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ARITHMETIC PROPERTIES OF DELANNOY NUMBERS AND SCHRÖDER NUMBERS

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Abstract. Define

$$D_n(x)=\sum_{k=0}^n\binom{n}{k}^2x^k(x+1)^{n-k}\quad\text{for } n=0,1,2,\ldots$$

and

$$s_n(x)=\sum_{k=1}^n\frac{1}{n}\binom{n}{k}\binom{n}{k-1}x^{k-1}(x+1)^{n-k}\quad\text{for } n=1,2,3,\ldots.$$

Then $D_n(1)$ is the n-th central Delannoy number D_n, and $s_n(1)$ is the n-th little Schröder number s_n. In this paper we obtain some surprising arithmetic properties of $D_n(x)$ and $s_n(x)$. We show that

$$\frac{1}{n}\sum_{k=0}^{n-1}D_k(x)s_{k+1}(x)\in\mathbb{Z}[x(x+1)]\quad\text{for all } n=1,2,3,\ldots.$$

Moreover, for any odd prime p and p-adic integer $x\not\equiv 0,-1\pmod p$, we establish the supercongruence

$$\sum_{k=0}^{p-1}D_k(x)s_{k+1}(x)\equiv 0\pmod{p^2}.$$

As an application we confirm Conjecture 5.5 in [S14a], in particular we prove that

$$\frac{1}{n}\sum_{k=0}^{n-1}T_kM_k(-3)^{n-1-k}\in\mathbb{Z}\quad\text{for all } n=1,2,3,\ldots,$$

where T_k is the k-th central trinomial coefficient and M_k is the k-th Motzkin number.

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1. Introduction

For $m,n\in\mathbb{N}=\{0,1,2,\ldots\}$, the Delannoy number

$$D_{m,n}:=\sum_{k\in\mathbb{N}}\binom{m}{k}\binom{n}{k}2^k \tag{1.1}$$

in combinatorics counts lattice paths from $(0,0)$ to (m,n) in which only east $(1,0)$, north $(0,1)$, and northeast $(1,1)$ steps are allowed (cf. R. P. Stanley [St99, p. 185]). The n-th central Delannoy number $D_n=D_{n,n}$ has another well-known expression:

$$D_n=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}=\sum_{k=0}^n\binom{n+k}{2k}\binom{2k}{k}. \tag{1.2}$$

For $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$, the n-th little Schröder number is given by

$$s_n:=\sum_{k=1}^n N(n,k)2^{k-1} \tag{1.3}$$

with the Narayana number $N(n,k)$ defined by

$$N(n,k):=\frac{1}{n}\binom{n}{k}\binom{n}{k-1}\in\mathbb{Z}.$$

(See [Gr, pp. 268–281] for certain combinatorial interpretations of the Narayana number $N(n,k)=N(n,n+1-k)$.) For $n\in\mathbb{N}$, the n-th large Schröder number is given by

$$S_n:=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}=\sum_{k=0}^n\binom{n+k}{2k}C_k, \tag{1.4}$$

where C_k denotes the Catalan number $\binom{2k}{k}/(k+1)=\binom{2k}{k}-\binom{2k}{k+1}$. It is well known that $S_n=2s_n$ for every $n=1,2,3,\ldots$. Both little Schröder numbers and large Schröder numbers have many combinatorial interpretations (cf. [St97] and [St99, pp. 178, 239-240]); for example, s_n is the number of ways to insert parentheses into an expression of $n+1$ terms with two or more items within a parenthesis, and S_n is the number of lattice paths from the point $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$ which never rise above the line $y=x$.

Surprisingly, the central Delannoy numbers and Schröder numbers arising naturally in enumerative combinatorics, have nice arithmetic properties. In 2011 the author [S11b] showed that

$$\sum_{k=1}^{p-1}\frac{D_k}{k^2}\equiv(-1)^{(p-1)/2}2E_{p-3}\pmod p\quad\text{and}\quad\sum_{k=1}^{p-1}\frac{S_k}{6^k}\equiv 0\pmod p$$

for any prime $p > 3$, where $E_0, E_1, E_2, \ldots$ are the Euler numbers. In 2014 the author [S14a] proved that

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)D_k^2 \in \mathbb{Z} \quad \text{for all } n = 1,2,3,\ldots,$$

and that

$$\sum_{k=0}^{p-1} D_k^2 \equiv \left(\frac{2}{p}\right) \pmod p \quad \text{for any odd prime } p,$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol.

Definition 1.1. We define

$$D_n(x) := \sum_{k=0}^{n}\binom{n}{k}^2 x^k(x+1)^{n-k} \qquad \text{for } n \in \mathbb{N}, \tag{1.5}$$

and

$$s_n(x) := \sum_{k=1}^{n} N(n,k)x^{k-1}(x+1)^{n-k} \qquad \text{for } n \in \mathbb{Z}^+. \tag{1.6}$$

Obviously $D_n(1) = D_n$ for $n \in \mathbb{N}$, and $s_n(1) = s_n$ for $n \in \mathbb{Z}^+$.

In this paper we obtain somewhat curious results involving the polynomials $D_n(x)$ and $s_n(x)$. Our first theorem is as follows.

Theorem 1.1. (i) *For any $n \in \mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1} D_k(x)s_{k+1}(x) = W_n(x(x+1)), \tag{1.7}$$

where

$$W_n(x) = \sum_{k=1}^{n} w(n,k)C_{k-1}x^{k-1} \in \mathbb{Z}[x] \tag{1.8}$$

with

$$w(n,k) = \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1} \in \mathbb{Z}. \tag{1.9}$$

(ii) *Let p be an odd prime. For any p-adic integer x, we have*

$$\sum_{k=0}^{p-1} D_k(x)s_{k+1}(x) \equiv \begin{cases} p(1-x(x+1)) \pmod{p^3} & \text{if } x \equiv 0,-1 \pmod p, \\ 2p^2 + \frac{2x+1}{x(x+1)}p^2(x^2 q_p(x) - (x+1)^2 q_p(x+1)) \pmod{p^3} & \text{otherwise,} \end{cases} \tag{1.10}$$

where $q_p(z)$ denotes the Fermat quotient $(z^{p-1}-1)/p$ for any p-adic integer $z \not\equiv 0 \pmod p$.

Remark 1.1. It is interesting to compare the new numbers $w(n,k)$ with the Narayana numbers $N(n,k)$.

Clearly, Theorem 1.1 in the case $x = 1$ yields the following consequence.

Corollary 1.1. *For any positive integer n, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1} D_k s_{k+1} = \sum_{k=1}^{n} w(n,k)C_{k-1}2^{k-1} \equiv 1 \pmod{2}. \tag{1.11}$$

Also, for any odd prime p we have

$$\sum_{k=0}^{p-1} D_k s_{k+1} \equiv 2p^2(1-3q_p(2)) \pmod{p^3}. \tag{1.12}$$

Remark 1.2. For the prime $p = 588811$, we have $q_p(2) \equiv 1/3 \pmod{p}$ and hence $\sum_{k=0}^{p-1} D_k s_{k+1} \equiv 0 \pmod{p^3}$. In 2016 J.-C. Liu [L] confirmed the author's conjecture (cf. [S11b, Conjecture 1.1]) that

$$\sum_{k=1}^{p-1} D_k S_k \equiv -2p\sum_{k=1}^{p-1}\frac{(-1)^k+3}{k} \pmod{p^4} \quad \text{for any prime } p>3.$$

From Theorem 1.1 we can deduce a novel combinatorial identity.

