

HANKEL-TYPE DETERMINANTS FOR SOME COMBINATORIAL SEQUENCES

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ABSTRACT. In this paper we confirm several conjectures of Sun on Hankel-type determinants for some combinatorial sequences including Franel numbers, Domb numbers and Apéry numbers. For any nonnegative integer n , define

$$f_n := \sum_{k=0}^n \binom{n}{k}^3, \quad D_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k},$$

$$b_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}, \quad A_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2.$$

For $n = 0, 1, 2, \dots$, we show that $6^{-n}|f_{i+j}|_{0 \leq i, j \leq n}$ and $12^{-n}|D_{i+j}|_{0 \leq i, j \leq n}$ are positive odd integers, and $10^{-n}|b_{i+j}|_{0 \leq i, j \leq n}$ and $24^{-n}|A_{i+j}|_{0 \leq i, j \leq n}$ are always integers.

1. INTRODUCTION

For a sequence $(a_n)_{n \geq 0}$ of complex numbers, its *Hankel matrix* is given by

$$H = [a_{i+j}]_{i, j \geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \\ a_2 & a_3 & a_4 & a_5 & \\ a_3 & a_4 & a_5 & a_6 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Hankel matrices are related to orthogonal polynomials, moment sequences and continued fractions, and they have been extensively studied in many branches of mathematics (see, e.g., [10, 16]). The Hankel-type determinants for the sequence a_0, a_1, a_2, \dots are those determinants $|a_{i+j}|_{0 \leq i, j \leq n}$ with $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, which are sometimes called Turánian determinants (cf. Karlin and Szegő [10]). For evaluations of Hankel-type determinants, LU decomposition, continued fractions and Dodgson condensation are some of the available tools that have been used with considerable success. See

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Krattenthaler [11, 12] for a wide range of techniques used to evaluate certain Hankel-type determinants.

In this paper we study positivity and divisibility properties of certain Hankel-type determinants for some well-known combinatorial sequences, and confirm several conjectures of Sun [21].

Recall that the Franel numbers are defined by

$$f_n := \sum_{k=0}^n \binom{n}{k}^3 \quad (n = 0, 1, 2, \dots).$$

For $r = 3, 4, 5, \dots$, the r -th order Franel numbers are given by

$$f_n^{(r)} := \sum_{k=0}^n \binom{n}{k}^r \quad (n = 0, 1, 2, \dots).$$

In 2013 Sun [19, 20] proved some fundamental congruences involving Franel numbers, for example, he showed that for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} (-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p}, \quad \sum_{k=0}^{p-1} \frac{(-1)^k}{k} f_k \equiv 0 \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \begin{cases} 2x - p/(2x) \pmod{p^2} & \text{if } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}) \text{ with } 3 \mid x - 1, \\ 3p/((p+1)/6) \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Our first theorem is about Hankel-type determinants for Franel numbers or generalized Franel numbers.

Theorem 1.1. *Let $n \in \mathbb{N}$ and $r \in \{3, 4, \dots\}$. Then $2^{-n} |f_{i+j}^{(r)}|_{0 \leq i, j \leq n}$ is an odd integer. Furthermore, $6^{-n} |f_{i+j}|_{0 \leq i, j \leq n}$ is a positive odd integer.*

The Domb numbers defined by

$$D_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} \quad (n \in \mathbb{N})$$

have various combinatorial interpretations, for example, D_n is the number of $2n$ -step polygons on the diamond lattice. The so-called Catalan-Larcombe-French numbers are given by

$$P_n = \sum_{k=0}^n \frac{\binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2}{\binom{n}{k}} \quad (n \in \mathbb{N}).$$

Both Domb numbers and Catalan-Larcombe-French numbers are related to Ramanujan-type series for $1/\pi$ (cf. [4, 5]).

Our second result is about Hankel-type determinants for Domb numbers as well as Catalan-Larcombe-French numbers.

Theorem 1.2. *For $n \in \mathbb{N}$, both $12^{-n}|D_{i+j}|_{0 \leq i, j \leq n}$ and $2^{-n(n+3)}|P_{i+j}|_{0 \leq i, j \leq n}$ are positive odd integers.*

The integers

$$b_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \quad (n = 0, 1, 2, \dots)$$

and

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \quad (n = 0, 1, 2, \dots)$$

are two kinds of Apéry numbers. They were first introduced by R. Apéry [1] in his proofs of the irrationality of $\zeta(2) = \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ and $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$. Such numbers are also related to Ramanujan-type series for $1/\pi$ (cf. [5]). For congruences involving Apéry numbers A_n ($n \in \mathbb{N}$), see [18].

