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TWO NEW KINDS OF NUMBERS AND RELATED DIVISIBILITY RESULTS

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Abstract. We mainly introduce two new kinds of numbers given by

$$R_n=\sum_{k=0}^n\binom nk\binom{n+k}k\frac1{2k-1}\qquad(n=0,1,2,\ldots)$$

and

$$S_n=\sum_{k=0}^n\binom nk^2\binom{2k}k(2k+1)\qquad(n=0,1,2,\ldots).$$

We find that such numbers have many interesting arithmetic properties. For example, if $p\equiv1\pmod 4$ is a prime with $p=x^2+y^2$ (where $x\equiv1\pmod 4$ and $y\equiv0\pmod 2$), then

$$R_{(p-1)/2}\equiv p-(-1)^{(p-1)/4}2x\pmod{p^2}.$$

Also,

$$\frac1{n^2}\sum_{k=0}^{n-1}S_k\in\mathbb Z\ \text{ and }\ \frac1n\sum_{k=0}^{n-1}S_k(x)\in\mathbb Z[x]\quad\text{for all } n=1,2,3,\ldots,$$

where $S_k(x)=\sum_{j=0}^k\binom kj^2\binom{2j}j(2j+1)x^j$. For any positive integers a and n, we show that, somewhat surprisingly,

$$\frac1{n^2}\sum_{k=0}^{n-1}(2k+1)\binom{n-1}k^a\binom{-n-1}k^a\in\mathbb Z\ \text{ and }\ \frac1n\sum_{k=0}^{n-1}\frac{\binom{n-1}k^a\binom{-n-1}k^a}{4k^2-1}\in\mathbb Z.$$

We also solve a conjecture of V.J.W. Guo and J. Zeng, and pose several conjectures for further research.

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1. Introduction

In combinatorics, the (large) Schröder numbers are given by

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{1}{k+1}\quad (n\in\mathbb{N}),\tag{1.1}$$

where $\mathbb{N}=\{0,1,2,\ldots\}$. They are integers since

$$C_k=\frac{1}{k+1}\binom{2k}{k}=\binom{2k}{k}-\binom{2k}{k+1}\in\mathbb{Z}\quad\text{for all } k\in\mathbb{N}.$$

Those C_n with $n\in\mathbb{N}$ are the well-known Catalan numbers. Both Catalan numbers and Schröder numbers have many combinatorial interpretations. For example, $S(n)$ is the number of lattice paths from the point $(0,0)$ to (n,n) with only allowed steps $(1,0)$, $(0,1)$ and $(1,1)$ which never rise above the line $y=x$.

We note that $(2k-1)\mid\binom{2k}{k}$ for all $k\in\mathbb{N}$. This is obvious for $k=0$. For each $k\in\mathbb{Z}^+=\{1,2,3,\ldots\}$, we have

$$\frac{\binom{2k}{k}}{2k-1}=\frac{2}{2k-1}\binom{2k-1}{k}=\frac{2}{k}\binom{2k-2}{k-1}=2C_{k-1}.$$

Motivated by this and (1.1), we introduce a new kind of numbers:

$$R_n:=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{1}{2k-1}=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{1}{2k-1}\quad (n\in\mathbb{N}).\tag{1.2}$$

Below are the values of $R_0,R_1,\ldots,R_{16}$ respectively:

$$-1,\ 1,\ 7,\ 25,\ 87,\ 329,\ 1359,\ 6001,\ 27759,\ 132689,\ 649815,$$
$$3242377,\ 16421831,\ 84196761,\ 436129183,\ 2278835681,\ 11996748255.$$

Applying the Zeilberger algorithm (cf. [PWZ, pp. 101-119]) via `Mathematica 9`, we get the following third-order recurrence for the new sequence $(R_n)_{n\geqslant 0}$:

$$(n+1)R_n-(7n+15)R_{n+1}+(7n+13)R_{n+2}-(n+3)R_{n+3}=0\quad\text{for } n\in\mathbb{N}.\tag{1.3}$$

In contrast, there is a second-order recurrence for Schröder numbers:

$$nS(n)-3(2n+3)S(n+1)+(n+3)S(n+2)=0\quad (n=0,1,2,\ldots).$$

So the sequence $(R_n)_{n\geqslant 0}$ looks more sophisticated than Schröder numbers.

For convenience, we also introduce the associated polynomials

$$R_n(x):=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{x^k}{2k-1}=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{x^k}{2k-1}\in\mathbb{Z}[x].\tag{1.4}$$

Note that $R_n = R_n(1)$ and $R_n(0) = -1$. Now we list $R_0(x), \ldots, R_5(x)$:

$$R_0(x) = -1,\ R_1(x) = 2x - 1,\ R_2(x) = 2x^2 + 6x - 1,$$
$$R_3(x) = 4x^3 + 10x^2 + 12x - 1,\ R_4(x) = 10x^4 + 28x^3 + 30x^2 + 20x - 1,$$
$$R_5(x) = 28x^5 + 90x^4 + 112x^3 + 70x^2 + 30x - 1.$$

Applying the Zeilberger algorithm via `Mathematica 9`, we get the following third-order recurrence for the polynomial sequence $(R_n(x))_{n\geqslant 0}$:

$$\begin{aligned}(n+1)R_n(x) - (4nx + 10x + 3n + 5)R_{n+1}(x) + (4nx + 6x + 3n + 7)R_{n+2}(x)\\ = (n+3)R_{n+3}(x).\end{aligned} \tag{1.5}$$

Let $p \equiv 1 \pmod 4$ be a prime. It is well-known that p can be written uniquely as a sum of two squares. Write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. In 1828 Gauss (cf. [BEW, (9.0.1)]) proved that

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p;$$

in 1986 Chowla, Dwork and Evans [CDE] showed further that

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

The key motivation to introduce the polynomials $R_n(x)$ ($x \in \mathbb{N}$) is our following result.

Theorem 1.1. (i) *Let $p \equiv 1 \pmod 4$ be a prime, and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then*

$$R_{(p-1)/2} - p \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(2k-1)(-16)^k} \equiv -2\left(\frac{2}{p}\right)x \pmod{p^2}, \tag{1.6}$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol. Also,

$$R_{(p-1)/2}(-2) + 2p\left(\frac{2}{p}\right) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(2k-1)8^k} \equiv \left(\frac{2}{p}\right)\frac{p}{2x} \pmod{p^2}, \tag{1.7}$$

$$R_{(p-1)/2}\left(-\frac{1}{2}\right) + \frac{p}{2}\left(\frac{2}{p}\right) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(2k-1)32^k} \equiv \frac{p}{4x} - x \pmod{p^2}. \tag{1.8}$$

(ii) *Let $p \equiv 3 \pmod 4$ be a prime. Then*

$$R_{(p-1)/2} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(2k-1)(-16)^k} \equiv -\frac{1}{2}\left(\frac{2}{p}\right)\binom{(p+1)/2}{(p+1)/4} \pmod p \tag{1.9}$$

and

$$R_{(p-1)/2}(-2) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(2k-1)8^k} \equiv -\frac{1}{2}\left(\frac{2}{p}\right)\binom{(p+1)/2}{(p+1)/4} \pmod p \tag{1.10}$$

We also have

$$R_{(p-1)/2}\left(-\frac{1}{2}\right) + \frac{p}{2}\left(\frac{2}{p}\right) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(2k-1)32^k} \equiv -\frac{p+1}{2^p+2}\binom{(p+1)/2}{(p+1)/4} \pmod{p^2}. \tag{1.11}$$

Our following theorem is motivated by (1.7).

Theorem 1.2. *Let $p = 2n+1$ be any odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{2k}{k+d}}{(2k-1)8^k} \equiv 0 \pmod p \tag{1.12}$$

for all $d \in \{0, \ldots, n\}$ with $d \equiv n \pmod 2$.

Remark 1.1. In contrast with (1.12), by induction we have

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}\binom{2k}{k+d}}{(2k-1)16^k} = \frac{2n+1}{(4d^2-1)16^n}\binom{2n}{n}\binom{2n}{n+d} \quad \text{for all } d, n \in \mathbb{N}.$$

Below is our third theorem.

Theorem 1.3. (i) *For any odd prime p, we have*

$$\sum_{k=0}^{p-1} R_k \equiv -p - \left(\frac{-1}{p}\right) \pmod{p^2}. \tag{1.13}$$

(ii) *For any positive integer n, we have*

$$R_n(-1) = -(2n+1) \tag{1.14}$$

and consequently

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{-n}{k}}{2k-1} = -2n. \tag{1.15}$$

Remark 1.2. Although there are many known combinatorial identities (cf. [G]), (1.15) seems new and concise.

Now we introduce another kind of new numbers:

$$S_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}(2k+1) \quad (n = 0, 1, 2, \ldots). \tag{1.16}$$

We also define the associated polynomials

$$S_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} (2k+1)x^k \quad (n = 0, 1, 2, \dots). \tag{1.17}$$

Here are the values of $S_0, S_1, \dots, S_{12}$ respectively:

$$1,\ 7,\ 55,\ 465,\ 4047,\ 35673,\ 316521,\ 2819295,\ 25173855,$$
$$225157881,\ 2016242265,\ 18070920255,\ 162071863425.$$

Now we list the polynomials $S_0(x), \dots, S_5(x)$:

$$\begin{aligned}
&S_0(x) = 1,\ S_1(x) = 6x + 1,\ S_2(x) = 30x^2 + 24x + 1,\\
&S_3(x) = 140x^3 + 270x^2 + 54x + 1,\\
&S_4(x) = 630x^4 + 2240x^3 + 1080x^2 + 96x + 1,\\
&S_5(x) = 2772x^5 + 15750x^4 + 14000x^3 + 3000x^2 + 150x + 1.
\end{aligned}$$

Applying the Zeilberger algorithm via `Mathematica 9`, we get the following recurrence for $(S_n)_{n\geqslant 0}$:

$$9(n+1)^2 S_n - (19n^2+74n+87)S_{n+1} + (n+3)(11n+29)S_{n+2} = (n+3)^2 S_{n+3}, \tag{1.18}$$

which looks more complicated than the recurrence relation (1.3) for $(R_n)_{n\geqslant 0}$. Also, the Zeilberger algorithm could yield a very complicated third-order recurrence for the polynomial sequence $(S_n(x))_{n\geqslant 0}$. Despite these complicated recurrences, we are able to establish the following result which looks interesting.

Theorem 1.4. (i) *For any positive integer n, we have*

$$\frac{1}{n^2}\sum_{k=0}^{n-1} S_k = \sum_{k=0}^{n-1} \binom{n-1}{k}^2 C_k \in \mathbb{Z} \tag{1.19}$$

and

$$\frac{1}{n}\sum_{k=0}^{n-1} S_k(x) \in \mathbb{Z}[x]. \tag{1.20}$$

(ii) *For any prime $p > 3$, we have*

$$\sum_{k=1}^{p-1} \frac{S_k}{k} \equiv p\sum_{k=1}^{p-1} \frac{S_k}{k^2} \equiv -\frac{p}{2}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2},$$

where $B_n(x)$ denotes the Bernoulli polynomial of degree n.