Corollary 1.2. *For any $n \in \mathbb{Z}^+$, we have*

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k-1}\frac{C_{k-1}}{(-4)^{k-1}} = \frac{\lfloor (n+1)/2\rfloor}{4^{n-1}}\binom{n}{\lfloor n/2\rfloor}^2. \tag{1.13}$$

Remark 1.3. If we let a_n denote the left-hand side of (1.13), then the Zeilberger algorithm (cf. [PWZ, pp. 101-119]) cannot find a closed form for a_n, and it only yields the following second-order recurrence relation:

$$(n+1)^2a_n + (2n+3)a_{n+1} - (n+1)(n+3)a_{n+2} = 0 \quad \text{for } n = 1,2,3,\ldots.$$

Now we give our second theorem which can be viewed as a supplement to Theorem 1.1.

Theorem 1.2. *Let p be any odd prime. Then*

$$\sum_{k=0}^{p-1} kD_k(x)s_{k+1}(x) \equiv 2(x(x+1))^{(p-1)/2} \pmod{p}. \tag{1.14}$$

In particular,

$$\sum_{k=0}^{p-1} kD_k s_{k+1} \equiv 2\left(\frac{2}{p}\right) \pmod{p}. \tag{1.15}$$

In the next section we are going to show Theorems 1.1-1.2 and Corollary 1.2. In Section 3 we will give applications of Theorems 1.1-1.2 to central trinomial coefficients, Motzkin numbers, and their generalizations. Section 4 contains two related conjectures.

Throughout this paper, for any polynomial $P(x)$ and $n \in \mathbb{N}$, we use $[x^n]P(x)$ to denote the coefficient of x^n in $P(x)$.

2. Proofs of Theorems 1.1-1.2 and Corollary 1.2

Recall the following definition given in [S12a] motivated by the large Schröder numbers.

Definition 2.1. For $n \in \mathbb{N}$ we set

$$S_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{x^k}{k+1} = \sum_{k=0}^{n}\binom{n+k}{2k}C_k x^k. \tag{2.1}$$

Lemma 2.1. *We have*

$$D_n(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k \quad \textit{for } n \in \mathbb{N}. \tag{2.2}$$

Also, for any $n \in \mathbb{Z}^+$ we have

$$D_{n+1}(x) - D_{n-1}(x) = 2x(2n+1)S_n(x) \tag{2.3}$$

and

$$(x+1)s_n(x) = S_n(x). \tag{2.4}$$

Proof. For $k, n \in \mathbb{N}$, we obviously have

$$\begin{aligned}[x^k]D_n(x) =& [x^k]\sum_{j=0}^{n}\binom{n}{j}^2 x^j(x+1)^{n-j} = \sum_{j=0}^{k}\binom{n}{j}^2\binom{n-j}{k-j}\\ =&\binom{n}{k}\sum_{j=0}^{k}\binom{n}{j}\binom{k}{k-j} = \binom{n}{k}\binom{n+k}{k}\end{aligned}$$

with the help of the Chu-Vandermonde identity (cf. [G, (3.1)]). This proves (2.2).

Now fix $n \in \mathbb{Z}^+$. For $k \in \mathbb{N}$, by (2.2) we clearly have

$$\begin{aligned}&[x^{k+1}](D_{n+1}(x) - D_{n-1}(x))\\ =&\binom{n+1+(k+1)}{2(k+1)}\binom{2(k+1)}{k+1} - \binom{n-1+(k+1)}{2(k+1)}\binom{2(k+1)}{k+1}\\ =&\frac{2n+1}{2k+1}\binom{n+k}{2k}\binom{2k+2}{k+1} = \frac{2(2n+1)}{k+1}\binom{n+k}{2k}\binom{2k}{k}\\ =&[x^{k+1}]2x(2n+1)S_n(x).\end{aligned}$$

So (2.3) follows.

For each $k\in\mathbb{N}$, it is apparent that

$$\begin{aligned}[x^k](x+1)s_n(x)=&[x^k](x+1)\sum_{j=1}^n N(n,j)x^{j-1}(x+1)^{n-j}\\ =&\sum_{0<j\leqslant k}N(n,j)\binom{n-j}{k-j}+\sum_{j=1}^{k+1}N(n,j)\binom{n-j}{k-j+1}\\ =&\frac{1}{n}\sum_{j=1}^{k+1}\binom{n}{j}\binom{n}{j-1}\binom{n-j+1}{k-j+1}\\ =&\frac{1}{n}\binom{n}{k}\sum_{j=0}^{k+1}\binom{n}{j}\binom{k}{k+1-j}\\ =&\frac{1}{n}\binom{n}{k}\binom{n+k}{k+1}=\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}=[x^k]S_n(x).\end{aligned}$$

This proves (2.4).

The proof of Lemma 2.1 is now complete. □

Remark 2.1. Note that the Legendre polynomial of degree n is given by

$$P_n(x)=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k.$$

Lemma 2.2. *Let $n\in\mathbb{Z}^+$. Then*

$$n(n+1)S_n(x)^2=\sum_{k=1}^n\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}x^{k-1}(x+1)^{k+1}\tag{2.5}$$

and

$$\frac{D_{n-1}(x)+D_{n+1}(x)}{2}S_n(x)=\sum_{k=0}^n\binom{n+k}{2k}\binom{2k}{k}^2\frac{2k+1}{(k+1)^2}x^k(x+1)^{k+1}.\tag{2.6}$$

Remark 2.2. The identities (2.5) and (2.6) are (2.1) and (3.6) of the author's paper [S12a] respectively.

Lemma 2.3. *For any $m,n\in\mathbb{Z}^+$ with $m\leqslant n$, we have the identity*

$$\begin{aligned}&\sum_{k=m}^n\binom{k+m}{2m}\left(2m+1-m(m+1)\frac{2k+1}{k(k+1)}\right)\\ &\quad=\frac{(n-m)(n+m+1)}{n+1}\binom{n+m}{2m}.\end{aligned}\tag{2.7}$$

Proof. When $n=m$, both sides of (2.7) vanish.

Let $m,n\in\mathbb{Z}^+$ with $n\geqslant m$. If (2.7) holds, then

$$\begin{aligned}&\sum_{k=m}^{n+1}\binom{k+m}{2m}\left(2m+1-m(m+1)\frac{2k+1}{k(k+1)}\right)\\ =&\frac{(n-m)(n+m+1)}{n+1}\binom{n+m}{2m}\\ &+\binom{n+1+m}{2m}\left(2m+1-m(m+1)\frac{2(n+1)+1}{(n+1)(n+2)}\right)\\ =&\binom{n+m+1}{2m}\left(\frac{(n-m)(n-m+1)}{n+1}+2m+1-\frac{m(m+1)(2n+3)}{(n+1)(n+2)}\right)\\ =&\binom{(n+1)+m}{2m}\frac{(n+1-m)((n+1)+m+1)}{(n+1)+1}.\end{aligned}$$

In view of the above, we have proved Lemma 2.3 by induction. □

Let A and B be integers. The Lucas sequence $u_n(A,B)$ $(n=0,1,2,\ldots)$ is defined by $u_0(A,B)=0$, $u_1(A,B)=1$, and

$$u_{n+1}(A,B)=Au_n(A,B)-Bu_{n-1}(A,B)\quad\text{for } n=1,2,3,\ldots.$$

It is well known that if $\Delta=A^2-4B\neq 0$ then

$$u_n(A,B)=\frac{\alpha^n-\beta^n}{\alpha-\beta}\quad\text{for all } n\in\mathbb{N},$$

where α and β are the two roots of the quadratic equation $x^2-Ax+B=0$ (so that $\alpha+\beta=A$ and $\alpha\beta=B$). It is also known (see, e.g., [S10, Lemma 2.3]) that for any odd prime p we have

$$u_p(A,B)\equiv\left(\frac{\Delta}{p}\right)\pmod p,\quad\text{and } u_{p-(\frac{\Delta}{p})}(A,B)\equiv 0\pmod p\text{ if } p\nmid B.$$

In particular, $F_p\equiv(\frac{5}{p})\pmod p$ and $p\mid F_{p-(\frac{5}{p})}$ for any odd prime p, where $F_n=u_n(1,-1)$ with $n\in\mathbb{N}$ is the n-th Fibonacci number.

