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DIVISIBILITY RESULTS ON FRANEL NUMBERS AND RELATED POLYNOMIALS

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Abstract. In this paper we establish some new divisibility results involving the Franel numbers $f_n = \sum_{k=0}^n \binom{n}{k}^3$ $(n = 0, 1, 2, \ldots)$ and the polynomials $g_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} x^k$ $(n = 0, 1, 2, \ldots)$. For example, we show that for any positive integer n we have

$$\frac{9}{2n^2(n+1)^2} \sum_{k=1}^n k^2(3k+1)(-1)^{n-k} f_k \in \{1, 2, 3, \ldots\}$$

and

$$\frac{2}{n(n+1)} \sum_{k=1}^n k^2(4k+3) g_k(2) \in \{1, 3, 5, \ldots\},$$

and for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} k^2(3k+1)(-1)^k f_k \equiv \frac{2}{9} p^2 \pmod{p^3} \ \text{ and } \sum_{k=0}^{p-1} k^2(4k+3) g_k(2) \equiv \frac{7}{2} p \pmod{p^2}.$$

1. Introduction

It is well known that

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

for all $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. In 1895 Franel introduced the Franel numbers

$$f_n := \sum_{k=0}^n \binom{n}{k}^3 \ (n \in \mathbb{N})$$

(cf. [10]) and noted the recurrence relation:

$$(n+1)^2 f_{n+1} = (7n(n+1)+2) f_n + 8n^2 f_{n-1} \ \text{ for all } n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}.$$

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In 1975, Barrucand [1] found that

$$\sum_{k=0}^{n}\binom{n}{k}f_k=g_n \quad \text{for all } n\in\mathbb{N},$$

where

$$g_n:=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}.$$

Callan [2] provided a combinatorial interpretation of this identity. In 1994, Strehl [11] obtained the following identity for Franel numbers:

$$f_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n} \qquad \text{for all } n\in\mathbb{N}. \tag{1.1}$$

In [16] Sun introduced the polynomials

$$g_n(x):=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}x^k \quad (n=0,1,2,\ldots)$$

and proved the following extension of Barrucand's identity

$$\sum_{k=0}^{n}\binom{n}{k}f_k(x)=g_n(x),$$

where

$$f_k(x):=\sum_{j=0}^{k}\binom{k}{j}^2\binom{2j}{k}x^j.$$

(Note that $g_n(x)$ also appeared in [14, Conjecture 7.9].) The polynomials $f_n(x)$ and $g_n(x)$ are also closely related to the Apéry polynomials (cf. [16]). In 2016, Guo, Mao and Pan [7] showed that

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)g_k(x)\in\mathbb{Z}[x] \qquad \text{for all } n\in\mathbb{Z}^+,$$

as conjectured by Sun [16].

In 2013 Sun [12, 13] studied congruences for Franel numbers systematically. Guo [5] confirmed a conjecture of Sun which states that

$$\frac{1}{2n^2}\sum_{k=0}^{n-1}(3k+2)(-1)^k f_k\in\mathbb{Z} \quad \text{for all } n\in\mathbb{Z}^+.$$

Our first theorem provides new congruences involving Franel numbers.

Theorem 1.1. *For any $n \in \mathbb{Z}^+$ we have*

$$\frac{9}{2n^2(n+1)^2}\sum_{k=1}^{n}k^2(3k+1)(-1)^{n-k}f_k \in \mathbb{Z}^+, \tag{1.2}$$

$$\frac{3}{n(n+1)^2}\sum_{k=1}^{n}(9k^3-6k^2-5k)(-1)^k f_k \in \mathbb{Z}, \tag{1.3}$$

$$\frac{3}{4n(n+1)^2}\sum_{k=1}^{n}(9k^3-15k^2-10k)(-1)^k f_k \in \mathbb{Z}, \tag{1.4}$$

$$\frac{3}{n(n+1)^2}\sum_{k=1}^{n}(9k^3+12k^2+5k)(-1)^{n-k} f_k \in \mathbb{Z}^+. \tag{1.5}$$