In 2012 Guo and Zeng [GZ, Corollary 5.6] employed q-binomial coefficients to prove that for any $a, b \in \mathbb{N}$ and positive integer n we have

$$\sum_{k=0}^{n-1} (-1)^{(a+b)k} \binom{n-1}{k}^a \binom{-n-1}{k}^b \equiv 0 \pmod{n}.$$

(Note that $\binom{-n-1}{k} = (-1)^k\binom{n+k}{k}$.) This, together with (1.15) and Theorem 1.4, led us to obtain the following result via a new method.

Theorem 1.5. (i) *Let* $a_1,\ldots,a_m$ *and* $n>0$ *be integers. Then*

$$\sum_{k=0}^{n-1}(\pm1)^k(2k+1)\prod_{i=1}^m\binom{a_in-1}{k}\equiv 0 \pmod n, \tag{1.21}$$

$$\sum_{k=0}^{n-1}(\pm1)^k(4k^3-1)\prod_{i=1}^m\binom{a_in-1}{k}\equiv 0 \pmod n. \tag{1.22}$$

Also,

$$\gcd(a_1+\cdots+a_m-1,2)\sum_{k=0}^{n-1}(-1)^{km}(2k+1)\prod_{i=1}^m\binom{a_in-1}{k}\equiv 0 \pmod{n^2}, \tag{1.23}$$

and

$$6\sum_{k=0}^{n-1}(-1)^{km}(3k^2+3k+1)\prod_{i=1}^m\binom{a_in-1}{k}\equiv 0 \pmod{n^2}. \tag{1.24}$$

Moreover,

$$\sum_{k=0}^{n-1}(-1)^k(4k^3-1)\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}\equiv 0 \pmod{n^2}, \tag{1.25}$$

and

$$\gcd(a_1+\cdots+a_m-1,2)\sum_{k=0}^{n-1}(3k^2+3k+1)\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k} \equiv 0 \pmod{n^3}. \tag{1.26}$$

(ii) *For any positive integers* a,b,n, *we have*

$$\frac1n\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}^a\binom{-n-1}{k}^a}{4k^2-1}\in\mathbb{Z},\quad \frac1n\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}^a\binom{-n-1}{k}^a}{\binom{k+2}{2}}\in\mathbb{Z}, \tag{1.27}$$

$$\frac1n\sum_{k=0}^{n-1}(-1)^k\left(1+\frac{2k}{4k^2-1}\right)\binom{n-1}{k}^a\binom{-n-1}{k}^a\in\mathbb{Z}, \tag{1.28}$$

$$\frac1n\sum_{k=0}^{n-1}(-1)^k\left(4-\frac{2k+3}{\binom{k+2}{2}}\right)\binom{n-1}{k}^a\binom{-n-1}{k}^a\in\mathbb{Z}, \tag{1.29}$$

and

$$\sum_{k=0}^{n-1}\frac{(-1)^{(a+b)k}}{4k^2-1}\binom{n-1}{k}^a\binom{-n-1}{k}^b\in\mathbb{Z}, \tag{1.30}$$

$$\sum_{k=0}^{n-1}\frac{(-1)^{(a+b-1)k}k}{4k^2-1}\binom{n-1}{k}^a\binom{-n-1}{k}^b\in\mathbb{Z}, \tag{1.31}$$

$$\sum_{k=0}^{n-1}\frac{(-1)^{(a+b)k}}{\binom{k+2}{2}}\binom{n-1}{k}^a\binom{-n-1}{k}^b\in\mathbb{Z}, \tag{1.32}$$

$$\sum_{k=0}^{n-1}\frac{(-1)^{(a+b-1)k}(2k+3)}{\binom{k+2}{2}}\binom{n-1}{k}^a\binom{-n-1}{k}^b\in\mathbb{Z}, \tag{1.33}$$

$$\sum_{k=0}^{n-1}\frac{(-1)^{(a+b)k}(3k+1)}{(2k+1)\binom{2k}{k}}\binom{n-1}{k}^a\binom{-n-1}{k}^b\in\mathbb{Z}, \tag{1.34}$$

$$\sum_{k=0}^{n-1}\frac{(-1)^{(a+b-1)k}(5k+3)}{(2k+1)\binom{2k}{k}}\binom{n-1}{k}^a\binom{-n-1}{k}^b\in\mathbb{Z}. \tag{1.35}$$

Remark 1.3. For any positive integer n, using (1.15) we can deduce that

$$\sum_{k=1}^{n-1}\frac{\binom{n-1}{k}\binom{-n-1}{k}}{4k^2-1}=\frac{1}{2}\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{-n}{k}}{2k-1}=-n.$$

An extension of (1.21) given in (4.5) confirms a conjecture of Guo and Zeng [GZ]. By (1.23), for any positive integers a,b,n we have the congruence

$$\gcd(a+b-1,2)\sum_{k=0}^{n-1}(-1)^{(a+b)k}(2k+1)\binom{n-1}{k}^a\binom{-n-1}{k}^b\equiv 0 \pmod{n^2}. \tag{1.36}$$

Corollary 1.1. *For $n\in\mathbb{N}$ define*

$$t_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\frac{1}{2k-1},$$

$$T_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2(2k+1),$$

$$T_n^+=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2(2k+1)^2,$$

$$T_n^-=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2(-1)^k(2k+1)^2.$$

Then, for any positive integer n, we have

$$\frac{1}{n^3}\sum_{k=0}^{n-1}(2k+1)t_k\in\mathbb{Z},\ \frac{1}{n^3}\sum_{k=0}^{n-1}(2k+1)T_k\in\mathbb{Z},\tag{1.37}$$

and

$$\frac{1}{n^4}\sum_{k=0}^{n-1}(2k+1)T_k^+\in\mathbb{Z},\ \frac{1}{n^3}\sum_{k=0}^{n-1}(2k+1)T_k^-\in\mathbb{Z}.\tag{1.38}$$

We will prove Theorems 1.1-1.3 in the next section. We are going to show Theorem 1.4 and a q-congruence related to (1.21) in Section 3. Section 4 is devoted to our proofs of Theorem 1.5 and Corollary 1.1 and some extensions. In Section 5 we pose several related conjectures for further research.

2. Proofs of Theorems 1.1-1.3

Lemma 2.1. *Let $p=2n+1$ be an odd prime. Then*

$$\begin{aligned}R_n(x)\equiv&\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{2k-1}\left(-\frac{x}{16}\right)^k\\ \equiv&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{2k-1}\left(-\frac{x}{16}\right)^k-p(-x)^{n+1}\pmod{p^2}.\end{aligned}\tag{2.1}$$

Proof. As pointed out in [S11, Lemma 2.2], for each $k=0,\ldots,n$ we have

$$\binom{(p-1)/2+k}{2k}=\frac{\prod_{0<j\leqslant k}(p^2-(2j-1)^2)}{(2k)!4^k}\equiv\frac{\binom{2k}{k}}{(-16)^k}\pmod{p^2}.$$

Recall that $(2k-1)\mid\binom{2k}{k}$ for all $k\in\mathbb{N}$. Therefore,

$$R_n(x)=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{x^k}{2k-1}\equiv\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{2k-1}\left(-\frac{x}{16}\right)^k\pmod{p^2}.$$

Clearly, $p\mid\binom{2k}{k}$ for all $k=n+1,\ldots,p-1$. Also,

$$\begin{aligned}\frac{\binom{p+1}{(p+1)/2}^2}{2\times(p+1)/2-1}\left(-\frac{x}{16}\right)^{(p+1)/2}=&\frac{4p\binom{p-1}{(p-3)/2}^2}{((p-1)/2)^2}\times\frac{(-x)^{(p+1)/2}}{4^{p+1}}\\ \equiv&p(-x)^{(p+1)/2}\pmod{p^2}.\end{aligned}$$

So the second congruence in (2.1) also holds. $\square$

Lemma 2.2. *For any nonnegative integer n, we have*

$$\sum_{k=0}^{n}\left((16-x)k^2-4\right)\frac{\binom{2k}{k}^2}{2k-1}x^{n-k}=\frac{4(n+1)^2}{2n+1}\binom{2n+1}{n}^2. \tag{2.2}$$

Proof. Let $P(x)$ denote the left-hand side of (2.2). Then

$$\begin{aligned}P(x)=&4\sum_{k=0}^{n}(4k^2-1)\frac{\binom{2k}{k}^2}{2k-1}x^{n-k}-\sum_{k=0}^{n}\frac{k^2\binom{2k}{k}^2}{2k-1}x^{n+1-k}\\=&4\sum_{k=0}^{n}(2k+1)\binom{2k}{k}^2x^{n-k}-4\sum_{k=1}^{n}(2k-1)\binom{2(k-1)}{k-1}^2x^{n-(k-1)}\\=&4(2n+1)\binom{2n}{n}^2=\frac{4(n+1)^2}{2n+1}\binom{2n+1}{n+1}^2.\end{aligned}$$

This concludes the proof. $\square$

Proof of Theorem 1.1. Applying Lemma 2.1 with $x=1,-2,-1/2$ we get the first congruence in each of (1.6)-(1.11).

Let p be an odd prime. For any p-adic integer $m\not\equiv 0 \pmod p$, by Lemma 2.2 we have

$$(16-m)\sum_{k=1}^{p-1}\frac{k^2\binom{2k}{k}^2}{(2k-1)m^k}-4\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(2k-1)m^k}\equiv 0 \pmod{p^2}$$

and hence

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(2k-1)m^k}\equiv&(16-m)\sum_{k=1}^{p-1}(2k-1)\frac{\binom{2(k-1)}{k-1}^2}{m^k}\\=&\left(\frac{16}{m}-1\right)\left(\sum_{j=0}^{p-1}(2j+1)\frac{\binom{2j}{j}^2}{m^j}-(2p-1)\frac{\binom{2p-2}{p-1}^2}{m^{p-1}}\right)\\\equiv&\left(\frac{16}{m}-1\right)\sum_{k=0}^{(p-1)/2}(2k+1)\frac{\binom{2k}{k}^2}{m^k} \pmod{p^2}.\end{aligned}$$

(Note that $\binom{2p-2}{p-1}=(2p-2)!/((p-1)!)^2\equiv 0 \pmod p$.) Taking $m=-16,8,32$ we obtain

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(2k-1)(-16)^k}\equiv-2\sum_{k=0}^{(p-1)/2}(2k+1)\frac{\binom{2k}{k}^2}{(-16)^k} \pmod{p^2}, \tag{2.3}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(2k-1)8^k}\equiv\sum_{k=0}^{(p-1)/2}(2k+1)\frac{\binom{2k}{k}^2}{8^k} \pmod{p^2}, \tag{2.4}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(2k-1)32^k}\equiv-\frac{1}{2}\sum_{k=0}^{(p-1)/2}(2k+1)\frac{\binom{2k}{k}^2}{32^k} \pmod{p^2}. \tag{2.5}$$

(i) Recall the condition $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0\pmod 2$. By [Su12a, Theorem 1.2],

$$\left(\frac{2}{p}\right)x\equiv\sum_{k=0}^{(p-1)/2}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2\equiv\sum_{k=0}^{(p-1)/2}\frac{k+1}{8^k}\binom{2k}{k}^2\pmod{p^2}.$$

The author [Su11, Conjecture 5.5] conjectured that

$$\left(\frac{2}{p}\right)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{8^k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{32^k}\equiv 2x-\frac{p}{2x}\pmod{p^2}$$

which was later confirmed by the author's brother Z.-H. Sun [S11], who also showed that

$$\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{32^k}\equiv 0\pmod{p^2}.$$

Combining these with (2.3)-(2.5), we immediately get the second congruences in (1.6)-(1.8).