Lemma 2.4. *Let p be an odd prime, and let x be any integer not divisible by p. Then*

$$W_p(x)\equiv\frac{4x+1}{2x}\left(\left(\frac{4x+1}{p}\right)-1\right)\pmod p. \tag{2.8}$$

Moreover, if $x\equiv -1/4\pmod p$ then

$$W_p(x)\equiv 2p\pmod{p^2}, \tag{2.9}$$

otherwise we have

$$\begin{aligned}W_p(x) \equiv&2p+\frac{4x+1}{2x}\left(1-x^{p-1}+(p+1)\left(\left(\frac{4x+1}{p}\right)-1\right)\right)\\&-\frac{4x+1}{4x^{2-(\frac{4x+1}{p})}}\left(2x+\left(\frac{4x+1}{p}\right)\right)u_{p-(\frac{4x+1}{p})}\left(2x+1,x^2\right)\pmod{p^2}.\end{aligned}\tag{2.10}$$

Proof. Clearly,

$$\binom{2p-1}{p-1}=\prod_{k=1}^{p-1}\left(1+\frac{p}{k}\right)\equiv 1+p\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k}+\frac{1}{p-k}\right)\equiv 1\pmod{p^2}.$$

(J. Wolstenholme [W] even showed that $\binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$ if $p>3$.) Thus

$$w(p,p)=\frac{1}{p}\binom{2p}{p-1}=\frac{2}{p+1}\binom{2p-1}{p-1}\equiv 2(1-p)\pmod{p^2},$$

and

$$C_{p-1}=\frac{1}{p}\binom{2p-2}{p-1}=\frac{1}{2p-1}\binom{2p-1}{p-1}\equiv -(2p+1)\pmod{p^2}.$$

For each $k=1,\ldots,p-1$, clearly

$$\begin{aligned}w(p,k)=&\frac{1}{k}\binom{p-1}{k-1}\binom{p+k-1}{k-1}\frac{p+k}{p+1}\\&=\left(1+\frac{p}{k}\right)\frac{1}{p+1}\prod_{0<j<k}\left(\frac{p-j}{j}\cdot\frac{p+j}{j}\right)\\&\equiv\left(\frac{1}{p+1}+\frac{p}{k}\right)(-1)^{k-1}\equiv(-1)^{k-1}\left(1-p+\frac{p}{k}\right)\pmod{p^2}.\end{aligned}$$

Therefore

$$\begin{aligned}
W_p(x) =&w(p,p)C_{p-1}x^{p-1}+\sum_{k=1}^{p-1}w(p,k)C_{k-1}x^{k-1}\\
\equiv&2(1-p)C_{p-1}x^{p-1}+\sum_{k=1}^{p-1}\left(1-p+\frac{p}{k}\right)C_{k-1}(-x)^{k-1}\\
\equiv&(2-2p)C_{p-1}x^{p-1}+(p+1)\bigg(\sum_{k=2}^{p}C_{k-1}(-x)^{k-1}+1-C_{p-1}(-x)^{p-1}\bigg)\\
&+p\sum_{k=1}^{p-1}\left(\frac{1}{k}-2\right)C_{k-1}(-x)^{k-1}\\
\equiv&p+1-(1-3p)(1+2p)x^{p-1}+(p+1)\sum_{k=1}^{p-1}C_k(-x)^k\\
&-\frac{p}{2}\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}(-x)^{k-1}\\
\equiv&2p+1-x^{p-1}+(p+1)\sum_{k=1}^{p-1}\frac{C_k}{m^k}-\frac{p}{2}m\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{km^k}\pmod{p^2},
\end{aligned}$$

where m is an integer with $m\equiv-1/x\pmod{p^2}$. By [S10, Theorem 1.1], we have

$$\begin{aligned}
\sum_{k=1}^{p-1}\frac{C_k}{m^k}\equiv&m^{p-1}-1-\frac{m-4}{2}\left(\left(\frac{\Delta}{p}\right)-1+u_{p-(\frac{\Delta}{p})}(m-2,1)\right)\pmod{p^2}\\
\equiv&-\frac{m-4}{2}\left(\left(\frac{\Delta}{p}\right)-1\right)\pmod{p},
\end{aligned}$$

where

$$\Delta:=m(m-4)\equiv-\frac{1}{x}\left(-\frac{1}{x}-4\right)=\frac{4x+1}{x^2}\pmod{p^2}.$$

So (2.8) follows.

If $m\not\equiv4\pmod{p}$, then by [S12b, Lemma 3.5] we have

$$\begin{aligned}
\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{km^k}\equiv&\frac{(-m)^{p-1}-1}{p}+\frac{m-4}{2}\left(\frac{\Delta}{p}\right)\frac{u_{p-(\frac{\Delta}{p})}(2-m,1)}{p}\\
=&\frac{m^{p-1}-1}{p}-\frac{m-4}{2}\left(\frac{\Delta}{p}\right)\frac{u_{p-(\frac{\Delta}{p})}(m-2,1)}{p}\pmod{p}
\end{aligned}$$

and hence from the above we deduce that

$$\begin{aligned}
W_p(x) \equiv& 2p+1-x^{p-1}\\
&+(p+1)\left(m^{p-1}-1-\frac{m-4}{2}\left(\left(\frac{\Delta}{p}\right)-1+u_{p-(\frac{\Delta}{p})}(m-2,1)\right)\right)\\
&-\frac{p}{2}m\left(\frac{m^{p-1}-1}{p}-\frac{m-4}{2}\left(\frac{\Delta}{p}\right)\frac{u_{p-(\frac{\Delta}{p})}(m-2,1)}{p}\right)\\
\equiv& 2p+1-x^{p-1}+\left(1-\frac{m}{2}\right)(m^{p-1}-1)-(p+1)\frac{m-4}{2}\left(\left(\frac{4x+1}{p}\right)-1\right)\\
&-\frac{m-4}{2}\left(1-\frac{m}{2}\left(\frac{4x+1}{p}\right)\right)u_{p-(\frac{4x+1}{p})}(m-2,1)\\
\equiv& 2p+1-x^{p-1}+\frac{2x+1}{2x}(1-x^{p-1})+(p+1)\frac{4x+1}{2x}\left(\left(\frac{4x+1}{p}\right)-1\right)\\
&+\frac{4x+1}{2x}\left(1+\frac{1}{2x}\left(\frac{4x+1}{p}\right)\right)u_{p-(\frac{4x+1}{p})}(m-2,1)\pmod{p^2}
\end{aligned}$$

which is equivalent to (2.10) since

$$(-x)^{k-1}u_k(m-2,1)=u_k(-x(m-2),(-x)^2)\equiv u_k(2x+1,x^2)\pmod{p^2}$$

for all $k\in\mathbb{N}$.