Now we state our last theorem.

Theorem 1.3. *For $n \in \mathbb{N}$, both $10^{-n}|b_{i+j}|_{0 \leq i, j \leq n}$ and $24^{-n}|A_{i+j}|_{0 \leq i, j \leq n}$ are always integers.*

Theorems 1.1-1.3 were originally conjectured by the second author [21]. We are not able to prove Sun's conjecture ([21]) that both $|b_{i+j}|_{0 \leq i, j \leq n}$ and $|A_{i+j}|_{0 \leq i, j \leq n}$ are always positive.

We will show Theorems 1.1-1.3 in Sections 2-4 respectively.

2. PROOF OF THEOREM 1.1

The binomial transformation of a sequence $(x_k)_{k \geq 0}$ of numbers is the sequence $(x'_n)_{n \geq 0}$ with

$$x'_n := \sum_{k=0}^n \binom{n}{k} x_k. \quad (2.1)$$

This often rises in combinatorics.

The following basic result is well-known and we will use it frequently.

Lemma 2.1. [13, Theorem 1] *Let $(x_k)_{k \geq 0}$ be a sequence of numbers. For any $n \in \mathbb{N}$, we have*

$$|x_{i+j}|_{0 \leq i, j \leq n} = |x'_{i+j}|_{0 \leq i, j \leq n}. \quad (2.2)$$

Let $A = [a_{n,k}]_{n,k \geq 0}$ be a matrix of real numbers. It is called *totally positive* (TP for short) if all its minors are nonnegative. Total positivity of matrices plays an important role in various branches of mathematics such as statistics, probability, mechanics, economics, and computer science (see, e.g., [14, 9]). A sequence $(a_k)_{k \geq 0}$ of numbers is called a *Stieltjes moment*

sequence if its Hankel matrix H is TP. It is well known that $(a_k)_{k \geq 0}$ is a Stieltjes moment sequence if and only if we can write a general term in the form

$$a_k = \int_0^{+\infty} x^k d\mu(x), \quad (2.3)$$

where μ is a non-negative measure on $[0, +\infty)$ (see, e.g., [14, Theorem 4.4]). To determine whether a sequence has the Stieltjes moment property or not is one of classical moment problems and it arises naturally in many branches of mathematics (cf. [16, 24]). There are many transformations and convolutions of sequences preserving Stieltjes moment sequences, see, e.g., [23]. We need the following lemma in this direction.

Lemma 2.2. [23] *If both $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ are Stieltjes moment sequences, then so is the sequence $(w_n)_{n \geq 0}$, where*

$$w_n := \sum_{k=0}^n \binom{n}{k}^2 x_k y_{n-k}.$$

The next result plays an important role in our proof.

Lemma 2.3. *Let k be a positive integer. Suppose that $(x_i)_{i \geq 0}$ is an integer sequence for which $x_0 = 1$, $2k \mid x_i$ for all $i \geq 1$, and $4k \mid x_i$ if and only if i is not a power of two. Then, for any $n \in \mathbb{N}$, the number $(2k)^{-n} |x_{i+j}|_{0 \leq i, j \leq n}$ is an odd integer.*

Proof. Since $2k \mid x_m$ for all $m = 1, 2, 3, \dots$, by the Laplace expansion of $|x_{i+j}|_{0 \leq i, j \leq n}$ according to the first row, it suffices to show that

$$|x_{i+j}/(2k)|_{1 \leq i, j \leq n} \equiv 1 \pmod{2}.$$

For any positive integer m , clearly $x_m/(2k)$ is congruent to 1 or 0 modulo 2 according as m is a power of two or not. Let B_n denote the $(0, 1)$ -matrix $[x_{i+j}/(2k) \pmod{2}]_{1 \leq i, j \leq n}$. It suffices to show the claim that

$$|B_n| \in \{\pm 1\}. \quad (2.4)$$

(2.4) is trivial for $n = 1, 2, 3$. Below we let $n \geq 4$. If $n + 1$ is a power of two, then the matrix B_n is an upper triangular matrix with the anti-diagonal line containing no 0, and thus $|B_n| = (-1)^{\binom{n}{2}} = -1$. If n is a

