n^s$ has a finite limit. From the equality we must have

$$1 + \sum_{\substack{\emptyset \subset I \subseteq \{1, \dots, k\} \\ \sum_{t \in I} 1/n_t \in \mathbb{Z}}} (-1)^{|I|} r^{\sum_{t \in I} 1/n_t} e^{2\pi i \sum_{t \in I} a_t/n_t} = (1-r)^m.$$

The last equality holds for any $r \geq 0$, and by comparing the coefficients we get

$$(2) \quad \sum_{\substack{\emptyset \subset I \subseteq \{1, \dots, k\} \\ \sum_{t \in I} 1/n_t = n}} (-1)^{|I|} e^{2\pi i \sum_{t \in I} a_t/n_t} = (-1)^n \binom{m}{n} \quad \text{for } n = 1, \dots, m.$$

Given $n \in \{1, \dots, m\}$ we have

$$\binom{m}{n} = \left| \sum_{\substack{\emptyset \subset I \subseteq \{1, \dots, k\} \\ \sum_{t \in I} 1/n_t = n}} (-1)^{|I|} e^{2\pi i \sum_{t \in I} a_t/n_t} \right| \leq \sum_{\substack{\emptyset \subset I \subseteq \{1, \dots, k\} \\ \sum_{t \in I} 1/n_t = n}} 1,$$

so there are at least $\binom{m}{n}$ subsets I of $\{1, \dots, k\}$ such that $\sum_{t \in I} 1/n_t$ equals n . This concludes the proof. ■

Observe that m sequences \mathbb{Z} form an exactly m times cover. This example shows that the lower bounds $\binom{m}{n}$ ($1 \leq n \leq m$) are best possible.

At the end we mention that in 1989 Ming-Zhi Zhang [7] obtained the following surprising result: If $\bigcup_{i=1}^k a_i + n_i \mathbb{Z} = \mathbb{Z}$ then $\sum_{i \in I} 1/n_i \in \mathbb{Z}^+$ for some $I \subseteq \{1, \dots, k\}$.

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References

- [1] J. Beebee, *Some trigonometric identities related to exact covers*, Proc. Amer. Math. Soc. **112** (1991), 329–338.
- [2] R. K. Guy, *Unsolved Problems in Number Theory*, pp. 140–141, Springer-Verlag, 1981.
- [3] Š. Porubský, *On m times covering systems of congruences*, Acta Arith. **29** (1976), 159–169.
- [4] Zhi-Wei Sun, *Several results on systems of residue classes*, Adv. in Math. (China) **18** (1989), 251–252.
- [5] Zhi-Wei Sun, *Systems of congruences with multipliers*, Nanjing Univ. J. Math. Bi-quarterly **6** (1989), 124–133.
- [6] Ming-Zhi Zhang, *On irreducible exactly m times covering system of residue classes*, J. Sichuan Univ. (Nat. Sci. Ed.) **28** (1991), 403–408.
- [7] Ming-Zhi Zhang, *A note on covering systems of residue classes*, J. Sichuan Univ. (Nat. Sci. Ed.) **26** (1989), 185–188, Special Issue.