When $m\equiv 4\pmod p$ (i.e., $x\equiv -1/4\pmod p$), we have

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{km^k}\equiv\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k4^k}\equiv 2q_p(2)\pmod p$$

by [ST, (1.12)], and hence

$$\begin{aligned}
W_p(x)\equiv& 2p+1-x^{p-1}+(p+1)\left(m^{p-1}-1+\frac{m-4}{2}\right)-\frac{p}{2}m2q_p(2)\\
\equiv& 2p+1-x^{p-1}+m^{p-1}-1+\frac{m-4}{2}-4(2^{p-1}-1)\\
\equiv& 2p+1-x^{p-1}+1-x^{p-1}-\frac{4x+1}{2x}-2(2^{2(p-1)}-1)\\
\equiv& 2\left(p+(4x+1)+(1-x^{p-1})+(1-4^{p-1})\right)\\
\equiv& 2\left(p+(4x+1)+1-(4x+1-1)^{p-1}\right)\\
\equiv& 2\left(p+(4x+1)+(p-1)(4x+1)\right)\equiv 2p\pmod{p^2}.
\end{aligned}$$

Therefore (2.9) is valid. □

Proof of Theorem 1.1. (i) Fix $n \in \mathbb{Z}^+$. For each $k \in \mathbb{Z}^+$, by (2.3) and Lemma 2.2 we have

$$\begin{aligned}
D_{k-1}(x)\frac{S_k(x)}{x+1} =& \frac{D_{k-1}(x)+D_{k+1}(x)}{2} \cdot \frac{S_k(x)}{x+1} - (2k+1)\frac{x}{x+1}S_k(x)^2 \\
=& \sum_{j=0}^{k} \binom{k+j}{2j}\binom{2j}{j}^2 \frac{2j+1}{(j+1)^2}(x(x+1))^j \\
& - \frac{2k+1}{k(k+1)} \sum_{j=0}^{k} \binom{k+j}{2j}\binom{2j}{j}\binom{2j}{j+1}(x(x+1))^j \\
=& \sum_{j=0}^{k} \binom{k+j}{2j}\binom{2j}{j}^2 \left(\frac{2j+1}{(j+1)^2} - \frac{2k+1}{k(k+1)} \cdot \frac{j}{j+1}\right)(x(x+1))^j \\
=& \sum_{j=0}^{k} \binom{k+j}{2j} C_j^2 \left(2j+1-j(j+1)\frac{2k+1}{k(k+1)}\right)(x(x+1))^j.
\end{aligned}$$

Combining this with (2.4) we obtain

$$D_{k-1}(x)s_k(x) = \sum_{j=0}^{k} \binom{k+j}{2j} C_j^2 \left(2j+1-j(j+1)\frac{2k+1}{k(k+1)}\right)(x(x+1))^j \tag{2.11}$$

for any $k \in \mathbb{Z}^+$. Therefore,

$$\begin{aligned}
\sum_{k=1}^{n} D_{k-1}(x)s_k(x) =& \sum_{k=1}^{n}\sum_{j=0}^{k} \binom{k+j}{2j} C_j^2 \left(2j+1-j(j+1)\frac{2k+1}{k(k+1)}\right)(x(x+1))^j \\
=& n + \sum_{j=1}^{n} C_j^2 (x(x+1))^j \sum_{k=j}^{n} \binom{k+j}{2j}\left(2j+1-j(j+1)\frac{2k+1}{k(k+1)}\right) \\
=& \sum_{j=0}^{n-1} C_j^2 (x(x+1))^j \frac{(n-j)(n+j+1)}{n+1}\binom{n+j}{2j}
\end{aligned}$$

with the help of Lemma 2.3. It follows that

$$\sum_{k=0}^{n-1} D_k(x)s_{k+1}(x) = \sum_{j=0}^{n-1} C_j(x(x+1))^j \frac{n}{j+1}\binom{n-1}{j}\binom{n+j+1}{j} = nW_n(x(x+1)).$$

For any $k \in \mathbb{Z}^+$, we have

$$w(n,k) = \frac{1}{n}\binom{n}{k}\binom{n+k}{k-1} = \frac{1}{n+1}\binom{n-1}{k-1}\binom{n+k}{k}$$

and hence $w(n,k) = (n+1)w(n,k) - nw(n,k) \in \mathbb{Z}$. So $W_n(x) \in \mathbb{Z}[x]$. This proves part (i) of Theorem 1.1.

(ii) Let x be any p-adic integer, and let y be an integer with $y \equiv x(x+1) \pmod{p^2}$. By (1.7) we have

$$\sum_{k=0}^{p-1} D_k(x)s_{k+1}(x) = pW_p(x(x+1)) \equiv pW_p(y) \pmod{p^3}. \tag{2.12}$$

If $y \equiv 0 \pmod p$, then

$$W_p(y) \equiv w(p,1) + w(p,2)y \equiv 1 - x(x+1) \pmod{p^2}.$$

If $x \equiv -1/2 \pmod p$ (i.e., $y \equiv -1/4 \pmod p$), then $W_p(y) \equiv 2p \pmod{p^2}$ by Lemma 2.4. Thus (1.10) holds for $x \equiv 0, -1, -1/2 \pmod p$.

Now assume that $x \not\equiv 0, -1, -1/2 \pmod p$, i.e., $y \not\equiv 0, -1/4 \pmod p$. Then

$$\left(\frac{4y+1}{p}\right) = \left(\frac{(2x+1)^2}{p}\right) = 1$$

and

$$\begin{aligned} u_{p-1}(2y+1, y^2) \equiv& u_{p-1}(x^2 + (x+1)^2, x^2(x+1)^2) \\ =& \frac{((x+1)^2)^{p-1} - (x^2)^{p-1}}{(x+1)^2 - x^2} \\ =& \frac{(x+1)^{p-1} + x^{p-1}}{2x+1}\left((x+1)^{p-1} - x^{p-1}\right) \\ \equiv& \frac{2}{2x+1}\left((x+1)^{p-1} - x^{p-1}\right) \pmod{p^2}. \end{aligned}$$

Combining this with Lemma 2.4, we obtain

$$\begin{aligned} W_p(y) \equiv& 2p + \frac{(2x+1)^2}{2x(x+1)}\left(1 - x^{p-1}(x+1)^{p-1}\right) \\ &- \frac{(2x+1)^2}{4x(x+1)}(2x(x+1)+1)\frac{2}{2x+1}\left((x+1)^{p-1} - x^{p-1}\right) \\ \equiv& 2p + \frac{(2x+1)^2}{2x(x+1)}\left(1 - x^{p-1} + 1 - (x+1)^{p-1}\right) \\ &- \frac{2x+1}{2x(x+1)}(2x^2+2x+1)\left((x+1)^{p-1} - x^{p-1}\right) \\ =& 2p + \frac{2x+1}{x(x+1)}\left(x^2(x^{p-1}-1) - (x+1)^2\left((x+1)^{p-1}-1\right)\right) \pmod{p^2}. \end{aligned}$$

Therefore (1.10) holds in light of (2.12).