Also, for any prime p we have the supercongruences

$$\sum_{k=1}^{p-1}k^2(3k+1)(-1)^k f_k \equiv \frac{2}{9}p^2 \pmod{p^3}, \tag{1.6}$$

$$\sum_{k=1}^{p-1}(9k^3-6k^2-5k)(-1)^k f_k \equiv \frac{11}{3}p^2 \pmod{p^3}, \tag{1.7}$$

$$\sum_{k=1}^{p-1}(9k^3-15k^2-10k)(-1)^k f_k \equiv \frac{20}{3}p^2 \pmod{p^3}, \tag{1.8}$$

$$\sum_{k=1}^{p-1}(9k^3+12k^2+5k)(-1)^k f_k \equiv -\frac{7}{3}p^2 \pmod{p^3}. \tag{1.9}$$

Our second theorem is related to the sequence $(g_n)_{n\geqslant 0}$ and related polynomials

$$S_n(x) := \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}(2k+1)x^k \quad (n=0,1,2,\ldots)$$

introduced by Sun [15].

Theorem 1.2. *For any $n \in \mathbb{Z}^+$ we have*

$$\frac{4}{3n(n+1)^2}\sum_{k=1}^{n}k(4k+3)g_k \in \mathbb{Z} \tag{1.10}$$

and

$$\frac{2}{n(n+1)}\sum_{k=0}^{n}kS_k(x) \in \mathbb{Z}[x]. \tag{1.11}$$

Remark 1.1. Guo and Liu [6] proved that $n^2 \mid 4\sum_{k=0}^{n-1}kS_k(1)$ for all $n \in \mathbb{Z}^+$ which was first conjectured by Sun [15].

Our third theorem involves the polynomials $g_n(x)$ $(n = 0, 1, 2, \ldots)$.

Theorem 1.3. *For any $n \in \mathbb{Z}^+$ we have*

$$\frac{2}{n(n+1)} \sum_{k=1}^{n} k(3k+2) g_k(-1) \in \mathbb{Z} \tag{1.12}$$

and

$$\frac{2}{n(n+1)} \sum_{k=1}^{n} k^2(4k+3) g_k(2) \in \{1, 3, 5, \ldots\}. \tag{1.13}$$

Also, for any prime $p > 3$ we have the supercongruences

$$\sum_{k=0}^{p-1} k(3k+2) g_k(-1) \equiv -\frac{5}{8} p \pmod{p^2} \tag{1.14}$$

and

$$\sum_{k=0}^{p-1} k^2(4k+3) g_k(2) \equiv \frac{7}{2} p \pmod{p^2}. \tag{1.15}$$

Remark 1.2. Sun [16] obtained the congruences

$$\sum_{k=1}^{p-1} \frac{g_k(-1)}{k} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{g_k(-1)}{k^2} \equiv 0 \pmod{p}$$

for any prime $p > 5$, which are similar to the classical Wolstenholme congruences.

To prove Theorems 1.1–1.3, we need a telescoping method for double summations developed by Chen, Hou and Mu [3]. Consider double sums of the form

$$S_n = \sum_{k=0}^{n-1} \sum_{l=0}^{k} F(k, l) \quad (n \in \mathbb{Z}^+),$$

where $F(k, l)$ is a bivariate hypergeometric term of k and l. Once we use the Maple package `APCI` (which can be downloaded from `http://www.combinatorics.net.cn/homepage/hou/apci.html`) to find two hypergeometric terms $G_1(k, l)$ and $G_2(k, l)$ such that

$$F(k, l) = \Delta_k \left(G_1(k, l) \right) + \Delta_l \left(G_2(k, l) \right), \tag{1.16}$$

where

$$G_1(k, l) = F(k, l) R_1(k, l) \quad \text{and} \quad G_2(k, l) = F(k, l) R_2(k, l)$$

with $R_1(k, l)$ and $R_2(k, l)$ being rational functions and

$$\Delta_k \left(G_1(k, l) \right) = G_1(k+1, l) - G_1(k, l)$$

and

$$\Delta_l \left(G_2(k, l) \right) = G_2(k, l+1) - G_2(k, l),$$

the sum S_n can be transformed into a single sum:

$$S_n = \sum_{l=0}^{n-1} \left(G_1(n,l) - G_1(l,l)\right) + \sum_{k=0}^{n-1} \left(G_2(k,k+1) - G_2(k,0)\right). \tag{1.17}$$

Via this telescoping method, Mu and Sun [8] confirmed several conjectures of Sun on congruences which could not be proved by other methods.