$$\left[\begin{array}{cc}
 \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ \vdots \\ 0 \end{array} & \begin{array}{c} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{array} \\
 \begin{array}{c} 1 \\ \vdots \\ \vdots \\ 0 \end{array} & \begin{array}{c} 1 \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ \vdots \\ 0 \end{array}
 \end{array} \right]
 \begin{array}{l}
 \left. \vphantom{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ \vdots \\ 0 \end{array}} \right\} n - 2t - 1 \\
 \left. \vphantom{\begin{array}{c} 1 \\ \vdots \\ \vdots \\ 0 \end{array}} \right\} 2t + 1 \\
 \left. \vphantom{\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ \vdots \\ 1 \\ \vdots \\ \vdots \\ 0 \end{array}} \right\} 2t + 1
 \end{array}$$

It follows that

$$|B_n| = |B_{n-2t-1}| \times \begin{vmatrix} & & & 1 \\ & 0 & & \\ & & \ddots & \\ & 1 & & 0 \\ 1 & & & \end{vmatrix}_{(2t+1) \times (2t+1)}.$$

So

$$|B_{n-2t-1}| \in \{\pm 1\} \Rightarrow |B_n| \in \{\pm 1\}.$$

Therefore (2.4) holds by induction, and this concludes our proof of Lemma 2.3. \square

We also need a result of N. J. Calkin [3].

Lemma 2.4. [3, Lemma 12] *For any positive integers r and n , we have $2^{\ell(n)} \mid f_n^{(r)}$, where $\ell(n)$ denotes the numbers of 1's in the binary expansion of n .*

Proof of Theorem 1.1. For any positive integer m , by Lemma 2.4 we have $2 \mid f_m^{(r)}$, and $4 \mid f_m^{(r)}$ if m is not a power of two. When m is a power of two, we have

$$\binom{m}{k} = \frac{m}{k} \binom{m-1}{k-1} \equiv 0 \pmod{2} \quad \text{for all } k = 1, \dots, m-1,$$

hence

$$f_m^{(r)} \equiv \binom{m}{0}^r + \binom{m}{m}^r = 2 \pmod{2^r}$$

and thus $4 \nmid f_m^{(r)}$. Applying Lemma 2.3 to the sequence $(f_m^{(r)})_{m \geq 0}$, we see that the number $2^{-n} |f_{i+j}^{(r)}|_{0 \leq i, j \leq n}$ is always an odd integer.

By Barrucand's identity (cf. [2]), for any $m \in \mathbb{N}$,

$$f'_m := \sum_{k=0}^m \binom{m}{k} f_k$$

coincides with

$$g_m := \sum_{k=0}^m \binom{m}{k}^2 \binom{2k}{k}.$$

(See also [17, A002893], and [22] for an extension.) Moreover, by Lemma 2.2, $(g_m)_{m \geq 0}$ is a Stieltjes moment sequence since $(\binom{2m}{m})_{m \geq 0}$ is a Stieltjes moment sequence (cf. [6]). Combining this with Lemma 2.1, we have

$$|f_{i+j}|_{0 \leq i, j \leq n} = |g_{i+j}|_{0 \leq i, j \leq n} \geq 0.$$

By Fermat's little theorem, $a^3 \equiv a \pmod{3}$ for any $a \in \mathbb{Z}$. Thus, for any positive integer m we have

$$g_m = \sum_{k=0}^m \binom{m}{k} f_k \equiv \sum_{k=0}^m \binom{m}{k} \sum_{j=0}^k \binom{k}{j} = \sum_{k=0}^m \binom{m}{k} 2^k = 3^m \equiv 0 \pmod{3}.$$

So $|g_{i+j}|_{0 \leq i, j \leq n}$ is divisibly by 3^n .

In view of the above, $6^{-n} |f_{i+j}|_{0 \leq i, j \leq n}$ is always a positive odd integer, as desired. \square

3. PROOF OF THEOREM 1.2

Lemma 3.1. *Let m and n be positive integers, and set*

$$D_n^{(m)} := \sum_{k=0}^n \binom{n}{k}^m \binom{2k}{k} \binom{2(n-k)}{n-k}.$$

Then $4 \mid D_n^{(m)}$. Also, $8 \mid D_n^{(m)}$ if and only if n is not a power of two.