So far we have completed the proof of Theorem 1.1. □

Proof of Corollary 1.2. It is known (cf. [G, (3.133)]) that

$$D_k\left(-\frac{1}{2}\right) = P_k(0) = \begin{cases} (-1)^{k/2}\binom{k}{k/2}/2^k & \text{if } 2 \mid k, \\ 0 & \text{otherwise.} \end{cases}$$

By (2.4) and [S11a, Lemma 4.3],

$$s_{k+1}\left(-\frac{1}{2}\right)=2S_{k+1}\left(-\frac{1}{2}\right)=\begin{cases}(-1)^{k/2}C_{k/2}/2^k & \text{if } 2\mid k,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore

$$\begin{aligned}&\sum_{k=0}^{n-1}D_k\left(-\frac{1}{2}\right)s_{k+1}\left(-\frac{1}{2}\right)\\ =&\sum_{\substack{0\leqslant k<n\\ 2\mid k}}\left(\frac{(-1)^{k/2}}{2^k}\right)^2\binom{k}{k/2}C_{k/2}=\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\frac{\binom{2j}{j}^2}{(j+1)16^j}.\end{aligned}$$

By induction,

$$\sum_{j=0}^{m}\frac{\binom{2j}{j}^2}{(j+1)16^j}=\frac{(2m+1)^2}{(m+1)16^m}\binom{2m}{m}^2\quad\text{for all } m\in\mathbb{N}.$$

So we have

$$\begin{aligned}&\sum_{k=0}^{n-1}D_k\left(-\frac{1}{2}\right)s_{k+1}\left(-\frac{1}{2}\right)\\ =&\frac{(2\lfloor (n-1)/2\rfloor+1)^2}{\lfloor (n+1)/2\rfloor 16^{\lfloor (n-1)/2\rfloor}}\binom{2\lfloor (n-1)/2\rfloor}{\lfloor (n-1)/2\rfloor}^2=\frac{\lfloor (n+1)/2\rfloor}{4^{n-1}}\binom{n}{\lfloor n/2\rfloor}^2.\end{aligned}$$

On the other hand, by applying (1.7) with $x=-1/2$ we obtain

$$\sum_{k=0}^{n-1}D_k\left(-\frac{1}{2}\right)s_{k+1}\left(-\frac{1}{2}\right)=nW_n\left(-\frac{1}{4}\right)=\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k-1}\frac{C_{k-1}}{(-4)^{k-1}}.$$

Combining these we get the desired identity (1.13). This concludes the proof. □

Lemma 2.5. *Let p be any odd prime. For each $j=0,\ldots,p$, we have*

$$C_j^2u_j\equiv\begin{cases}2 \pmod p & \text{if } j=(p-1)/2,\\ 0 \pmod p & \text{otherwise,}\end{cases}\tag{2.13}$$

where

$$u_j:=\sum_{j<k\leqslant p}(k-1)\binom{k+j}{2j}\left(2j+1-j(j+1)\frac{2k+1}{k(k+1)}\right).\tag{2.14}$$

Proof. Clearly, $u_p=0$ and

$$u_{p-1}=(p-1)\binom{p+(p-1)}{2(p-1)}\left(2(p-1)+1-(p-1)p\frac{2p+1}{p(p+1)}\right)\equiv 0\pmod p.$$

If $(p-1)/2 < j < p-1$, then $C_j = (2j)!/(j!(j+1)!) \equiv 0 \pmod p$ and hence $C_j^2 u_j \equiv 0 \pmod p$ since pu_j is a p-adic integer.

Note that

$$C_{(p-1)/2} = \frac{2}{p+1}\binom{p-1}{(p-1)/2} \equiv 2(-1)^{(p-1)/2} \pmod p.$$

As

$$\binom{k+(p-1)/2}{p-1} \equiv 0 \pmod p \quad \text{for all } k = \frac{p+1}{2},\dots,p,$$

we have

$$\begin{aligned}
u_{(p-1)/2} \equiv& \sum_{k=p-1}^{p}(k-1)\binom{k+(p-1)/2}{p-1}\left(2\cdot\frac{p-1}{2}+1-\frac{p-1}{2}\cdot\frac{p+1}{2}\cdot\frac{2k+1}{k(k+1)}\right)\\
\equiv& \frac14\sum_{k=p-1}^{p}(k-1)\binom{k+(p-1)/2}{p-1}\frac{2k+1}{k(k+1)}\\
=&\frac{p-2}{4}\binom{p-1+(p-1)/2}{(p-1)/2}\frac{2(p-1)+1}{(p-1)p}+\frac{p-1}{4}\binom{p+(p-1)/2}{(p+1)/2}\frac{2p+1}{p(p+1)}\\
=&\frac{p-2}{4}\cdot\frac{2p-1}{p-1}\cdot\frac{\prod_{1<r\leqslant(p-1)/2}(p-1+r)}{((p-1)/2)!}\\
&+\frac{p-1}{4}\cdot\frac{2p+1}{p+1}\cdot\frac{\prod_{r=1}^{(p-1)/2}(p+r)}{((p+1)/2)!}\\
\equiv& -\frac12\cdot\frac{1}{(p-1)/2}-\frac14\cdot\frac{1}{(p+1)/2}\equiv\frac12 \pmod p.
\end{aligned}$$

Obviously,

$$u_0 = \sum_{k=1}^{p}(k-1) = \frac{p(p-1)}{2} \equiv 0 \pmod p.$$

Applying the Zeilberger algorithm via `Mathematica 9`, we find that

$$(j+2)u_j + 2(2j+1)u_{j+1} = \frac{f(p,j)\binom{p+j}{2j}}{2(j+1)(j+2)(2j+3)}$$

for all $j = 0,\dots,p-1$, where

$$f(p,j) := (p-j)(p+j+1)\left((2j+3)^2p^2-(2j^2+8j+7)p-(j+1)(j+2)\right).$$

This implies that for $0 \leqslant j < (p-3)/2$ we have

$$u_j \equiv 0 \pmod p \Longrightarrow u_{j+1} \equiv 0 \pmod p.$$

Thus $u_j \equiv 0 \pmod p$ for all $j = 0,\dots,(p-3)/2$.

Combining the above, we immediately obtain the desired (2.13). □

Proof of Theorem 1.2. In view of (2.11),

$$\begin{aligned}
&\sum_{k=1}^{p}(k-1)D_{k-1}(x)s_k(x)\\
&=\sum_{k=1}^{p}(k-1)\sum_{j=0}^{k}\binom{k+j}{2j}C_j^2\left(2j+1-j(j+1)\frac{2k+1}{k(k+1)}\right)(x(x+1))^j\\
&=\sum_{k=1}^{p}(k-1)+\sum_{j=1}^{p}(x(x+1))^jC_j^2u_j=\frac{p(p-1)}{2}+\sum_{j=1}^{p}C_j^2u_j(x(x+1))^j,
\end{aligned}$$

where u_j is given by (2.14). Thus, by applying Lemma 2.5 we find that

$$\frac{1}{p}\left(\sum_{k=0}^{p-1}kD_k(x)s_{k+1}(x)-2(x(x+1))^{(p-1)/2}\right)\in\mathbb{Z}_p[x(x+1)], \tag{2.15}$$

where $\mathbb{Z}_p$ denotes the ring of p-adic integers. Therefore (1.14) holds. (1.14) with $x=1$ gives (1.15). This concludes the proof. □

3. Applications to central trinomial coefficients and Motzkin numbers

Let $n\in\mathbb{N}$ and $b,c\in\mathbb{Z}$. The n-th generalized central trinomial coefficient $T_n(b,c)$ is defined to be $[x^n](x^2+bx+c)^n$, the coefficient of x^n in the expansion of $(x^2+bx+c)^n$. It is easy to see that

$$T_n(b,c)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k. \tag{3.1}$$

Note that $T_n(1,1)$ is the central trinomial coefficient T_n and $T_n(2,1)$ is the central binomial coefficient $\binom{2n}{n}$. Also, $T_n(3,2)$ coincides with the central Delannoy number D_n. Sun [S14a] also defined the generalized Motzkin number $M_n(b,c)$ by

$$M_n(b,c)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_kb^{n-2k}c^k. \tag{3.2}$$