(ii) Now we consider the case $p\equiv 3\pmod 4$. By [Su13a, Theorem 1.3],

$$\sum_{k=0}^{(p-1)/2}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2\equiv\sum_{k=0}^{(p-1)/2}\frac{2k}{(-16)^k}\binom{2k}{k}^2\equiv\frac14\left(\frac{2}{p}\right)\binom{(p+1)/2}{(p+1)/4}\pmod p$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{2k+1}{8^k}\binom{2k}{k}^2\equiv\sum_{k=0}^{(p-1)/2}\frac{2k}{8^k}\binom{2k}{k}^2\equiv-\frac12\left(\frac{2}{p}\right)\binom{(p+1)/2}{(p+1)/4}\pmod p.$$

Combining this with (2.3) and (2.4), we obtain the second congruences in (1.9) and (1.10).

Z.-H. Sun [S11, Theorem 2.2] confirmed the author's conjectural congruence

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{32^k}\equiv 0\pmod{p^2}.$$

He also showed [S11, Theorem 2.3] that

$$\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{32^k}\equiv\left(\frac{2}{p}\right)\frac{p+1}{4\times 2^{(p-1)/2}}\binom{(p+1)/2}{(p+1)/4}\pmod{p^2}.$$

Observe that

$$\begin{aligned}2^{p-1}+1=&2+\left(\left(\frac{2}{p}\right)2^{(p-1)/2}+1\right)\left(\left(\frac{2}{p}\right)2^{(p-1)/2}-1\right)\\ \equiv&2+2\left(\left(\frac{2}{p}\right)2^{(p-1)/2}-1\right)=2\left(\frac{2}{p}\right)2^{(p-1)/2}\pmod{p^2}.\end{aligned}$$

Therefore,

$$\sum_{k=0}^{(p-1)/2}\frac{2k+1}{32^k}\binom{2k}{k}^2\equiv\frac{p+1}{2^{p-1}+1}\binom{(p+1)/2}{(p+1)/4}\pmod{p^2}.$$

Combining this with (2.5) we obtain the second congruence in (1.11).

The proof of Theorem 1.1 is now complete. □

Proof of Theorem 1.2. Clearly $\binom{2k}{k}/(2k-1)\equiv 0\pmod p$ if $n+1<k<p$. Thus

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{2k}{k+n}}{(2k-1)8^k}\equiv&\sum_{k=n}^{n+1}\frac{\binom{2k}{k}\binom{2k}{k+n}}{(2k-1)8^k}\\ =&\frac{\binom{p-1}{n}}{(2n-1)8^n}+\frac{\binom{p+1}{n+1}\binom{p+1}{p}}{p8^{n+1}}\\ \equiv&\frac{1}{2}(-1)^{n+1}\left(\frac{8}{p}\right)+\frac{2\frac{p}{n}\binom{p-1}{n-1}(p+1)}{p8^{n+1}}\equiv 0\pmod p.\end{aligned}$$

So (1.12) holds for $d=n$.

Define

$$u_m(d)=\sum_{k=0}^{m}\frac{\binom{2k}{k}\binom{2k}{k+d}}{(2k-1)8^k}\quad\text{for } d,m\in\mathbb{N}.$$

Applying the Zeilberger algorithm via `Mathematica 9`, we get the recurrence

$$(2d-1)u_m(d)+(2d+5)u_m(d+2)=(d+1)\frac{\binom{2m}{m}\binom{2m+2}{m+d+2}}{(m+1)8^m}.$$

If $0\leqslant d\leqslant n-2$, then

$$\frac{\binom{2(p-1)}{p-1}\binom{2p}{p+d+1}}{8^{p-1}p}=\frac{\frac{p}{2p-1}\binom{2p-1}{p}\frac{2p}{p+d+1}\binom{2p-1}{p+d}}{8^{p-1}p}\equiv 0\pmod p$$

and hence

$$(2d-1)u_{p-1}(d)\equiv-(2d+5)u_{p-1}(d+2)\pmod p,$$

therefore

$$u_{p-1}(d+2)\equiv 0\pmod p\implies u_{p-1}(d)\equiv 0\pmod p.$$

In view of the above, we have proved the desired result by induction. □

Lemma 2.3. *For any integers $k>0$ and $n\geqslant 0$, we have the identity*

$$(-1)^k\frac{\binom{n}{k}\binom{-n}{k}}{\binom{2k-1}{k}}=\frac{2n}{n+k}\binom{n+k}{2k}=\binom{n+k}{2k}+\binom{n+k-1}{2k}. \tag{2.6}$$

Proof. Observe that

$$\begin{aligned}(-1)^k\binom{n}{k}\binom{-n}{k}=&\binom{n}{k}\binom{n+k-1}{k}=\binom{n}{k}\binom{n+k}{k}\frac{n}{n+k}\\=&\binom{n+k}{2k}\binom{2k}{k}\frac{n}{n+k}=\frac{2n}{n+k}\binom{n+k}{2k}\binom{2k-1}{k}\end{aligned}$$

and

$$\frac{2n}{n+k}\binom{n+k}{2k}=\left(1+\frac{n-k}{n+k}\right)\binom{n+k}{2k}=\binom{n+k}{2k}+\binom{n+k-1}{2k}.$$

So (2.6) follows. $\square$

Proof of Theorem 1.3. (i) It is known that

$$\sum_{n=0}^{m}\binom{n+l}{l}=\binom{l+m+1}{l+1}\quad\text{for all}\quad l,m\in\mathbb{N}$$

(cf. [G, (1.49)]). Thus

$$\begin{aligned}\sum_{n=0}^{p-1}R_n=&\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n+k}{2k}\frac{\binom{2k}{k}}{2k-1}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2k-1}\sum_{n=k}^{p-1}\binom{n+k}{2k}\\=&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2k-1}\binom{p+k}{2k+1}=\sum_{k=0}^{p-1}\frac{p}{(2k+1)(2k-1)}\prod_{0<j\leqslant k}\frac{p^2-j^2}{j^2}\\\equiv&p\sum_{k=0}^{p-1}\frac{(-1)^k}{4k^2-1}=-p+p\sum_{k=1}^{(p-1)/2}\left(\frac{(-1)^k}{4k^2-1}+\frac{(-1)^{p-k}}{4(p-k)^2-1}\right)\\\equiv&-p+p\left(\frac{(-1)^{(p-1)/2}}{4((p-1)/2)^2-1}+\frac{(-1)^{(p+1)/2}}{4((p+1)/2)^2-1}\right)\\\equiv&-p+\left(\frac{-1}{p}\right)\left(\frac{1}{p-2}-\frac{1}{p+2}\right)\equiv-p-\left(\frac{-1}{p}\right)\pmod{p^2}.\end{aligned}$$

(ii) For any positive integer n, clearly

$$\begin{aligned}&R_n(-1)-R_{n-1}(-1)\\=&\sum_{k=0}^{n}\left(\binom{n+k}{2k}-\binom{n-1+k}{2k}\right)\binom{2k}{k}\frac{(-1)^k}{2k-1}\\=&\sum_{k=1}^{n}\binom{n-1+k}{2k-1}(-1)^k2C_{k-1}=-2\sum_{j=0}^{n-1}\binom{n+j}{2j+1}(-1)^jC_j\end{aligned}$$

and hence $R_n(-1) - R_{n-1}(-1) = -2$ with the help of [Su12b, (2.6)]. Thus, by induction, (1.14) holds for all $n \in \mathbb{N}$.

In view of (2.6) and (1.14), for each positive integer n we have

$$\begin{aligned}2\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{-n}{k}}{2k-1} =&\sum_{k=0}^{n}\left(\binom{n+k}{2k}+\binom{n+k-1}{2k}\right)\binom{2k}{k}\frac{(-1)^k}{2k-1}\\ =&\sum_{k=0}^{n}\left(\binom{n}{k}\binom{n+k}{k}+\binom{n-1}{k}\binom{n-1+k}{k}\right)\frac{(-1)^k}{2k-1}\\ =&R_n(-1)+R_{n-1}(-1)=-(2n+1)-(2n-1)=-4n\end{aligned}$$

and hence (1.15) holds.

The proof of Theorem 1.3 is now complete. □

3. Proof of Theorem 1.4 and a q-congruence related to (1.21)

Proof of (1.19). Define

$$h_n := \sum_{k=0}^{n}\binom{n}{k}^2 C_k \quad \text{for } n = 0, 1, 2, \ldots .$$

We want to show that $\sum_{k=0}^{n-1} S_k = n^2 h_{n-1}$ for any positive integer n. This is trivial for $n = 1$. So, it suffices to show that

$$S_n = (n+1)^2 h_n - n^2 h_{n-1} = \sum_{k=0}^{n}((n+1)^2-(n-k)^2)\binom{n}{k}^2 C_k$$

for all $n = 1, 2, 3, \ldots$. Define $v_n = \sum_{k=0}^{n}((n+1)^2-(n-k)^2)\binom{n}{k}^2 C_k$ for $n \in \mathbb{N}$. It is easy to check that $v_n = S_n$ for $n = 0, 1, 2$. Via the Zeilberger algorithm we find the recurrence

$$9(n+1)^2 v_n - (19n^2+74n+87)v_{n+1} + (n+3)(11n+29)v_{n+2} = (n+3)^2 v_{n+3}.$$

This, together with (1.18), implies that $v_n = S_n$ for all $n \in \mathbb{N}$. □

For each integer n we set

$$[n]_q = \frac{1-q^n}{1-q},$$

which is the usual q-analogue of n. For any $n \in \mathbb{Z}$, we define

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{j=0}^{k-1}[n-j]_q}{\prod_{j=1}^{k}[j]_q} \quad \text{for } k = 1, 2, 3, \ldots .$$

Obviously $\lim_{q\to 1}\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. It is easy to see that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \qquad \text{for all } k, n = 1, 2, 3, \ldots .$$

By this recursion, $\begin{bmatrix} n \\ k \end{bmatrix}_q \in \mathbb{Z}[q]$ for all $k, n \in \mathbb{N}$. For any integers a, b and $n > 0$, clearly

$$a \equiv b \pmod{n} \implies [a]_q \equiv [b]_q \pmod{[n]_q}.$$

Let n be a positive integer. The cyclotomic polynomial

$$\Phi_n(q) := \prod_{\substack{a=1 \\ (a,n)=1}}^{n} \left(q - e^{2\pi i a/n}\right) \in \mathbb{Z}[q]$$

is irreducible in the ring $\mathbb{Z}[q]$. It is well-known that

$$q^n - 1 = \prod_{d \mid n} \Phi_d(q).$$

Note that $\Phi_1(q) = q - 1$.