Proof. For each $k = 1, 2, 3, \dots$ we obviously have

$$\binom{2k}{k} = 2 \binom{2k-1}{k-1} \equiv 0 \pmod{2}.$$

Thus

$$\begin{aligned}
& \sum_{\substack{k=0 \\ 2k \neq n}}^n \binom{n}{k}^m \binom{2k}{k} \binom{2(n-k)}{n-k} \\
&= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \left(\binom{n}{k}^m + \binom{n}{n-k}^m \right) \binom{2k}{k} \binom{2(n-k)}{n-k} \\
&= 2 \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{k}^m \binom{2k}{k} \binom{2(n-k)}{n-k} \\
&\equiv 2 \binom{n}{0}^m \binom{2 \times 0}{0} \binom{2n}{n} = 4 \binom{2n-1}{n-1} \pmod{8}.
\end{aligned}$$

If $n = 2k$ for some positive integer k , then

$$\binom{n}{k}^m \binom{2k}{k} \binom{2(n-k)}{n-k} = \binom{2k}{k}^{m+2} = 2^{m+2} \binom{2k-1}{k-1}^{m+2} \equiv 0 \pmod{8}.$$

So we always have

$$D_n^{(m)} \equiv 4 \binom{2n-1}{n-1} \pmod{8}.$$

Now we show that $\binom{2n-1}{n-1}$ is odd if and only if n is a power of two. Clearly, $\binom{2-1}{1-1} = 1$ is odd. For $n > 1$, we can write $n-1$ as $\sum_{i=0}^k \delta_i 2^i$ with $\delta_0, \dots, \delta_k \in \{0, 1\}$ and $\delta_k = 1$, hence

$$\begin{aligned}
\binom{2n-1}{n-1} &= \binom{\delta_k \times 2^{k+1} + \delta_{k-1} \times 2^k + \dots + \delta_0 2 + 1}{\delta_k 2^k + \dots + \delta_1 2 + \delta_0} \\
&\equiv \binom{\delta_k}{0} \binom{\delta_{k-1}}{\delta_k} \dots \binom{\delta_0}{\delta_1} \binom{1}{\delta_0} \pmod{2}
\end{aligned}$$

by Lucas' congruence (cf. [8]), and thus

$$\binom{2n-1}{n-1} \equiv 1 \pmod{2} \iff \delta_0 = \delta_1 = \dots = \delta_k = 1 \iff n \text{ is a power of two.}$$

Combining the above, we immediately obtain the desired result. \square

Lemma 3.2. [15, Pólya and Szegő] *If both $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ are Stieltjes moment sequences, then so is the sequence $(z_n)_{n \geq 0}$, where*

$$z_n = \sum_{k=0}^n \binom{n}{k} x_k y_{n-k}.$$

Lemma 3.3. *We have $D_n \equiv 1 \pmod{3}$ for all $n = 0, 1, 2, \dots$*

Proof. We use induction on n .

Clearly, both $D_0 = 1$ and $D_1 = 4$ are congruent to 1 modulo 3.

Now let $n > 1$ be an integer and assume that $D_m \equiv 1 \pmod{3}$ for all $m = 0, \dots, n-1$.

Case 1. $3 \mid n$.

In this case, by Lucas' theorem, for any $k = 0, \dots, n$ we have

$$\binom{n}{k} \equiv \begin{cases} \binom{n/3}{k/3} \pmod{3} & \text{if } 3 \mid k, \\ 0 \pmod{3} & \text{if } 3 \nmid k. \end{cases}$$

Thus

$$\begin{aligned} D_n &\equiv \sum_{j=0}^{n/3} \binom{n/3}{j}^2 \binom{6j}{3j} \binom{2(n-3j)}{n-3j} \\ &\equiv \sum_{j=0}^{n/3} \binom{n/3}{j}^2 \binom{2j}{j} \binom{2(n/3-j)}{n/3-j} = D_{n/3} \equiv 1 \pmod{3} \end{aligned}$$

with the help of the induction hypothesis.

Case 2. $3 \nmid n$.

It is known that

$$n^3 D_n = 2(2n-1)(5n^2 - 5n + 2)D_{n-1} - 64(n-1)^3 D_{n-2}$$

(cf. [17, A002895]) which can be obtained via Zeilberger's algorithm. So we have

$$\begin{aligned} n^3 D_n &\equiv (n+1)(-n^2 + n + 2)D_{n-1} - (n-1)^3 D_{n-2} \\ &\equiv -(n+1)^3 D_{n-1} - (n-1)^3 D_{n-2} \pmod{3}. \end{aligned}$$

By Fermat's little theorem, $a^3 \equiv a \pmod{3}$ for all $a \in \mathbb{Z}$. Thus, by applying the induction hypothesis we obtain

$$n D_n \equiv -(n+1) - (n-1) \equiv n \pmod{3}$$

and hence $D_n \equiv 1 \pmod{3}$ as desired.