Note that $M_n(1,1)$ is the usual Motzkin number M_n (whose combinatorial interpretations can be found in [St99, Ex. 6.38]) and $M_n(2,1)$ is the Catalan number C_{n+1}. Also, $M_n(3,2)$ coincides with the little Schröder number s_{n+1}. The author [S14a, S14b] deduced some congruences involving $T_n(b,c)$ and $M_n(b,c)$, and proposed in [S14b] some conjectural series for $1/\pi$ involving $T_n(b,c)$ such as

$$\begin{aligned}\sum_{k=0}^\infty\frac{66k+17}{(2^{11}3^3)^k}T_k(10,11^2)^3=&\frac{540\sqrt2}{11\pi},\\ \sum_{k=0}^\infty\frac{126k+31}{(-80)^{3k}}T_k(22,21^2)^3=&\frac{880\sqrt5}{21\pi},\\ \sum_{k=0}^\infty\frac{3990k+1147}{(-288)^{3k}}T_k(62,95^2)^3=&\frac{432}{95\pi}(195\sqrt{14}+94\sqrt2).\end{aligned}$$

Now we point out that $T_n(b,c)$ and $M_n(b,c)$ are actually related to the polynomials $D_n(x)$ and $s_{n+1}(x)$.

Lemma 3.1. *Let $b,c\in\mathbb{Z}$ with $d=b^2-4c\neq0$. For any $n\in\mathbb{N}$ we have*

$$T_n(b,c)=(\sqrt d)^nD_n\left(\frac{b/\sqrt d-1}{2}\right)\tag{3.3}$$

and

$$M_n(b,c)=(\sqrt d)^ns_{n+1}\left(\frac{b/\sqrt d-1}{2}\right).\tag{3.4}$$

Proof. In view of (3.1) and (3.2),

$$\frac{T_n(b,c)}{(\sqrt d)^n}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom n{2k}\binom{2k}k\left(\frac b{\sqrt d}\right)^{n-2k}\left(\frac cd\right)^k$$

and

$$\frac{M_n(b,c)}{(\sqrt d)^n}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom n{2k}C_k\left(\frac b{\sqrt d}\right)^{n-2k}\left(\frac cd\right)^k.$$

Note that

$$\left(\frac b{\sqrt d}\right)^2-4\frac cd=1.$$

So, it suffices to show the polynomial identities

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom n{2k}\binom{2k}k(2x+1)^{n-2k}(x(x+1))^k=D_n(x)\tag{3.5}$$

and

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom n{2k}C_k(2x+1)^{n-2k}(x(x+1))^k=s_{n+1}(x).\tag{3.6}$$

It is easy to verify (3.5) and (3.6) for $n = 0, 1, 2$. Let $u_n(x)$ denote the left-hand side or the right-hand side of (3.5). By the Zeilberger algorithm (cf. [PWZ, pp. 101-119]), we have the recurrence

$$(n+2)u_{n+2}(x) = (2x+1)(2n+3)u_{n+1}(x) - (n+1)u_n(x) \quad \text{for } n = 0, 1, 2, \ldots .$$

Thus (3.5) is valid by induction. Let $v_n(x)$ denote the left-hand side or the right-hand side of (3.6). By the Zeilberger algorithm, we have the recurrence

$$(n+4)v_{n+2}(x) = (2x+1)(2n+5)v_{n+1}(x) - (n+1)v_n(x) \quad \text{for } n = 0, 1, 2, \ldots .$$

So (3.6) also holds by induction.

The proof of Lemma 3.1 is now complete. □

With the help of Theorem 1.1, we are able to confirm Conjecture 5.5 of the author [S14a] by proving the following result.

Theorem 3.1. *Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$.*

(i) For any $n \in \mathbb{Z}^+$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} T_k(b,c)M_k(b,c)d^{n-1-k} = \sum_{k=1}^{n} w(n,k)C_{k-1}c^{k-1}d^{n-k} \in \mathbb{Z}. \tag{3.7}$$

Moreover, for any odd prime p not dividing cd, we have

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)M_k(b,c)}{d^k} \equiv \frac{pb^2}{2c}\left(\left(\frac{d}{p}\right)-1\right) \pmod{p^2}, \tag{3.8}$$

and furthermore

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{T_k(b,c)M_k(b,c)}{d^k}\\ \equiv&\frac{pb^2}{2c}\left(\left(\frac{d}{p}\right)-1\right)+\frac{p^2}{2c}\left(b^2\left(q_p(d)-q_p(c)+\left(\frac{d}{p}\right)\right)-d\right)\\ &-\frac{pb^2}{4c^{2-(\frac{d}{p})}}\left(2c+d\left(\frac{d}{p}\right)\right)u_{p-(\frac{d}{p})}(b^2-2c,c^2) \pmod{p^3}.\end{aligned} \tag{3.9}$$

(ii) For any odd prime p not dividing d, we have

$$\sum_{k=0}^{p-1}\frac{kT_k(b,c)M_k(b,c)}{d^k} \equiv 2\left(\frac{cd}{p}\right) \pmod{p}. \tag{3.10}$$

Proof. (i) Let's first prove (3.7) for any $n \in \mathbb{Z}^+$.

We first consider the case $d=0$, i.e., $c=b^2/4$. In this case, for any $k\in\mathbb{N}$ we have

$$T_k(b,c)=\left(\frac{b}{2}\right)^k T_k(2,1)=\left(\frac{b}{2}\right)^k\binom{2k}{k}$$

and

$$M_k(b,c)=\left(\frac{b}{2}\right)^k M_k(2,1)=\left(\frac{b}{2}\right)^k C_{k+1}.$$

Thus

$$\begin{aligned}&\frac{1}{n}\sum_{k=0}^{n-1}T_k(b,c)M_k(b,c)d^{n-1-k}\\=&\frac{T_{n-1}(b,c)M_{n-1}(b,c)}{n}=\frac{1}{n}\left(\frac{b^2}{4}\right)^{n-1}\binom{2(n-1)}{n-1}C_n\\=&c^{n-1}C_{n-1}C_n=w(n,n)C_{n-1}c^{n-1}\end{aligned}$$

and hence (3.7) is valid.

Now assume that $d\neq0$. By Lemma 3.1 and (1.7), we have

$$\begin{aligned}&\frac{1}{n}\sum_{k=0}^{n-1}\frac{T_k(b,c)M_k(b,c)}{d^k}\\=&\frac{1}{n}\sum_{k=0}^{n-1}D_k\left(\frac{b/\sqrt{d}-1}{2}\right)s_{k+1}\left(\frac{b/\sqrt{d}-1}{2}\right)\\=&W_n\left(\frac{b/\sqrt{d}-1}{2}\cdot\frac{b/\sqrt{d}+1}{2}\right)=W_n\left(\frac{b^2/d-1}{4}\right)=W_n\left(\frac{c}{d}\right)\end{aligned}$$

and hence (3.7) holds in view of (1.8).

Below we suppose that p is an odd prime not dividing cd. From the above, we have

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)M_k(b,c)}{d^k}=pW_p\left(\frac{c}{d}\right)\equiv pW_p(x)\pmod{p^3},\tag{3.11}$$

where x is an integer with $x\equiv c/d\pmod{p^2}$. As $p\nmid d$ and $d(4x+1)\equiv4c+d=b^2\pmod{p^2}$, we have

$$\left(\frac{4x+1}{p}\right)=\left(\frac{d^2(4x+1)}{p}\right)=\left(\frac{b^2d}{p}\right).$$

In view of Lemma 2.4,

$$W_p(x)\equiv\frac{4c/d+1}{2c/d}\left(\left(\frac{4x+1}{p}\right)-1\right)\equiv\frac{b^2}{2c}\left(\left(\frac{d}{p}\right)-1\right)\pmod{p}.$$

Combining this with (3.11) we immediately obtain (3.8).