Lemma 3.1 (q-Lucas Theorem (cf. [O]))**.** *Let $a, b, d, s, t \in \mathbb{N}$ with $s < d$ and $t < d$. Then*

$$\begin{bmatrix} ad+s \\ bd+t \end{bmatrix}_q \equiv \binom{a}{b} \begin{bmatrix} s \\ t \end{bmatrix}_q \pmod{\Phi_d(q)}. \tag{3.1}$$

Lemma 3.2. *Let n be a positive integer and let $k \in \mathbb{N}$ with $k < (n-1)/2$. Then*

$$\sum_{h=0}^{n-1} q^h \begin{bmatrix} h \\ k \end{bmatrix}_q^2 \equiv 0 \pmod{\Phi_n(q)}. \tag{3.2}$$

Proof. Note that

$$\sum_{h=0}^{n-1} q^h \begin{bmatrix} h \\ k \end{bmatrix}_q^2 = \sum_{m=0}^{n-1-k} q^{k+m} \begin{bmatrix} k+m \\ m \end{bmatrix}_q^2$$

and

$$\begin{aligned}\begin{bmatrix} k+m \\ m \end{bmatrix}_q =& \prod_{j=1}^{m} \frac{1-q^{k+j}}{1-q^j} = \prod_{j=1}^{m} \left(q^{k+j} \frac{q^{-k-j}-1}{1-q^j} \right) \\ =&(-1)^m q^{km+m(m+1)/2} \prod_{j=1}^{m} \frac{1-q^{-k-j}}{1-q^j} = (-1)^m q^{km+m(m+1)/2} \begin{bmatrix} -k-1 \\ m \end{bmatrix}_q.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{h=0}^{n-1} q^h \begin{bmatrix} h \\ k \end{bmatrix}_q^2 =& \sum_{m=0}^{n-1-k} q^{k+m} q^{2km+m(m+1)} \begin{bmatrix} -k-1 \\ m \end{bmatrix}_q^2 \\ \equiv& q^{-k^2-k-1} \sum_{m=0}^{n-1-k} q^{(k+m+1)^2} \begin{bmatrix} n-k-1 \\ m \end{bmatrix}_q^2 \\ \equiv& q^{-k(k+1)-1} \sum_{m=0}^{n-1-k} q^{(n-k-m-1)^2} \begin{bmatrix} n-k-1 \\ m \end{bmatrix}_q \begin{bmatrix} n-k-1 \\ n-k-1-m \end{bmatrix}_q \\ =& q^{-k(k+1)-1} \begin{bmatrix} 2(n-k-1) \\ n-k-1 \end{bmatrix}_q \pmod{\Phi_n(q)}\end{aligned}$$

with the help of the q-Chu-Vandermonde identity (cf. [AAR, p. 542]). As $2(n-1-k) \geqslant n > n-k-1$, $\left[{2(n-k-1) \atop n-k-1}\right]_q$ is divisible by $\Phi_n(q)$. Therefore (3.2) holds. $\square$

Theorem 3.1. *For any integers $n > k \geqslant 0$, we have*

$$[2k+1]_q \begin{bmatrix} 2k \\ k \end{bmatrix}_q \sum_{h=0}^{n-1} q^h \begin{bmatrix} h \\ k \end{bmatrix}_q^2 \equiv 0 \pmod{[n]_q} \tag{3.3}$$

and hence

$$(2k+1)\binom{2k}{k}\sum_{h=0}^{n-1}\binom{h}{k}^2 \equiv 0 \pmod{n}. \tag{3.4}$$

Proof. Clearly (3.3) with $q \to 1$ yields (3.4), and (3.3) holds trivially in the case $n = 1$ and $k = 0$. Below we only need to prove (3.3) for $n > 1$.

As the polynomials $\Phi_2(q), \Phi_3(q), \ldots$ are pairwise coprime and

$$[n]_q = \prod_{\substack{d \mid n \\ d>1}} \Phi_d(q), \tag{3.5}$$

it suffices to show

$$[2k+1]_q \begin{bmatrix} 2k \\ k \end{bmatrix}_q \sum_{h=0}^{n-1} q^h \begin{bmatrix} h \\ k \end{bmatrix}_q^2 \equiv 0 \pmod{\Phi_d(q)} \tag{3.6}$$

for any divisor $d > 1$ of n. Set $m = n/d$ and write $k = bd + t$ with $b, t \in \mathbb{N}$ and $t < d$. If $t < (d-1)/2$, then by applying Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}\sum_{h=0}^{n-1} q^h \begin{bmatrix} h \\ k \end{bmatrix}_q^2 &= \sum_{a=0}^{m-1}\sum_{s=0}^{d-1} q^{ad+s} \begin{bmatrix} ad+s \\ bd+t \end{bmatrix}_q^2 \\ &\equiv \sum_{a=0}^{m-1}\sum_{s=0}^{d-1} q^s \binom{a}{b}^2 \begin{bmatrix} s \\ t \end{bmatrix}_q^2 = \sum_{a=0}^{m-1}\binom{a}{b}^2 \sum_{s=0}^{d-1} q^s \begin{bmatrix} s \\ t \end{bmatrix}_q^2 \equiv 0 \pmod{\Phi_d(q)}.\end{aligned}$$

If $t = (d-1)/2$, then

$$[2k+1]_q = [2bd+2t+1]_q = [(2b+1)d]_q \equiv 0 \pmod{[d]_q}.$$

When $d/2 \leqslant t < d$, by Lemma 3.1 we have

$$\begin{bmatrix} 2k \\ k \end{bmatrix}_q = \begin{bmatrix} (2b+1)d+2t-d \\ bd+t \end{bmatrix}_q \equiv \binom{2b+1}{b}\begin{bmatrix} 2t-d \\ t \end{bmatrix}_q = 0 \pmod{\Phi_d(q)}.$$

So (3.6) holds, and this completes the proof. $\square$

Proof of (1.20). In light of (3.4),

$$\begin{aligned}\frac{1}{n}\sum_{h=0}^{n-1}S_h(x)=&\frac{1}{n}\sum_{h=0}^{n-1}\sum_{k=0}^{h}\binom{h}{k}^2\binom{2k}{k}(2k+1)x^k\\=&\sum_{k=0}^{n-1}\frac{x^k}{n}(2k+1)\binom{2k}{k}\sum_{h=0}^{n-1}\binom{h}{k}^2\in\mathbb{Z}[x].\end{aligned}$$

This concludes the proof. □

Proof of Theorem 1.4(ii). Let $p>3$ be a prime. By a well-known result of Wolstenholme [W],

$$\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0 \pmod{p^2}\quad\text{and}\quad\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0\pmod p.$$

Clearly,

$$\begin{aligned}\sum_{n=1}^{p-1}\frac{S_n}{n^2}=&\sum_{n=1}^{p-1}\frac{1}{n^2}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}(2k+1)\\\equiv&\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2k+1}{k^2}\sum_{n=k}^{p-1}\binom{n-1}{k-1}^2\\=&\sum_{k=1}^{p-1}\frac{2k+1}{k^3}2(2k-1)\binom{2(k-1)}{k-1}\sum_{h=0}^{p-1}\binom{h}{k-1}^2\\&-\sum_{k=1}^{p-1}\frac{2k+1}{k^2}\binom{2k}{k}\binom{p-1}{k-1}^2\\\equiv&-\sum_{k=1}^{p-1}\frac{2k+1}{k^2}\binom{2k}{k}\pmod p\end{aligned}$$

with the help of Theorem 3.1. Note that

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod{p^2}\quad\text{and}\quad\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}\equiv\frac{1}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)\pmod p$$

by [ST] and [MT] respectively. Therefore,

$$\sum_{n=1}^{p-1}\frac{S_n}{n^2}\equiv-2\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}-\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}\equiv-\frac{1}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)\pmod p.$$

Observe that

$$\begin{aligned}\sum_{n=1}^{p-1}\frac{S_n}{n} &=\sum_{n=1}^{p-1}\frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}(2k+1)\\ &\equiv\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2k+1}{k}\sum_{n=k}^{p-1}\binom{n-1}{k-1}\binom{n}{k}\\ &\equiv\sum_{k=1}^{p-1}\binom{2k}{k}\frac{2k+1}{k}\left(\sum_{n=k}^{p-1+k}\binom{n-1}{k-1}\binom{n}{k}-\binom{p-1}{k-1}\binom{p}{k}\right)\\ &=\sum_{k=1}^{p-1}\frac{2k+1}{k}\binom{2k}{k}\left(\sum_{j=0}^{p-1}\binom{k+j-1}{j}\binom{k+j}{j}-\frac{p}{k}\binom{p-1}{k-1}^2\right)\\ &\equiv\sum_{k=1}^{p-1}\frac{2k+1}{k}\binom{2k}{k}\sum_{j=0}^{p-1}\binom{-k}{j}\binom{-k-1}{j}-\sum_{k=1}^{p-1}\frac{2k+1}{k}\binom{2k}{k}\frac{p}{k}\\ &=\sum_{k=1}^{p-1}\frac{2k+1}{k^2}k\binom{2k}{k}\sum_{j=0}^{p-1}\binom{-k}{j}\binom{-k-1}{j}-p\sum_{k=1}^{p-1}\frac{2k+1}{k^2}\binom{2k}{k} \pmod{p^2}.\end{aligned}$$

By [Su16, Lemma 3.4],

$$k\binom{2k}{k}\sum_{j=0}^{p-1}\binom{-k}{j}\binom{-k-1}{j}\equiv p \pmod{p^2}\quad\text{for all } k=1,\ldots,p-1.$$

So we have

$$\begin{aligned}\sum_{n=1}^{p-1}\frac{S_n}{n} &\equiv p\sum_{k=1}^{p-1}\frac{2k+1}{k^2}-p\sum_{k=1}^{p-1}\frac{2k+1}{k^2}\binom{2k}{k}\\ &\equiv-\frac{p}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2}.\end{aligned}$$

This concludes the proof of Theorem 1.4(ii). □

Now we present a q-congruence related to (1.21).