In view of the above, we have completed the proof Lemma 3.3. \square

Lemma 3.4. *For any positive integer n , we have $D_n'' \equiv 0 \pmod{3}$.*

Proof. By Lemma 3.3, we have

$$D_k' = \sum_{j=0}^k \binom{k}{j} D_j \equiv \sum_{j=0}^k \binom{k}{j} \equiv 2^k \pmod{3}$$

for all $k \in \mathbb{N}$. Thus

$$D_n'' = \sum_{k=0}^n \binom{n}{k} D_k' \equiv \sum_{k=0}^n \binom{n}{k} 2^k \equiv 3^n \equiv 0 \pmod{3}.$$

This completes the proof. \square

Proof of Theorem 1.2. (i) By Lemma 3.1, the sequence $(D_m)_{m \geq 0}$ satisfies the conditions of Lemma 2.3 with $k = 2$. Thus, $4^{-n}|D_{i+j}|_{0 \leq i, j \leq n}$ is always an odd integer. On the other hand, it follows from Lemmas 2.1 and 3.4 that $3^{-n}|D_{i+j}|_{0 \leq i, j \leq n}$ is also an integer. Hence $12^{-n}|D_{i+j}|_{0 \leq i, j \leq n}$ is odd. Moreover, by Lemma 2.2, we know that $(D_m)_{m \geq 0}$ is a Stieltjes moment sequence since $((\binom{2m}{m})_{m \geq 0})$ is a Stieltjes moment sequence. So $12^{-n}|D_{i+j}|_{0 \leq i, j \leq n}$ is always a positive odd integer.

(ii) By Lemma 3.2, we know that $(D_m^{(1)})_{m \geq 0}$ is a Stieltjes moment sequence since $((\binom{2m}{m})_{m \geq 0})$ is a Stieltjes moment sequence. So $|D_{i+j}^{(1)}|_{0 \leq i, j \leq n}$ is always nonnegative. Hence, with the help of Lemma 3.1 and Lemma 2.3, $4^{-n}|D_{i+j}^{(1)}|_{0 \leq i, j \leq n}$ is a positive odd integer. It is known (cf. [17, A053175]) that

$$P_m = 2^m \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{2k} \binom{2k}{k}^2 4^{m-2k} = 2^m D_m^{(1)}$$

for any $m \in \mathbb{N}$. Therefore,

$$\begin{aligned} \frac{|P_{i+j}|_{0 \leq i, j \leq n}}{2^{n(n+3)}} &= \frac{|2^{i+j} D_{i+j}^{(1)}|_{0 \leq i, j \leq n}}{2^{n(n+3)}} = \frac{\prod_{i=1}^n 2^i \times \prod_{j=1}^n 2^j}{2^{n(n+3)}} |D_{i+j}^{(1)}|_{0 \leq i, j \leq n} \\ &= \frac{2^{n(n+1)}}{2^{n(n+3)}} |D_{i+j}^{(1)}|_{0 \leq i, j \leq n} = \frac{|D_{i+j}^{(1)}|_{0 \leq i, j \leq n}}{4^n}, \end{aligned}$$

which is a positive odd integer. \square

4. PROOF OF THEOREM 1.3

Lemma 4.1. *For any positive integer n , we have*

$$b'_n \equiv 0 \pmod{2} \quad \text{and} \quad b''_n \equiv 0 \pmod{5}.$$

Proof. For any positive integer m , we have

$$b_m = \sum_{k=0}^m \binom{m}{k} \binom{m+k}{2k} \binom{2k}{k} = 1 + \sum_{k=1}^m \binom{m}{k} \binom{m+k}{2k} 2 \binom{2k-1}{k-1} \equiv 1 \pmod{2}.$$

Thus

$$b'_n = \sum_{k=0}^n \binom{n}{k} b_k \equiv \sum_{k=0}^n \binom{n}{k} = 2^n \equiv 0 \pmod{2}.$$

If $b_k \equiv 3^k \pmod{5}$ for all $k \in \mathbb{N}$, then

$$\begin{aligned} b''_n &= \sum_{m=0}^n \binom{n}{m} \sum_{k=0}^m \binom{m}{k} b_k \\ &\equiv \sum_{m=0}^n \binom{n}{m} \sum_{k=0}^m \binom{m}{k} 3^k = \sum_{m=0}^n \binom{n}{m} 4^m = 5^n \equiv 0 \pmod{5}. \end{aligned}$$