Now we show (3.9). If $x \equiv -1/4 \pmod p$ (i.e., $p \mid b$), then by (2.9) we have

$$W_p(x) \equiv 2p \equiv \frac{p}{2c}(4c - b^2) \pmod{p^2}$$

and hence (3.9) holds by (3.11). Below we assume $p \nmid b$. Then

$$\left(\frac{(2x+1)^2 - 4x^2}{p}\right) = \left(\frac{4x+1}{p}\right) = \left(\frac{b^2 d}{p}\right) = \left(\frac{d}{p}\right),$$

$p \mid u_{p-(\frac{d}{p})}(2x+1, x^2)$ and

$$\begin{aligned}&d^{p-1-(\frac{d}{p})} u_{p-(\frac{d}{p})}\left(2x+1, x^2\right)\\ =&u_{p-(\frac{d}{p})}(d(2x+1), d^2x^2)\\ \equiv&u_{p-(\frac{d}{p})}(2c+d, c^2) = u_{p-(\frac{d}{p})}(b^2-2c, c^2) \pmod{p^2}.\end{aligned}$$

So, applying (2.10) we get

$$\begin{aligned}W_p(x) \equiv&2p + \frac{4c/d+1}{2c/d}\left(1 - \left(\frac{c}{d}\right)^{p-1} + (p+1)\left(\left(\frac{d}{p}\right) - 1\right)\right)\\ &- \frac{4c/d+1}{4(c/d)^{2-(\frac{d}{p})}}\left(2\frac{c}{d} + \left(\frac{d}{p}\right)\right) d^{(\frac{d}{p})} u_{p-(\frac{d}{p})}(b^2-2c, c^2)\\ \equiv&2p + \frac{b^2}{2c}\left(d^{p-1} - 1 - (c^{p-1}-1) + (p+1)\left(\left(\frac{d}{p}\right) - 1\right)\right)\\ &- \frac{b^2}{4c^{2-(\frac{d}{p})}}\left(2c + d\left(\frac{d}{p}\right)\right) u_{p-(\frac{d}{p})}(b^2-2c, c^2) \pmod{p^2}.\end{aligned}$$

This, together with (3.11), yields the desired (3.9).

(ii) Fix an odd prime p not dividing d. Let $x = b/\sqrt{d} - 1$. Then

$$x(x+1) = \frac{b/\sqrt{d}-1}{2} \cdot \frac{b/\sqrt{d}+1}{2} = \frac{b^2/d - 1}{4} = \frac{c}{d}$$

is a p-adic integer. Thus, with the help of (2.15), we have

$$\sum_{k=0}^{p-1} kD_k(x)s_{k+1}(x) \equiv 2\left(\frac{c}{d}\right)^{(p-1)/2} \equiv 2\left(\frac{cd}{p}\right) \pmod p.$$

Combining this with Lemma 3.1, we immediately obtain (3.10).

In view of the above, we have proved Theorem 3.1. □

Let ω denote the primitive cubic root $(-1+\sqrt{-3})/2$ of unity. Then $\omega + \bar\omega = -1$ and $\omega\bar\omega = 1$. So,

$$u_n(-1,1) = \frac{\omega^n - \bar\omega^n}{\omega - \bar\omega} = 0 \ \text{ and } \ u_n(3,9) = \frac{(-3\omega)^n - (-3\bar\omega)^n}{(-3\omega) - (-3\bar\omega)} = 0$$

for any $n \in \mathbb{N}$ with $3 \mid n$. In view of this, Theorem 3.1 in the cases $b = c \in \{1, 3\}$ yields the following consequence.

Corollary 3.1. *For any positive integer n, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}T_kM_k(-3)^{n-1-k}=\sum_{k=1}^{n}w(n,k)C_{k-1}(-3)^{n-k}\in\mathbb{Z} \tag{3.12}$$

and

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{T_k(3,3)M_k(3,3)}{(-3)^k}=\sum_{k=1}^{n}(-1)^{k-1}w(n,k)C_{k-1}\in\mathbb{Z}. \tag{3.13}$$

Moreover, for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}\frac{T_kM_k}{(-3)^k}\equiv\frac{p}{2}\left(\left(\frac{p}{3}\right)-1\right)+\frac{p^2}{2}\left(q_p(3)+\left(\frac{p}{3}\right)+3\right)\pmod{p^3}, \tag{3.14}$$

$$\sum_{k=0}^{p-1}\frac{T_k(3,3)M_k(3,3)}{(-3)^k}\equiv\frac{3p}{2}\left(\left(\frac{p}{3}\right)-1\right)+\frac{p^2}{2}\left(3\left(\frac{p}{3}\right)+1\right)\pmod{p^3}, \tag{3.15}$$

$$\sum_{k=0}^{p-1}\frac{kT_kM_k}{(-3)^k}\equiv2\left(\frac{p}{3}\right)\pmod{p}\ \text{and}\ \sum_{k=0}^{p-1}\frac{kT_k(3,3)M_k(3,3)}{(-3)^k}\equiv2\left(\frac{-1}{p}\right)\pmod{p}. \tag{3.16}$$

Remark 3.1. Let $p>3$ be a prime. In the case $p\equiv2\pmod{3}$, the author (cf. [S14a, Conjecture 5.6]) even conjectured that

$$\sum_{k=0}^{p-1}\frac{T_k(3,3)M_k(3,3)}{(-3)^k}\equiv p^3-p^2-3p\pmod{p^4}$$

which is stronger than (3.15). The author's conjectural supercongruences (cf. [S14a, Conjecture 1.1(ii)])

$$\sum_{k=0}^{p-1}M_k^2\equiv(2-6p)\left(\frac{p}{3}\right)\pmod{p^2},\quad\sum_{k=0}^{p-1}kM_k^2\equiv(9p-1)\left(\frac{p}{3}\right)\pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}T_kM_k\equiv\frac{4}{3}\left(\frac{p}{3}\right)+\frac{p}{6}\left(1-9\left(\frac{p}{3}\right)\right)\pmod{p^2}$$

remain open. We also observe that

$$\sum_{k=0}^{p-1}kT_kM_k\equiv\left(\frac{-1}{p}\right)-\frac{5}{3}\left(\frac{p}{3}\right)\pmod{p}.$$

The Lucas numbers $L_0, L_1, L_2, \ldots$ are given by

$$L_0 = 2,\ L_1 = 1,\ \text{and } L_{n+1} = L_n + L_{n-1} \text{ for } n = 1, 2, 3, \ldots.$$

It is easy to see that $L_n = 2F_{n+1} - F_n = 2F_{n-1} + F_n$ for all $n \in \mathbb{Z}^+$. Thus, for any odd prime $p \neq 5$ we have

$$L_{p-(\frac{p}{5})} = 2F_p - \left(\frac{p}{5}\right) F_{p-(\frac{p}{5})} \equiv 2\left(\frac{p}{5}\right) \pmod p$$

and hence

$$u_{p-(\frac{p}{5})}(3,1) = \frac{(\alpha^2)^{p-(\frac{p}{5})} - (\beta^2)^{p-(\frac{p}{5})}}{\alpha^2 - \beta^2} = F_{p-(\frac{p}{5})} L_{p-(\frac{p}{5})} \equiv 2\left(\frac{p}{5}\right) F_{p-(\frac{p}{5})} \pmod {p^2},$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Note also that

$$u_n(3 \times 5, 5^2) = 5^{n-1} u_n(3,1) = 5^{n-1} F_n L_n \quad \text{for any } n \in \mathbb{N}.$$

Thus Theorem 3.1 with $(b,c) = (1,-1), (5,5)$ leads to the following corollary.