Theorem 3.2. *Let $a,b\in\mathbb{N}$, and let n be a positive integer. For each $a'\in\{a,a-1\}$, we have*

$$\sum_{k=0}^{n-1}(-1)^{a'k}q^{a'k(k+1)/2-k}[2k+1]_q\begin{bmatrix}n-1\\k\end{bmatrix}_q^a\begin{bmatrix}n+k\\k\end{bmatrix}_q^b\equiv 0 \pmod{[n]_q}. \tag{3.7}$$

Therefore

$$\sum_{k=0}^{n-1}(\pm1)^k(2k+1)\binom{n-1}{k}^a\binom{n+k}{k}^b\equiv 0 \pmod{n}. \tag{3.8}$$

Proof. (3.8) follows from (3.7) with $q\to 1$. Note that (3.7) is trivial for $n=1$.

Below we assume $n>1$ and want to prove (3.7). In view of (3.5), it suffices to show that the left-hand side of (3.7) is divisible by $\Phi_d(q)$ for any divisor $d>1$ of n. Write $n=dm$. By Lemma 3.1,

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^{a'k}q^{a'k(k+1)/2-k}[2k+1]_q\begin{bmatrix}n-1\\ k\end{bmatrix}_q^a\begin{bmatrix}n+k\\ k\end{bmatrix}_q^b\\ =&\sum_{j=0}^{m-1}\sum_{r=0}^{d-1}(-1)^{a'(jd+r)}q^{a'(jd+r)(jd+r+1)/2-(jd+r)}\bigg([2(jd+r)+1]_q\\ &\times\begin{bmatrix}(m-1)d+d-1\\ jd+r\end{bmatrix}_q^a\begin{bmatrix}(m+j)d+r\\ jd+r\end{bmatrix}_q^b\bigg)\\ \equiv&\sum_{j=0}^{m-1}(-1)^{a'jd}q^{a'jd(jd+1)/2}\\ &\times\sum_{r=0}^{d-1}(-1)^{a'r}q^{a'r(r+1)/2-r}[2r+1]_q\binom{m-1}{j}^a\begin{bmatrix}d-1\\ r\end{bmatrix}_q^a\binom{m+j}{j}^b\begin{bmatrix}r\\ r\end{bmatrix}_q^b\\ =&\sum_{j=0}^{m-1}(-1)^{a'jd}q^{a'jd(jd+1)/2}\binom{m-1}{j}^a\binom{m+j}{j}^b\\ &\times\sum_{r=0}^{d-1}(-1)^{a'r}q^{a'r(r+1)/2-r}[2r+1]_q\begin{bmatrix}d-1\\ r\end{bmatrix}_q^a\pmod{\Phi_d(q)}.\end{aligned}$$

For each $r=0,\ldots,d-1$, we have

$$\begin{aligned}\begin{bmatrix}d-1\\ r\end{bmatrix}_q&=\prod_{0<s\leqslant r}\frac{1-q^{d-s}}{1-q^s}=\prod_{0<s\leqslant r}\left(q^{-s}\ \frac{q^s-1+(1-q^d)}{1-q^s}\right)\\ &\equiv(-1)^rq^{-r(r+1)/2}\pmod{\Phi_d(q)}.\end{aligned}$$

So, by the above, it suffices to show that

$$\sum_{r=0}^{d-1}(-1)^{a'r}q^{a'r(r+1)/2-r}[2r+1]_q\left((-1)^rq^{-r(r+1)/2}\right)^a\equiv 0\pmod{\Phi_d(q)}.$$

As $a'\in\{a,a-1\}$, this reduces to

$$\sum_{r=0}^{d-1}q^{-r}[2r+1]_q\equiv 0\equiv\sum_{r=0}^{d-1}(-1)^rq^{-r(r+1)/2-r}[2r+1]_q\pmod{\Phi_d(q)}.\tag{3.9}$$

It is clear that

$$\sum_{r=0}^{d-1}q^{-r}[2r+1]_q=\sum_{r=0}^{d-1}q^{-r}\frac{1-q^{2r+1}}{1-q}\equiv\sum_{r=0}^{d-1}\frac{q^{d-r}-q^{r+1}}{1-q}=0\pmod{\Phi_d(q)}.$$

Also,

$$\begin{aligned}
&\sum_{r=0}^{d-1}(-1)^r q^{-(r^2+3r)/2}\frac{1-q^{2r+1}}{1-q}\\
=&\frac{1}{1-q}\sum_{r=0}^{d-1}(-1)^r\left(q^{-r(r+3)/2}-q^{-(r-2)(r+1)/2}\right)\\
=&\frac{1}{1-q}\left(\sum_{r=0}^{d-1}(-1)^r q^{-r(r+3)/2}-\sum_{r=-2}^{d-3}(-1)^r q^{-r(r+3)/2}\right)\\
=&\frac{1}{1-q}\left((-1)^{d-1}q^{-(d-1)(d+2)/2}+(-1)^{d-2}q^{-(d-2)(d+1)/2}\right)\\
=&\frac{(-1)^{d-1}}{1-q}\left(q^{1-d(d+1)/2}-q^{1-d(d-1)/2}\right)=(-1)^{d-1}q^{1-d(d+1)/2}[d]_q
\end{aligned}$$

and hence the second congruence in (3.9) holds too. This concludes the proof. □

4. Proofs of Theorem 1.5 and Corollary 1.1 and some extensions

Theorem 4.1. *Let $a_1,\dots,a_m\in\mathbb{Z}$ and $b_1,\dots,b_m\in\mathbb{N}$. Let $f:\mathbb{N}\to\mathbb{Z}$ be a function with $k\mid f(k)$ for all $k\in\mathbb{N}$. Let n be a positive integer and set $d=\gcd(a_1,\dots,a_m,b_1,\dots,b_m,n)$. Then we have*

$$\sum_{k=0}^{n-1}\bar f(k)\prod_{i=1}^m\binom{a_i-1}{b_i+k}\equiv 0\pmod d,\tag{4.1}$$

where $\bar f(k)=f(k+1)-(-1)^m f(k)$. If $k^2\mid f(k)$ for all $k\in\mathbb{N}$, then

$$\sum_{k=0}^{n-1}\bar f(k)\prod_{i=1}^m\binom{a_i-1}{b_i+k}\equiv(-1)^m\left(\sum_{i=1}^m a_i\right)\sum_{0<k<n}\frac{f(k)}{k}\prod_{i=1}^m\binom{a_i-1}{b_i+k}\pmod{d^2}.\tag{4.2}$$

Proof. Clearly $f(0)=0$. Observe that

$$\begin{aligned}
&\sum_{k=0}^{n-1}\bar f(k)\prod_{i=1}^m\binom{a_i-1}{b_i+k}\\
=&\sum_{k=0}^{n-1}f(k+1)\prod_{i=1}^m\binom{a_i-1}{b_i+k}-(-1)^m\sum_{k=0}^{n-1}f(k)\prod_{i=1}^m\binom{a_i-1}{b_i+k}\\
=&\sum_{k=1}^{n}f(k)\prod_{i=1}^m\binom{a_i-1}{b_i+k-1}-(-1)^m\sum_{k=0}^{n-1}f(k)\prod_{i=1}^m\binom{a_i-1}{b_i+k}\\
=&f(n)\prod_{i=1}^m\binom{a_i-1}{b_i+n-1}+\sum_{0<k<n}f(k)d_k,
\end{aligned}$$

where

$$d_k := \prod_{i=1}^{m}\left(\binom{a_i}{b_i+k} - \binom{a_i-1}{b_i+k}\right) - (-1)^m \prod_{i=1}^{m}\binom{a_i-1}{b_i+k}$$

can be written as $\sum_{i=1}^m c_{i,k}\binom{a_i}{b_i+k}$ with $c_{i,k}\in\mathbb{Z}$. Since $k \mid f(k)$ and

$$k\binom{a_i}{b_i+k} = a_i\binom{a_i-1}{b_i+k-1} - b_i\binom{a_i}{b_i+k} \equiv 0 \pmod d \tag{4.3}$$

for all $k=1,2,3,\ldots$, we derive (4.1) from the above.

Now we assume $k^2 \mid f(k)$ for all $k\in\mathbb{N}$. For any $0<k<n$, if $1\leqslant i<j\leqslant m$ then

$$f(k)\binom{a_i}{b_i+k}\binom{a_j}{b_j+k} = \frac{f(k)}{k^2}\left(k\binom{a_i}{b_i+k}\right)\left(k\binom{a_j}{b_j+k}\right) \equiv 0 \pmod{d^2},$$

thus we may use (4.3) to deduce that

$$\begin{aligned}
f(k)d_k \equiv& f(k)\sum_{i=1}^m \binom{a_i}{b_i+k}\prod_{j\neq i}\left(-\binom{a_j-1}{b_j+k}\right)\\
=&\frac{f(k)}{k}\sum_{i=1}^m\left(a_i\binom{a_i-1}{b_i+k-1} - b_i\binom{a_i}{b_i+k}\right)(-1)^{m-1}\prod_{j\neq i}\binom{a_j-1}{b_j+k}\\
=&\frac{f(k)}{k^2}\sum_{i=1}^m\left(-ka_i\binom{a_i-1}{b_i+k} + (a_i-b_i)k\binom{a_i}{b_i+k}\right)(-1)^{m-1}\prod_{j\neq i}\binom{a_j-1}{b_j+k}\\
\equiv&\frac{f(k)}{k}(a_1+\cdots+a_m)(-1)^m\prod_{i=1}^m\binom{a_i-1}{b_i+k} \pmod{d^2}.
\end{aligned}$$

Therefore, (4.2) follows. $\square$

Corollary 4.1. *Let $a_1,\ldots,a_m\in\mathbb{Z}$ and $b_1,\ldots,b_m\in\mathbb{N}$. Let n be any positive integer and set $d=\gcd(a_1,\ldots,a_m,b_1,\ldots,b_m,n)$. Then we have*

$$\sum_{k=0}^{n-1}(-1)^{km}\prod_{i=1}^m\binom{a_i-1}{b_i+k} \equiv 0 \pmod d, \tag{4.4}$$

$$\sum_{k=0}^{n-1}(\pm1)^k(2k+1)\prod_{i=1}^m\binom{a_i-1}{b_i+k} \equiv 0 \pmod d, \tag{4.5}$$

$$\sum_{k=0}^{n-1}(\pm1)^k(4k^3-1)\prod_{i=1}^m\binom{a_i-1}{b_i+k} \equiv 0 \pmod d. \tag{4.6}$$

Also,

$$\gcd\left(\frac{a_1+\cdots+a_m}{d}-1,2\right)\sum_{k=0}^{n-1}(-1)^{km}(2k+1)\prod_{i=1}^m\binom{a_i-1}{b_i+k} \equiv 0 \pmod{d^2} \tag{4.7}$$

and

$$6\sum_{k=0}^{n-1}(-1)^{km}(3k^2+3k+1)\prod_{i=1}^{m}\binom{a_i-1}{b_i+k}\equiv 0 \pmod{d^2}. \tag{4.8}$$

Proof. Clearly, $(-1)^{km+m}=(-1)^{(k+1)m}(k+1)-(-1)^m((-1)^{km}k)$,

$$\begin{aligned}(\pm1)^k(2k+1)=&(\pm1)^{(k+1)-1}(k+1)\pm(\pm1)^{k-1}k,\\ =&(\pm1)^{(k+1)-1}(k+1)^2\mp(\pm1)^{k-1}k^2,\end{aligned}$$

and

$$\begin{aligned}(\pm1)^k(4k^3-1)=&(\pm1)^{(k+1)-1}(k+1)^2(2(k+1)-3)\pm(\pm1)^{k-1}k^2(2k-3)\\ =&(-1)^{(k+1)-1}\left((k+1)^2k^2-(k+1)\right)\mp(\pm1)^{k-1}\left(k^2(k-1)^2-k\right).\end{aligned}$$

So (4.4)-(4.6) follow from the first assertion in Theorem 4.1.