It remains to prove $b_k \equiv 3^k \pmod{5}$ by induction on $k \in \mathbb{N}$. Note that $b_k = 3^k$ for $k = 0, 1$. Let $k > 1$ be an integer. It is well known that (cf. [17, A005258])

$$k^2 b_k = (11k^2 - 11k + 3)b_{k-1} + (k-1)^2 b_{k-2}.$$

If $b_{k-1} \equiv 3^{k-1} \pmod{5}$ and $b_{k-2} \equiv 3^{k-2} \pmod{5}$, then

$$k^2 b_k \equiv (3(k^2 - k + 3) + (k-1)^2)3^{k-2} \equiv 9k^2 3^{k-2} = k^2 3^k \pmod{5}$$

and hence $b_k \equiv 3^k \pmod{5}$ if $5 \nmid k$. When $5 \mid k$, by Lucas' theorem, for each $j \in \{0, \dots, k\}$ we have

$$\binom{k}{j}^2 \binom{k+j}{j} \equiv \begin{cases} \binom{k/5}{j/5}^2 \binom{(k+j)/5}{j/5} \pmod{5} & \text{if } 5 \mid j, \\ 0 \pmod{5} & \text{if } 5 \nmid j. \end{cases}$$

So, if $5 \mid k$ and $b_{k/5} \equiv 3^{k/5} \pmod{5}$, then

$$b_k \equiv \sum_{i=0}^{k/5} \binom{k/5}{i}^2 \binom{k/5+i}{i}^2 = b_{k/5} \equiv 3^{k/5} \equiv (3^5)^{k/5} = 3^k \pmod{5}.$$

This concludes our proof. \square

Lemma 4.2. *For each $n = 3, 4, \dots$, we have $A'_n \equiv 0 \pmod{24}$.*

Proof. From Gessel [7], we have $A_{2k} \equiv 1 \pmod{8}$, $A_{2k+1} \equiv 5 \pmod{8}$ and $A_k \equiv (-1)^k \pmod{3}$ for all $k \in \mathbb{N}$. This implies that $A_k \equiv 3 - 2(-1)^k \pmod{24}$ for all $k \in \mathbb{N}$. Therefore,

$$A'_n = \sum_{k=0}^n \binom{n}{k} A_k \equiv \sum_{k=0}^n \binom{n}{k} (3 - 2(-1)^k) = 3 \times 2^n \equiv 0 \pmod{24}.$$

This concludes the proof. \square

Proof of Theorem 1.3. (i) By Lemma 2.1 and Lemma 4.1, $10^{-n} |b_{i+j}|_{0 \leq i, j \leq n}$ is always an integer.

(ii) Let $x_n = A'_n = \sum_{k=0}^n \binom{n}{k} A_k$. Then $x_0 = 1$, $x_1 = 6$, $x_2 = 84$, $x_3 = 1680 = 70 \times 24$. By Lemma 2.1, we have

$$24^{-n} |A_{i+j}|_{0 \leq i, j \leq n} = 24^{-n} |x_{i+j}|_{0 \leq i, j \leq n}.$$

If a term X in the Laplace expansion of $|x_{i+j}|_{0 \leq i, j \leq n}$ only contains one entry in $\{x_0, x_1, x_2\}$, then by Lemma 4.2 we have $X \equiv 0 \pmod{24^n}$. So, we only need to consider all terms containing two or three entries in $\{x_0, x_1, x_2\}$. Clearly, $x_0 x_2 = 84$, $x_1 x_1 = 36$, $x_1 x_2 \equiv 0 \pmod{24}$, $x_2 x_2 \equiv 0 \pmod{24}$ and $x_2 x_2 x_2 \equiv 0 \pmod{24^2}$. Thus, among all the terms in the Laplace expansion of $|x_{i+j}|_{0 \leq i, j \leq n}$, it suffices to consider those containing $x_0 x_2$ and $x_1 x_1$. Hence, for some integer $M \equiv 0 \pmod{24^{n-1}}$ we have

$$|x_{i+j}|_{0 \leq i, j \leq n} \equiv (x_0 x_2 - x_1 x_1) M \equiv 0 \pmod{24^n}.$$

This concludes the proof. \square

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