We are going to prove Theorems 1.1–1.3 in Sections 2–4 respectively.

2. Proof of Theorem 1.1

We just prove (1.2) and (1.6) in details. Formulas (1.3)–(1.5) and (1.7)–(1.9) can be proved in a similar way.

Lemma 2.1. *For any $n \in \mathbb{Z}^+$, we have*

$$(-1)^n \sum_{k=1}^{n} k^2(3k+1)(-1)^k f_k > 0. \tag{2.1}$$

Proof. Let a_n denote the left-hand side of the inequality (2.1). Then $a_1 = 8$, $a_2 = 272$, and

$$a_{n+1} - a_{n-1} = -n^2(3n+1)f_n + (n+1)^2(3(n+1)+1)f_{n+1} > 0$$

for all $n = 2, 3, \ldots$. So the desired result follows. □

Proofs of (1.2) *and* (1.6). Let $n \in \mathbb{Z}^+$. In view of (1.1), we let

$$F(k,l) = k^2(3k+1)(-1)^k \binom{k}{l}^2 \binom{2l}{k}$$

be the summand and use the command `MZeil` of the package `APCI` in Maple to obtain

$$G_1(k,l) = \frac{(-1)^k}{9}(k-l)(k-1)(9k^2 - 12kl - 17k + 4l)\binom{k}{l}^2\binom{2l}{k}$$

and

$$G_2(k,l) = \frac{(-1)^k}{9(k+1)} l(k-2l)(k-2l+1)(3kl + 14k - l + 1)\binom{k}{l}\binom{k+1}{l}\binom{2l}{k}$$

for which (1.16) holds. In light of (1.17) and noting that $G_1(l,l) = G_2(k,k+1) = G_2(k,0) = 0$ we arrive at

$$\sum_{k=0}^{n-1}\sum_{l=0}^{k} F(k,l) = \frac{(-1)^n(n-1)}{9}\sum_{l=0}^{n-1}(n-l)(9n^2 - 12nl - 17n + 4l)\binom{n}{l}^2\binom{2l}{n}.$$

Thus we obtain that

$$\begin{aligned}
&(-1)^n\frac{9}{2}\sum_{k=0}^{n-1}k^2(3k+1)(-1)^kf_k\\
=&\frac{n-1}{2}\sum_{l=0}^{n-1}(n-l)(9n^2-12nl-17n+4l)\binom{n}{l}^2\binom{2l}{n}\\
=&n(n-1)\sum_{l=1}^{n-1}(9n^2-12nl-17n+4l)\binom{n-1}{l}\binom{n-1}{l-1}\binom{2l-1}{n-1}\\
\equiv&-n\sum_{l=1}^{n-1}4l\binom{n-1}{l}\binom{n-1}{l-1}\binom{2l-1}{n-1}\\
=&-2n^2\sum_{l=1}^{n-1}\binom{n-1}{l}\binom{n-1}{l-1}\binom{2l}{n}\\
\equiv&0 \pmod{n^2},
\end{aligned}$$

and hence

$$\frac{9}{2}\sum_{k=0}^{n}k^2(3k+1)(-1)^kf_k\equiv 0 \pmod{(n+1)^2}.$$

Obviously $\gcd(n,n+1)=1$ and

$$\frac{9}{2}n^2(3n+1)f_n\equiv 0 \pmod{n^2}.$$

So we finally get (1.2) with the help of Lemma 2.1.

We can easily verify (1.6) for $p=2,3$. Below we let $p>3$ be a prime. By the previous argument,

$$\begin{aligned}
&-\frac{9}{2}\sum_{k=0}^{p-1}k^2(3k+1)(-1)^kf_k\\
=&\frac{p-1}{2}\sum_{l=0}^{p-1}(p-l)(9l^2-12pl-17p+4l)\binom{p}{l}^2\binom{2l}{p}\\
=&\frac{p-1}{2}\sum_{l=(p+1)/2}^{p-1}(p-l)(9p^2-12pl-17p+4l)\frac{p^2}{l^2}\binom{p-1}{l-1}^2\binom{2l}{p}\\
\equiv&2p^2\sum_{k=1}^{(p-1)/2}\binom{2(p-k)}{p}=2p^2\sum_{k=1}^{(p-1)/2}\prod_{j=1}^{p-2k}\frac{p+j}{j}\equiv-p^2 \pmod{p^3}.
\end{aligned}$$