Now we prove (4.7). Let $f(k)=(-1)^{km}k^2$ for all $k\in\mathbb{N}$. Then

$$f(k+1)-(-1)^m f(k)=(-1)^{(k+1)m}(2k+1).$$

Applying the second assertion in Theorem 4.1, we get

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^{km+m}(2k+1)\prod_{i=1}^{m}\binom{a_i-1}{b_i+k}\\ \equiv&(-1)^m(a_1+\cdots+a_m)\sum_{k=0}^{n-1}(-1)^{km}k\prod_{i=1}^{m}\binom{a_i-1}{b_i+k}\pmod{d^2}\end{aligned}$$

and hence

$$\begin{aligned}&\gcd\left(\frac{a_1+\cdots+a_m}{d}-1,2\right)\sum_{k=0}^{n-1}(-1)^{km}(2k+1)\prod_{i=1}^{m}\binom{a_i-1}{b_i+k}\\ \equiv&\frac{(a_1+\cdots+a_m)/d}{\gcd((a_1+\cdots+a_m)/d,2)}d\sum_{k=0}^{n-1}(-1)^{km}((2k+1)-1)\prod_{i=1}^{m}\binom{a_i-1}{b_i+k}\pmod{d^2}.\end{aligned}$$

Combining this with (4.4) and (4.5), we immediately obtain the desired (4.7).

It remains to show (4.8). Let $g(k)=(-1)^{km}k^3$ for all $k\in\mathbb{N}$. Then

$$g(k+1)-(-1)^m g(k)=(-1)^{(k+1)m}(3k^2+3k+1).$$

Applying the second assertion in Theorem 4.1, we obtain

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^{km+m}(3k^2+3k+1)\prod_{i=1}^{m}\binom{a_i-1}{b_i+k}\\ \equiv&(-1)^m(a_1+\cdots+a_m)\sum_{k=0}^{n-1}(-1)^{km}k^2\prod_{i=1}^{m}\binom{a_i-1}{b_i+k}\pmod{d^2}\\ \equiv&0\pmod{d}\end{aligned}$$

and hence

$$\begin{aligned}&6\sum_{k=0}^{n-1}(-1)^{km}(3k^2+3k+1)\prod_{i=1}^m\binom{a_i-1}{b_i+k}\\ \equiv&\frac{a_1+\cdots+a_m}{d}d\sum_{k=0}^{n-1}(-1)^{km}(2(3k^2+3k+1)-3(2k+1)+1)\prod_{i=1}^m\binom{a_i-1}{b_i+k}\\ \equiv&0\pmod{d^2}\end{aligned}$$

with the use of (4.4) and (4.5). Thus (4.8) holds.

The proof of Corollary 4.1 is now complete. □

Remark 4.1. (4.4) was first established by Guo and Zeng [GZ, Theorem 5.5] via q-binomial coefficients, while (4.5) was conjectured by them in [GZ, Conjecture 5.8].

Theorem 4.2. *Let $a_1,\ldots,a_m\in\mathbb{Z}$, and let $f:\mathbb{N}\to\mathbb{Z}$ be a function with $k^3\mid f(k)$ for all $k\in\mathbb{N}$. Then, for any positive integer n, we have*

$$\begin{aligned}&\sum_{k=0}^{n-1}\Delta f(k)\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}\\ \equiv&n^2(a_1^2+\cdots+a_m^2)\sum_{0<k<n}\frac{f(k)}{k^2}\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}\pmod{n^3},\end{aligned}\tag{4.9}$$

where $\Delta f(k)=f(k+1)-f(k)$.

Proof. Note that $f(0)=0$ and

$$\begin{aligned}&\sum_{k=0}^{n-1}(f(k+1)-f(k))\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}\\ =&\sum_{k=1}^{n}f(k)\prod_{i=1}^m\binom{a_in-1}{k-1}\binom{-a_in-1}{k-1}-\sum_{k=0}^{n-1}f(k)\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}\\ =&f(n)\prod_{i=1}^m\binom{a_in-1}{n-1}\binom{-a_in-1}{n-1}+\sum_{0<k<n}f(k)d_k(n)-f(0),\end{aligned}$$

where

$$d_k(n):=\prod_{i=1}^m\binom{a_in-1}{k-1}\binom{-a_in-1}{k-1}-\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}.$$

Since

$$\begin{aligned}&\binom{a_in-1}{k}\binom{-a_in-1}{k}-\binom{a_in}{k}\binom{-a_in}{k}\\ =&\frac{a_in-k}{k}\binom{a_in-1}{k-1}\frac{-a_in-k}{k}\binom{-a_in-1}{k-1}-\frac{a_in}{k}\binom{a_in-1}{k-1}\frac{-a_in}{k}\binom{-a_in-1}{k-1}\\ =&\left(\frac{k^2-(a_in)^2}{k^2}+\frac{(a_in)^2}{k^2}\right)\binom{a_in-1}{k-1}\binom{-a_in-1}{k-1}=\binom{a_in-1}{k-1}\binom{-a_in-1}{k-1}\end{aligned}$$

and

$$k^3\binom{a_in}{k}\binom{-a_in}{k}\binom{a_jn}{k} = (a_in)(-a_in)a_jn\binom{a_in-1}{k-1}\binom{-a_in}{k-1}\binom{a_in}{k-1},$$

for $0 < k < n$ we have

$$\begin{aligned}k^3d_k(n) =&k^3\prod_{i=1}^m\left(\binom{a_in-1}{k}\binom{-a_in-1}{k}-\binom{a_in}{k}\binom{-a_in}{k}\right)\\&-k^3\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}\\\equiv&-k^3\sum_{i=1}^m\binom{a_in}{k}\binom{-a_in}{k}\prod_{j\neq i}\binom{a_jn-1}{k}\binom{-a_jn-1}{k}\\=&n^2\sum_{i=1}^m a_i^2k\left(\binom{a_in}{k}-\binom{a_in-1}{k}\right)\left(\binom{-a_in}{k}-\binom{-a_in-1}{k}\right)\\&\times\prod_{j\neq i}\binom{a_jn-1}{k}\binom{-a_jn-1}{k}\\\equiv&n^2(a_1^2+\cdots+a_m^2)k\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}\pmod{n^3}.\end{aligned}$$

Therefore (4.9) follows from the above. □

Lemma 4.1. *For any $k,n\in\mathbb{N}$, we have*

$$\frac{k}{\binom{2k-1}{k}}\binom{n}{k}\binom{-n}{k}\equiv 0\pmod{n}. \tag{4.10}$$

Proof. The assertion holds trivially for $k=0$, below we assume $k>0$. In view of (2.6),

$$(-1)^k\binom{n}{k}\binom{-n}{k}=\binom{2k-1}{k}\frac{2n}{n+k}\binom{n+k}{2k}=\binom{2k-1}{k}\frac{n}{k}\binom{n+k-1}{2k-1}$$

and thus (4.10) follows. □

Theorem 4.3. *Let $a_1,\ldots,a_m$ be positive integers with $\min\{a_1,\ldots,a_m\}=1$, and let f be a function from $\mathbb{N}$ to the field $\mathbb{Q}$ of rational numbers. Let n be any positive integer.*

(i) *If $\binom{2k-1}{k}f(k)\in\mathbb{Z}$ for all $k\in\mathbb{N}$, then we have*

$$\sum_{k=0}^{n-1}\Delta f(k)\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}\in\mathbb{Z}. \tag{4.11}$$

(ii) *If* $\binom{2k-1}{k}f(k)\in k\mathbb{Z}$ *for all* $k\in\mathbb{N}$, *then we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}\Delta f(k)\prod_{i=1}^{m}\binom{a_in-1}{k}\binom{-a_in-1}{k}\in\mathbb{Z}. \tag{4.12}$$

Proof. As in the proof of Theorem 4.2, by Abel's partial summation we have

$$\begin{aligned}&\sum_{k=0}^{n-1}\Delta f(k)\prod_{i=1}^{m}\binom{a_in-1}{k}\binom{-a_in-1}{k}\\ =&f(n)\prod_{i=1}^{m}\binom{a_in-1}{n-1}\binom{-a_in-1}{n-1}+\sum_{0<k<n}f(k)d_k(n)-f(0),\end{aligned}\tag{4.13}$$

where

$$\begin{aligned}d_k(n):=&\prod_{i=1}^{m}\left(\binom{a_in-1}{k}\binom{-a_in-1}{k}-\binom{a_in}{k}\binom{-a_in}{k}\right)\\ &-\prod_{i=1}^{m}\binom{a_in-1}{k}\binom{-a_in-1}{k}\end{aligned}$$

can be written as $\sum_{i=1}^m\binom{a_in}{k}\binom{-a_in}{k}c_{i,k}(n)$ with $c_{i,k}(n)\in\mathbb{Z}$.

(i) By Lemma 2.3, $\binom{2k-1}{k}\mid\binom{a_in}{k}\binom{-a_in}{k}$ for any $i=1,\ldots,m$ and $k=0,\ldots,n$. If $f(k)\binom{2k-1}{k}\in\mathbb{Z}$ for all $k\in\mathbb{N}$, then

$$f(0)\in\mathbb{Z},\quad f(n)\binom{-n-1}{n-1}=f(n)(-1)^{n-1}\binom{2n-1}{n}\in\mathbb{Z},$$

and $f(k)d_k(n)\in\mathbb{Z}$ for all $0<k<n$, thus (4.11) follows (4.13).

(ii) By Lemma 4.1, for any $i=1,\ldots,m$ and $k=0,\ldots,n$ we have

$$\frac{k}{\binom{2k-1}{k}}\binom{a_in}{k}\binom{-a_in}{k}\equiv 0\pmod{n}.$$

If $\binom{2k-1}{k}f(k)\in k\mathbb{Z}$ for all $k\in\mathbb{N}$, then $f(0)=0$,

$$(-1)^{n-1}f(n)\binom{-n-1}{n-1}=f(n)\binom{2n-1}{n}\equiv 0\pmod{n},$$

and $f(k)d_k(n)\equiv 0\pmod{n}$ for all $0<k<n$, therefore (4.12) follows from (4.13).