Corollary 3.2. *For any $n \in \mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1} T_k(1,-1)M_k(1,-1)5^{n-1-k} = \sum_{k=1}^{n}(-1)^{k-1}w(n,k)C_{k-1}5^{n-k} \in \mathbb{Z} \tag{3.17}$$

and

$$\frac{1}{n}\sum_{k=0}^{n-1} \frac{T_k(5,5)M_k(5,5)}{5^k} = \sum_{k=1}^{n} w(n,k)C_{k-1} \in \mathbb{Z}. \tag{3.18}$$

Also, for any prime $p \neq 2, 5$ we have the congruences

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{T_k(1,-1)M_k(1,-1)}{5^k} \equiv & \frac{p}{2}\left(1 - \left(\frac{p}{5}\right)\right) + \frac{p^2}{2}\left(5 - \left(\frac{p}{5}\right) - q_p(5)\right) \\ & + \frac{p}{2}\left(5 - 2\left(\frac{p}{5}\right)\right) F_{p-(\frac{p}{5})} \pmod{p^3},\end{aligned} \tag{3.19}$$

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{T_k(5,5)M_k(5,5)}{5^k} \equiv & \frac{5p}{2}\left(\left(\frac{p}{5}\right) - 1\right) + \frac{p^2}{2}\left(5\left(\frac{p}{5}\right) - 1\right) \\ & - \frac{5p}{2}\left(1 + 2\left(\frac{p}{5}\right)\right) F_{p-(\frac{p}{5})} \pmod{p^3},\end{aligned} \tag{3.20}$$

and

$$\left(\frac{-5}{p}\right)\sum_{k=0}^{p-1} \frac{kT_k(1,-1)M_k(1,-1)}{5^k} \equiv \sum_{k=0}^{p-1} \frac{kT_k(5,5)M_k(5,5)}{5^k} \equiv 2 \pmod p. \tag{3.21}$$

4. Two related conjectures

In view of (1.8) and (1.9), we can easily see that

$$\begin{aligned}&W_1(x)=1,\ W_2(x)=2x+1,\ W_3(x)=10x^2+5x+1,\\&W_4(x)=70x^3+42x^2+9x+1.\end{aligned}$$

Applying the Zeilberger algorithm (cf. [PWZ, pp. 101-119]) via `Mathematica 9`, we obtain the following third-order recurrence with $n\in\mathbb{Z}^+$:

$$\begin{aligned}&(n+3)^2(n+4)(2n+3)W_{n+3}(x)\\=&(n+3)(2n+5)(4x(2n+3)^2+3n^2+11n+10)W_{n+2}(x)\\&-(n+1)(2n+3)(4x(2n+5)^2+3n^2+13n+14)W_{n+1}(x)\\&+n(n+1)^2(2n+5)W_n(x).\end{aligned}\tag{4.1}$$

(This is a verified result, not a conjecture.)

For any $n\in\mathbb{Z}^+$, we clearly have $w(n,n)=C_n$. For the polynomial

$$w_n(x):=\sum_{k=1}^n w(n,k)x^{k-1},\tag{4.2}$$

we have the relation

$$w_n(-1-x)=(-1)^{n-1}w_n(x)\tag{4.3}$$

since

$$\sum_{k=m}^n(-1)^{n-k}\binom{k-1}{m-1}w(n,k)=w(n,m)\quad\text{ for all } m=1,\ldots,n,\tag{4.4}$$

which can be deduced with the help of the Chu-Vandermonde identity in the following way:

$$\begin{aligned}&\sum_{k=m}^n(-1)^{n-k}\binom{k-1}{m-1}w(n,k)\\=&\binom{n-1}{m-1}\sum_{k=m}^n\frac{(-1)^{n-k}}{k}\binom{n-m}{n-k}\binom{-n-2}{k-1}(-1)^{k-1}\\=&\frac{(-1)^n}{n+1}\binom{n-1}{m-1}\sum_{k=m}^n\binom{n-m}{n-k}\binom{-n-1}{k}\\=&\frac{(-1)^n}{n+1}\binom{n-1}{m-1}\binom{-m-1}{n}=w(n,m).\end{aligned}$$

Via the Zeilberger algorithm we obtain the recurrence

$$(n+3)w_{n+2}(x)=(2x+1)(2n+3)w_{n+1}(x)-nw_n(x)\quad\text{for } n=1,2,3,\ldots\tag{4.5}$$

As $w_2(x)=2x+1$, this recurrence implies that $w_{2n}(-1/2)=0$ and hence $w_{2n}(x)/(2x+1)\in\mathbb{Z}[x]$ for all $n\in\mathbb{Z}^+$. We also note that

$$\sum_{n=1}^{\infty} w_n(x)y^n=\frac{1-y-2xy-\sqrt{(y-1)^2-4xy}}{2x(x+1)y}, \tag{4.6}$$

while

$$\sum_{n=0}^{\infty} S_n y^n=\frac{1-y-\sqrt{y^2-6y+1}}{2y}.$$

Now we pose two conjectures for further research.

Conjecture 4.1. *For any integer $n>1$, all the polynomials*

$$w_{2n-1}(x),\quad \frac{w_{2n}(x)}{2x+1}\quad \text{and}\quad W_n(x)$$

are irreducible over the field of rational numbers.

Conjecture 4.2. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$f_n(x):=\frac{1}{n}\sum_{k=0}^{n-1} D_k(x)R_k(x)\in\mathbb{Z}[x], \tag{4.7}$$

where

$$R_k(x):=\sum_{l=0}^{k}\binom{k}{l}\binom{k+l}{l}\frac{x^l}{2l-1}=\sum_{l=0}^{k}\binom{k+l}{2l}\binom{2l}{l}\frac{x^l}{2l-1}. \tag{4.8}$$

Also, $f_2(x), f_3(x), \ldots$ are all irreducible over the field of rational numbers, and

$$f_n(1)=\frac{1}{n}\sum_{k=0}^{n-1} D_kR_k\equiv(-1)^n \pmod{32}$$

for each $n\in\mathbb{Z}^+$, where $R_k=R_k(1)$.

(ii) *Let p be any odd prime. Then*

$$\sum_{k=0}^{p-1} D_kR_k\equiv\begin{cases}-p+8p^2q_p(2)-2p^3E_{p-3} \pmod{p^4} & \text{if } p\equiv 1 \pmod 4,\\ -5p \pmod{p^3} & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{4.9}$$

Also,

$$\sum_{k=1}^{p-1}\frac{D_kR_k}{k}\equiv\left(4-\left(\frac{-1}{p}\right)\right)q_p(2) \pmod p, \tag{4.10}$$

$$\sum_{k=1}^{p-1} kD_kR_k\equiv\frac{1}{2}+\frac{3}{2}p\left(1-2\left(\frac{-1}{p}\right)\right) \pmod{p^2}, \tag{4.11}$$

and

$$\sum_{k=1}^{p-1} kD_k(x)R_k(x)\equiv\frac{x^{p-1}}{2} \pmod p.$$

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