This proves (1.6). □

3. Proof of Theorem 1.2

Lemma 3.1. [9, p.132] *For all $n \in \mathbb{N}$ we have*

$$3\sum_{k=0}^{n} k\binom{n}{k}^2\binom{2k}{k} = 2ng_n.$$

Lemma 3.2. *Let n be a positive integer and k, l be integers with $n \geqslant k \geqslant l \geqslant 0$. Then we have*

$$\binom{n}{k}\binom{k}{l}\binom{2l}{k+1} \equiv 0 \pmod{n}. \tag{3.1}$$

Proof. Clearly, (3.1) holds if and only if

$$\begin{aligned}&\nu_p((n-1)!) + \nu_p((2l)!)\\ \geqslant&\nu_p((n-k)!) + \nu_p(l!) + \nu_p((k-l)!) + \nu_p((k+1)!) + \nu_p((2l-k-1)!)\end{aligned}$$

for any prime p, where $\nu_p(q)$ denotes the p-adic order of $q \in \mathbb{Z}^+$. Since $\nu_p(q!) = \sum_{s\geqslant 1}\lfloor q/p^s\rfloor$ for all $q \in \mathbb{Z}^+$, it suffices to prove for each $m = 2, 3, \ldots$ the inequality

$$\left\lfloor\frac{n-1}{m}\right\rfloor + \left\lfloor\frac{2l}{m}\right\rfloor \geqslant \left\lfloor\frac{n-k}{m}\right\rfloor + \left\lfloor\frac{l}{m}\right\rfloor + \left\lfloor\frac{k-l}{m}\right\rfloor + \left\lfloor\frac{k+1}{m}\right\rfloor + \left\lfloor\frac{2l-k-1}{m}\right\rfloor. \tag{3.2}$$

This can be easily verified if $n-k, l, k-l$ are all divisible by m.

Now we suppose that one of $n-k, l, k-l$ is not divisible by m. Note that $\lfloor x/m\rfloor = \lfloor (x-1)/m\rfloor$ for any integer $x \not\equiv 0 \pmod{m}$. Also, $\lfloor x\rfloor + \lfloor y\rfloor + \lfloor z\rfloor \leqslant \lfloor x+y+z\rfloor$ for any real numbers x, y, z. So

$$\left\lfloor\frac{l}{m}\right\rfloor + \left\lfloor\frac{n-k}{m}\right\rfloor + \left\lfloor\frac{k-l}{m}\right\rfloor \leqslant \left\lfloor\frac{n-1}{m}\right\rfloor$$

and

$$\left\lfloor\frac{k+1}{m}\right\rfloor + \left\lfloor\frac{2l-k-1}{m}\right\rfloor \leqslant \left\lfloor\frac{2l}{m}\right\rfloor.$$

Adding these two inequalities we obtain (3.2). This concludes the proof. □

Proof of Theorem 1.2. We first show (1.10). In view of Lemma 3.1,

$$\sum_{k=0}^{n-1} 2k(2kg_k) = \sum_{k=0}^{n-1} 3k\sum_{l=0}^{k}\binom{k}{l}^2\binom{2l}{l}2l$$

and hence

$$\sum_{k=0}^{n-1} k(4k+3)g_k = 3\sum_{k=0}^{n-1} kS_k(1). \tag{3.3}$$

Since $n^2 \mid 4\sum_{k=0}^{n-1} kS_k(1)$ as conjectured by Sun [15] and proved in [6, Theorem 1.1], we have

$$\frac{4}{3n^2}\sum_{k=0}^{n-1} k(4k+3)g_k = \frac{4}{n^2}\sum_{k=0}^{n-1} kS_k(1) \in \mathbb{Z}. \tag{3.4}$$

Substituting $n+1$ for n, we obtain

$$\frac{4}{3(n+1)^2}\sum_{k=1}^{n} k(4k+3)g_k \in \mathbb{Z}. \tag{3.5}$$

By Lemma 3.1,

$$\frac{4}{3}n(4n+3)g_n \equiv \frac{16}{3}n^2 g_n \equiv 0 \pmod{n}.$$

Combining this with (3.4)–(3.5), we have proved (1.10).