The proof of Theorem 4.3 is now complete. $\square$

Theorem 4.4. *Let a, b and n be positive integers. For any function $f:\mathbb{N}\to\mathbb{Q}$ with $f(k)\binom{2k-1}{k}\in\mathbb{Z}$ for all $k\in\mathbb{N}$, we have*

$$\sum_{k=0}^{n-1}(f(k+1)-(-1)^{a+b}f(k))\binom{n-1}{k}^a\binom{-n-1}{k}^b\in\mathbb{Z}. \tag{4.14}$$

Proof. Clearly Theorem 4.3(i) implies (4.14) in the case $a=b$. To handle the general case, we need some new ideas.

By Abel's partial summation,

$$\begin{aligned}&\sum_{k=0}^{n-1}(f(k+1)-(-1)^{a+b}f(k))\binom{n-1}{k}^a\binom{-n-1}{k}^b\\ =&\sum_{k=1}^{n}f(k)\binom{n-1}{k-1}^a\binom{-n-1}{k-1}^b-(-1)^{a+b}\sum_{k=0}^{n-1}f(k)\binom{n-1}{k}^a\binom{-n-1}{k}^b\\ =&f(n)\binom{-n-1}{n-1}^b+\sum_{k=0}^{n-1}f(k)\left(\binom{n}{k}-\binom{n-1}{k}\right)^a\left(\binom{-n}{k}-\binom{-n-1}{k}\right)^b\\ &-(-1)^{a+b}\sum_{k=0}^{n-1}f(k)\binom{n-1}{k}^a\binom{-n-1}{k}^b.\end{aligned}$$

Note that $\binom{-n-1}{n-1}=(-1)^{n-1}\binom{2n-1}{n}$. For each $k=0,\ldots,n-1$, we have $\binom{2k-1}{k}\mid\binom{n}{k}\binom{-n}{k}$ by (2.6), and

$$\begin{aligned}\binom{\pm n}{k}\binom{\mp n-1}{k}=&(-1)^k\binom{\pm n}{k}\binom{\pm n+k}{k}\\ =&(-1)^k\binom{\pm n+k}{2k}\binom{2k}{k}=(-1)^k2\binom{\pm n+k}{2k}\binom{2k-1}{k},\end{aligned}$$

therefore

$$\left(\binom{n}{k}-\binom{n-1}{k}\right)^a\left(\binom{-n}{k}-\binom{-n-1}{k}\right)^b-(-1)^{a+b}\binom{n-1}{k}^a\binom{n-1}{k}^b$$

is divisible by $\binom{2k-1}{k}$. As $f(k)\binom{2k-1}{k}\in\mathbb{Z}$ for all $k=0,\ldots,n$, combining the above we obtain (4.14). $\square$

Proof of Theorem 1.5. (i) (1.21)-(1.24) are special cases of (4.5)-(4.8) respectively. For the function $f(k)=(-1)^{k-1}k^2(2k-3)$, we clearly have $\Delta f(k)=(-1)^k(4k^3-1)$ for all $k\in\mathbb{N}$. So, (1.25) follows from the last part of Theorem 4.1. As $3k^2+3k+1=(k+1)^3-k^3$, Theorem 4.2 implies that

$$\begin{aligned}&\sum_{k=0}^{n-1}(3k^2+3k+1)\prod_{i=1}^{m}\binom{a_in-1}{k}\binom{-a_in-1}{k}\\ \equiv&n^2(a_1^2+\ldots+a_n^2)\sum_{k=0}^{n-1}k\prod_{i=1}^{m}\binom{a_in-1}{k}\binom{-a_in-1}{k}\pmod{n^3}.\end{aligned}$$

By Corollary 4.1,

$$\sum_{k=0}^{n-1}((2k+1)-1)\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}\equiv 0\pmod{n}.$$

Therefore

$$\begin{aligned}&\gcd(a_1+\cdots+a_m-1,2)\sum_{k=0}^{n-1}(3k^2+3k+1)\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}\\ \equiv&n^2\frac{a_1^2+\cdots a_m^2}{\gcd(a_1+\cdots+a_m,2)}\sum_{k=0}^{n-1}((2k+1)-1)\prod_{i=1}^m\binom{a_in-1}{k}\binom{-a_in-1}{k}\\ \equiv&0\pmod{n^3}.\end{aligned}$$

This proves (1.26).

(ii) Now let a,b,n be positive integers. Note that

$$\frac{2}{k+1}\binom{2k-1}{k}=\frac{\binom{2k}{k}}{k+1}=C_k\quad\text{and}\quad\frac{\binom{2k-1}{k}}{2k-1}=\begin{cases}C_{k-1}&\text{if }k>0,\\-1&\text{if }k=0.\end{cases}$$

For $k\in\mathbb{N}$, define

$$f_1(k)=\frac{k}{2k-1},\ f_2(k)=\frac{(-1)^kk}{2k-1},\ f_3(x)=\frac{2k}{k+1},\ f_4(x)=\frac{(-1)^k2k}{k+1}.$$

Then $f_i(k)\binom{2k-1}{k}\in k\mathbb{Z}$ for all $i=1,\ldots,4$. Clearly,

$$\begin{aligned}\Delta f_1(k)=&\frac{k+1}{2k+1}-\frac{k}{2k-1}=-\frac{1}{4k^2-1},\\ \Delta f_2(k)=&\frac{(-1)^{k+1}(k+1)}{2k+1}-\frac{(-1)^kk}{2k-1}=(-1)^{k-1}\left(1+\frac{2k}{4k^2-1}\right),\\ \Delta f_3(k)=&\frac{2(k+1)}{k+2}-\frac{2k}{k+1}=\frac{1}{\binom{k+2}{2}},\\ \Delta f_4(k)=&\frac{(-1)^{k+1}2(k+1)}{k+2}-\frac{(-1)^k2k}{k+1}=(-1)^{k-1}\left(4-\frac{2k+3}{\binom{k+2}{2}}\right).\end{aligned}$$

Applying Theorem 4.3(ii) with $f=f_1,\ldots,f_4$, we immediately get (1.27)-(1.29).

Write $m=a+b$. For $k\in\mathbb{N}$, define

$$\begin{aligned}f_5(k)=\frac{(-1)^{km}}{2k-1},\ f_6(k)=\frac{(-1)^{k(m-1)}}{2k-1},\ f_7(k)=\frac{(-1)^{km}2}{k+1},\\ f_8(k)=\frac{(-1)^{k(m-1)}2}{k+1},\ f_9(k)=\frac{(-1)^{km}}{\binom{2k-1}{k}},\ f_{10}(k)=\frac{(-1)^{k(m-1)}}{\binom{2k-1}{k}}.\end{aligned}$$

Then $f_i(k)\binom{2k-1}{k} \in \mathbb{Z}$ for all $i = 5, \ldots, 10$. Let $\bar{f}_i(k) = f_i(k+1) - (-1)^m f_i(k)$ for $i = 5, \ldots, 10$. Observe that

$$\begin{aligned}
\bar{f}_5(k) =& \frac{(-1)^{(k+1)m}}{2k+1} - (-1)^m \frac{(-1)^{km}}{2k-1} = (-1)^{(k-1)m} \frac{-2}{4k^2-1},\\
\bar{f}_6(k) =& \frac{(-1)^{(k+1)(m-1)}}{2k+1} - (-1)^m \frac{(-1)^{k(m-1)}}{2k-1} = (-1)^{(k-1)(m-1)} \frac{4k}{4k^2-1},\\
\bar{f}_7(k) =& \frac{(-1)^{(k+1)m}2}{k+2} - (-1)^m \frac{(-1)^{km}2}{k+1} = (-1)^{(k-1)m} \frac{-1}{\binom{k+2}{2}},\\
\bar{f}_8(k) =& \frac{(-1)^{(k+1)(m-1)}2}{k+2} - (-1)^m \frac{(-1)^{k(m-1)}2}{k+1} = (-1)^{(k-1)(m-1)} \frac{2k+3}{\binom{k+2}{2}},\\
\bar{f}_9(k) =& \frac{(-1)^{(k+1)m}}{\binom{2k+1}{k+1}} - (-1)^m \frac{(-1)^{km}}{\binom{2k-1}{k}} = (-1)^{(k-1)m} \frac{-(3k+1)}{(2k+1)\binom{2k}{k}},
\end{aligned}$$

and

$$\bar{f}_{10}(k) = \frac{(-1)^{(k+1)(m-1)}}{\binom{2k+1}{k+1}} - (-1)^m \frac{(-1)^{k(m-1)}}{\binom{2k-1}{k}} = \frac{(-1)^{(k-1)(m-1)}(5k+3)}{(2k+1)\binom{2k}{k}}.$$

Theorem 4.4 with $f = f_5, \ldots, f_{10}$ clearly yields (1.30)-(1.35).

The proof of Theorem 1.5 is now complete. $\square$

Lemma 4.2. *Let $a_0, a_1, \ldots$ be a sequence of complex numbers, and define*

$$\tilde{a}_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 a_k \quad \text{for } n \in \mathbb{N}. \tag{4.15}$$

Then, for any positive integer n, we have

$$\frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1)\tilde{a}_k = \sum_{k=0}^{n-1} \frac{a_k}{2k+1} \binom{n-1}{k}^2 \binom{n+k}{k}^2. \tag{4.16}$$

Proof. By [Su12b, Lemma 2.1],

$$\sum_{m=0}^{n-1} (2m+1)\binom{m+k}{2k}^2 = \frac{(n-k)^2}{2k+1}\binom{n+k}{2k}^2 \quad \text{for all } k \in \mathbb{N}.$$

Thus

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)\tilde{a}_m &= \sum_{m=0}^{n-1}(2m+1)\sum_{k=0}^{m}\binom{m+k}{2k}^2\binom{2k}{k}^2 a_k \\ &= \sum_{k=0}^{n-1}\binom{2k}{k}^2 a_k \sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}^2 \\ &= \sum_{k=0}^{n-1}\binom{2k}{k}^2 \frac{a_k}{2k+1}(n-k)^2\binom{n+k}{2k}^2 \\ &= \sum_{k=0}^{n-1}\frac{a_k}{2k+1}(n-k)^2\binom{n}{k}^2\binom{n+k}{k}^2 \\ &= n^2\sum_{k=0}^{n-1}\frac{a_k}{2k+1}\binom{n-1}{k}^2\binom{n+k}{k}^2.\end{aligned}$$

This proves (4.16). □

Proof of Corollary 1.1. By Lemma 4.2 and (1.27), we have

$$\frac{1}{n^3}\sum_{k=0}^{n-1}(2k+1)t_k = \frac{1}{n}\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}^2\binom{n+k}{k}^2}{4k^2-1} \in \mathbb{Z}.$$

In light of Lemma 4.2,

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)T_k = \sum_{k=0}^{n-1}\binom{n-1}{k}^2\binom{n+k}{k}^2.$$