Next, we show (1.11). Since

$$\sum_{k=0}^{n} kS_k(x) = \sum_{k=0}^{n} k\sum_{l=0}^{k}\binom{k}{l}^2\binom{2l}{l}(2l+1)x^l = \sum_{l=0}^{n}(2l+1)\binom{2l}{l}x^l\sum_{k=l}^{n} k\binom{k}{l}^2,$$

it suffices to prove that

$$\frac{2}{n}(2l+1)\binom{2l}{l}\sum_{k=0}^{n-1} k\binom{k}{l}^2 \in \mathbb{Z} \tag{3.6}$$

for any $l = 0, \ldots, n-1$. With helps of the Chu-Vandermonde identity (cf. [4, (3.1)]) and the identity

$$\sum_{k=0}^{n-1} k\binom{k}{j} = \frac{n(j+1)-1}{n+1}\binom{n+1}{j+2},$$

we have

$$\begin{aligned}
\sum_{k=l}^{n-1} k\binom{k}{l}^2 &= \sum_{k=l}^{n-1} k\binom{k}{l}\sum_{j=0}^{l}\binom{k-l}{j}\binom{l}{l-j} \\
&= \sum_{k=l}^{n-1} k\sum_{j=0}^{l}\binom{k}{l+j}\binom{l+j}{j}\binom{l}{j} \\
&= \sum_{j=0}^{l}\frac{n(l+j+1)-1}{n+1}\binom{l+j}{j}\binom{l}{j}\binom{n+1}{j+l+2} \\
&\equiv -\sum_{j=0}^{l}\binom{l}{j}\binom{l+j}{j}\binom{n+1}{j+l+2} \pmod{n}.
\end{aligned}$$

For each $j = 0, \ldots, l$, clearly

$$\begin{aligned}&2(2l+1)\binom{2l}{l}\binom{l}{j}\binom{n+1}{j+l+2}\\=&(n+1)\binom{n}{j+l+1}\binom{2l+2}{j+l+2}\binom{j+l+1}{l+1} \equiv 0 \pmod n\end{aligned}$$

with the help of Lemma 3.2. Thus (3.6) holds and this ends the proof of (1.11). $\square$

4. Proof of Theorem 1.3

Lemma 4.1. *Let n be a positive integer. Then*

$$\sum_{k=0}^{n-1} k(3k+2)g_k(-1) = \frac{n}{4}\sum_{l=0}^{n-1}\frac{(-1)^{n-l}}{2l+1}\binom{n-1}{2l}\binom{n}{2l}\binom{2l}{l}P(n,l), \tag{4.1}$$

where

$$P(n,l) = 3n^3 - 18n^2l + 30nl^2 - 14n^2 + 48nl - 50l^2 + 11n - 30l.$$

Also,

$$\sum_{k=0}^{n-1} k^2(4k+3)g_k(2) = -\frac{n}{6}\sum_{l=0}^{n-1}\frac{2^l}{(l+1)^2}\binom{n-1}{l}\binom{n}{l}\binom{2l}{l}Q(n,l), \tag{4.2}$$

where

$$\begin{aligned}Q(n,l) =&176n^4l - 240n^3l^2 - 70n^2l^3 + 98nl^4 + 88n^4 - 452n^3l + 278n^2l^2 + 49nl^3\\&- 168n^3 + 374n^2l - 192nl^2 + 114n^2 - 196nl + 98l^2 - 55n + 77l + 21.\end{aligned}$$

Proof. (i) We first prove (4.1). By Sun [17, Remark 5.9],

$$g_k(-1) = \sum_{l=0}^{k}\binom{k}{2l}^2\binom{2l}{l}(-1)^{k-l} \quad \text{for all } k = 0, 1, 2, \ldots.$$

Thus we set

$$F(k,l) := k(3k+2)\binom{k}{2l}^2\binom{2l}{l}(-1)^{k-l}$$

and find that (1.16) holds for

$$G_1(k,l) = \frac{(k-2l)}{4(2l+1)}\binom{k}{2l}^2\binom{2l}{l}(-1)^{k-l}P(k,l)$$

and

$$G_2(k,l) = -\frac{l^3(3k-5)}{k+1}\binom{k}{2l}\binom{k+1}{2l}\binom{2l}{l}(-1)^{k-l}.$$

Therefore, by (1.17) we have

$$\sum_{k=0}^{n-1} k(3k+2)g_k(-1) = \sum_{k=0}^{n-1}\sum_{l=0}^{k} F(k,l) = \sum_{l=0}^{n-1} G_1(n,l),$$

which yields (4.1).