By [GZ, (1.9)] or (4.4),

$$\sum_{k=0}^{n-1}\binom{n-1}{k}^2\binom{n+k}{k}^2 \equiv 0 \pmod{n}.$$

So we have $\sum_{k=0}^{n-1}(2k+1)T_k \equiv 0 \pmod{n^3}$. By Lemma 4.2 and (1.36) and (1.21),

$$\frac{1}{n^4}\sum_{k=0}^{n-1}(2k+1)T_k^+ = \frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)\binom{n-1}{k}^2\binom{n+k}{k}^2 \in \mathbb{Z}$$

and

$$\frac{1}{n^3}\sum_{k=0}^{n-1}(2k+1)T_k^- = \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2k+1)\binom{n-1}{k}^2\binom{n+k}{k}^2 \in \mathbb{Z}.$$

Therefore both (1.37) and (1.38) hold. This concludes the proof. □

5. Some related conjectures

Conjecture 5.1. *Let $p \equiv 3 \pmod 4$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(2k-1)8^k} \equiv -\left(\frac{2}{p}\right) \frac{p+1}{2^{p-1}+1} \binom{(p+1)/2}{(p+1)/4} \pmod{p^2} \tag{5.1}$$

and

$$3\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{2k}{k+1}}{(2k-1)8^k} \equiv p + \left(\frac{2}{p}\right) \frac{2p}{\binom{(p+1)/2}{(p+1)/4}} \pmod{p^2}. \tag{5.2}$$

Conjecture 5.2. (i) *The sequence $(R_{n+1}/R_n)_{n\geqslant 3}$ is strictly increasing to the limit $3+2\sqrt{2}$, and the sequence $(\sqrt[n+1]{R_{n+1}}/\sqrt[n]{R_n})_{n\geqslant 5}$ is strictly decreasing.*

(ii) *The sequence $(S_{n+1}/S_n)_{n\geqslant 3}$ is strictly increasing to the limit 9, and the sequence $(\sqrt[n+1]{S_{n+1}}/\sqrt[n]{S_n})_{n\geqslant 1}$ is strictly decreasing.*

Remark 5.1. The author [Su13b] made many similar conjectures for some well-known integer sequences.

Conjecture 5.3. *For any positive integer n, both $R_n(x)$ and $S_n(x)$ are irreducible over the field of rational numbers.*

Conjecture 5.4. *For any $n \in \mathbb{Z}^+$, the number $\frac{3}{n}\sum_{k=0}^{n-1} R_k^2$ is always an odd integer; moreover,*

$$\frac{3}{n}\sum_{k=0}^{n-1} R_k(x)^2 \in \mathbb{Z}[x] \quad \text{and} \quad \frac{1}{n}\sum_{k=0}^{n-1}(2k+1)R_k^2 \in \mathbb{Z}. \tag{5.3}$$

Also, for any odd prime p we have

$$\sum_{k=0}^{p-1} R_k^2 \equiv \frac{p}{3}\left(11 - 4\left(\frac{-1}{p}\right)\right) \pmod{p^2} \tag{5.4}$$

and

$$\sum_{k=0}^{p-1}(2k+1)R_k^2 \equiv 4p\left(\frac{-1}{p}\right) - p^2 \pmod{p^3}. \tag{5.5}$$

Remark 5.2. For any positive integer n, we can easily deduce that

$$\frac{3}{n}\sum_{k=0}^{n-1}(2k+1)R_k(x) = \sum_{k=0}^{n-1}(n-k)\binom{n+k}{2k}\binom{2k}{k}\left(\frac{2}{2k-1} - \frac{1}{k+1}\right)x^k \in \mathbb{Z}[x]. \tag{5.6}$$

Conjecture 5.5. *We have*

$$\frac{4}{n^2}\sum_{k=0}^{n-1} kS_k \in \mathbb{Z} \quad \textit{for all } n = 1, 2, 3, \ldots. \tag{5.7}$$

Also, for any prime p we have

$$\sum_{k=0}^{p-1} kS_k \equiv \frac{p^2}{8}\left(5 - 9\left(\frac{p}{3}\right)\right) \pmod{p^3}. \tag{5.8}$$

Conjecture 5.6. *For $n \in \mathbb{N}$ define*

$$\begin{aligned} s_n &:= \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\frac{1}{2k-1},\\ S_n^+ &:= \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}(2k+1)^2,\\ S_n^- &:= \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}(2k+1)^2(-1)^k. \end{aligned}$$

Then, for any positive integer n, we have

$$\frac{1}{n^2}\sum_{k=0}^{n-1} s_k \in \mathbb{Z}, \quad \frac{1}{n^2}\sum_{k=0}^{n-1} S_k^+ \in \mathbb{Z} \quad \textit{and} \quad \frac{1}{n^2}\sum_{k=0}^{n-1} S_k^- \in \mathbb{Z}. \tag{5.9}$$

Remark 5.3. For any positive integer n, we can easily deduce $\sum_{k=0}^{n-1} S_k^{\pm} \equiv 0 \pmod{n}$ with the help of (3.4). We also conjecture that $\sum_{k=0}^{p-1} s_k \equiv -(9(\frac{p}{3})+1)p^2/2 \pmod{p^3}$ for any prime p.

Conjecture 5.7. *For $n \in \mathbb{N}$ define*

$$s_n(q) := \sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q^2\begin{bmatrix} 2k \\ k\end{bmatrix}_q\frac{q^k}{[2k-1]_q}.$$

Then, for any positive integer n, we have

$$\frac{1+q}{2}\sum_{k=0}^{n-1} q^k s_k(q) \equiv 0 \pmod{[n]_q^2}. \tag{5.10}$$

Remark 5.4. (5.10) is a q-analogue of the conjectural congruence $\sum_{k=0}^{n-1} s_k \equiv 0 \pmod{n^2}$. We could prove (5.10) modulo $[n]_q$.

Conjecture 5.8. *Let m be any positive integer.*

(i) *Define*

$$S_n^{(m)}(x) := \sum_{k=0}^n \binom{n}{k}^m \frac{(km+1)!}{(k!)^m} x^k \quad \textit{for} \;\; n = 0, 1, 2, \ldots .$$

Then, for any positive integer n, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} S_k^{(m)}(x) \in \mathbb{Z}[x], \tag{5.11}$$

i.e.,

$$\frac{(km+1)!}{(k!)^m} \sum_{h=k}^{n-1} \binom{h}{k}^m \equiv 0 \pmod{n} \quad \textit{for all} \;\; k = 0, \ldots, n-1. \tag{5.12}$$

(ii) *Define*

$$S_n^{(m)}(x;q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q^m \frac{\prod_{j=1}^{km+1} [j]_q}{(\prod_{0<j\leqslant k} [j]_q)^m} x^k \quad \textit{for} \;\; n = 0, 1, 2, \ldots . \tag{5.13}$$

Then, for any integer $n > 0$, all the coefficients of the polynomial $\sum_{k=0}^{n-1} q^k S_k^{(m)}(x;q)$ in x are divisible by $[n]_q$ in the ring $\mathbb{Z}[q]$, i.e.,

$$\frac{\prod_{j=1}^{km+1} [j]_q}{(\prod_{0<j\leqslant k} [j]_q)^m} \sum_{h=k}^{n-1} q^h \begin{bmatrix} h \\ k \end{bmatrix}_q^m \equiv 0 \pmod{[n]_q} \quad \textit{for all } k = 0, \ldots, n-1. \tag{5.14}$$

Remark 5.5. (a) Note that $S_n^{(2)}(x) = S_n(x)$, and (5.11) and (5.12) are extensions of (1.20) and (3.4) respectively. Part (ii) of Conjecture 5.8 presents a q-analogue of the first part, and our Theorem 3.1 confirms it for $m = 2$. Conjecture 5.8 for $m = 1$ is easy, and we are also able to prove Conjecture 5.8 in the case $m = 3$.

(b) The congruence in (5.12) for $k = 1$ states that

$$(m+1)! \sum_{h=1}^{n-1} h^m \equiv 0 \pmod{n}.$$

This is easy since

$$\frac{1}{n} \sum_{h=0}^{n-1} h^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m-k}$$

(cf. [IR, p. 230]) and $(k+1)! B_k \in \mathbb{Z}$ by the von Staudt-Clausen theorem (cf. [IR, p. 233]).

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References

[AAR] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.

[BEW] B. C. Berndt, R. J. Evans and K. S. Williams, *Gauss and Jacobi Sums*, John Wiley & Sons, 1998.

[CDE] S. Chowla, B. Dwork and R. J. Evans, *On the mod p^2 determination of $\binom{(p-1)/2}{(p-1)/4}$*, J. Number Theory **24** (1986), 188–196.

[G] H.W. Gould, Combinatorial Identities, Morgantown Printing and Binding Co., 1972.

[GZ] V.J.W. Guo and J. Zeng, *New congruences for sums involving Apéry numbers or central Delannoy numbers*, Int. J. Number Theory **8** (2012), 2003–2016.

[IR] K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory (Graduate Texts in Math.; 84), 2nd ed., Springer, New York, 1990.

[MT] S. Mattarei and R. Tauraso, *Congruences for central binomial sums and finite polylogarithms*, J. Number Theory **133** (2013), 131–157.

[O] G. Olive, *Generalized powers*, Amer. Math. Monthly **72** (1965), 619–627.

[PWZ] M. Petkovšek, H. S. Wilf and D. Zeilberger, $A = B$, A K Peters, Wellesley, 1996.

[S11] Z.-H. Sun, *Congruences concerning Legendre polynomials*, Proc. Amer. Math. Soc. **139** (2011), 1915–1929.

[Su11] Z.-W. Sun, *On congruences related to central binomial coefficients*, J. Number Theory **131** (2011), 2219–2238.

[Su12a] Z.-W. Sun, *On sums involving products of three binomial coefficients*, Acta Arith. **156** (2012), 123–141.

[Su12b] Z.-W. Sun, *On sums of Apéry polynomials and related congruences*, J. Number Theory **132** (2012), 2673–2699.

[Su13a] Z.-W. Sun, *Supercongruences involving products of two binomial coefficients*, Finite Fields Appl. **22** (2013), 24–44.

[Su13b] Z.-W. Sun, *Conjectures involving arithmetical sequences*, in: Number Theory: Arithmetic in Shangri-La (Shanghai, 2011), S. Kanemitsu et al. (eds.), World Sci., Hackensack, NJ, 2013, pp. 244-258.

[Su16] Z.-W. Sun, *Congruences involving $g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k$*, Ramanujan J. **40** (2016), 511–533.

[ST] Z.-W. Sun and R. Tauraso, *New congruences for central binomial coefficients*, Adv. Appl. Math. **45** (2010), 125–148.

[W] J. Wolstenholme, *On certain properties of prime numbers*, Quart. J. Appl. Math. **5** (1862), 35–39.