(ii) Next we show (4.2). For

$$F(k,l) = k^2(4k+3)\binom{k}{l}^2\binom{2l}{l}2^l,$$

we find that (1.16) holds with

$$G_1(k,l) = -\frac{k2^l}{6(l+1)^2}\binom{k-1}{l}\binom{k}{l}\binom{2l}{l}Q(k,l)$$

and

$$\begin{aligned} G_2(k,l) = &\frac{kl2^l}{6(k+1)}\binom{k}{l}\binom{k+1}{l}\binom{2l}{l} \\ &\times (44k^2l + 6kl^2 - 14l^3 + 66k^2 - 14kl + 43l^2 + 66k + 29l). \end{aligned}$$

Then (4.2) is valid since

$$\sum_{k=0}^{n-1} k^2(4k+3)g_k(2) = \sum_{k=0}^{n-1}\sum_{l=0}^{k} F(k,l) = \sum_{l=0}^{n-1} G_1(n,l)$$

by (1.17). □

Proof of Theorem 1.3. (i) We first prove (1.12) for any $n \in \mathbb{Z}^+$. In view of (4.1) and noting that

$$\frac{n}{2(2l+1)}\binom{n}{2l}\binom{2l}{l}l = \frac{ln}{n+1}\binom{n+1}{2l+1}\binom{2l-1}{l} \equiv 0 \pmod{n}$$

for all $l = 0, \ldots, n-1$, we have

$$\begin{aligned} &2\sum_{k=0}^{n-1} k(3k+2)g_k(-1) \\ \equiv&\frac{n(3n^3-14n^2+11n)}{2(n+1)}\sum_{l=0}^{n-1}(-1)^{n-l}\binom{n-1}{2l}\binom{n+1}{2l+1}\binom{2l}{l} \equiv 0 \pmod{n}. \end{aligned} \tag{4.3}$$

Substituting $n+1$ for n in the above formula we then get

$$2\sum_{k=0}^{n} k(3k+2)g_k(-1) \equiv 0 \pmod{n+1}. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain the desired (1.12).

Next we show (1.14) for any prime $p > 3$. In light of (4.1),

$$\sum_{k=0}^{p-1} k(3k+2)g_k(-1) = \frac{p}{4}\sum_{l=0}^{p-1} \frac{(-1)^{p-l}}{2l+1}\binom{p-1}{2l}\binom{p}{2l}\binom{2l}{l}P(p,l).$$

Note that $P(p,0) \equiv 0 \pmod p$ and also

$$\binom{p}{2l} = \frac{p}{2l}\binom{p-1}{2l-1} \equiv 0 \pmod p \quad \text{for all } l = 1,\ldots,\frac{p-3}{2}.$$

Thus

$$\begin{aligned}\sum_{k=0}^{p-1} k(3k+2)g_k(-1) \equiv& \frac{p}{4}\cdot\frac{(-1)^{p-(p-1)/2}}{p}\binom{p-1}{p-1}\binom{p}{p-1}\binom{p-1}{(p-1)/2}P\left(p,\frac{p-1}{2}\right)\\ =& \frac{p}{4}(-1)^{(p+1)/2}\binom{p-1}{(p-1)/2}P\left(p,\frac{p-1}{2}\right)\\ \equiv& -\frac{p}{4}\left(-50\left(\frac{p-1}{2}\right)^2 - 30\frac{p-1}{2}\right) \equiv -\frac{5}{8}p \pmod{p^2}.\end{aligned}$$

(ii) Now we show (1.13) for any $n \in \mathbb{Z}^+$. By (4.2) we have

$$\begin{aligned}2\sum_{k=0}^{n-1} k^2(4k+3)g_k(2) \equiv& -\frac{n}{3}\sum_{l=0}^{n-1}\frac{2^l}{(l+1)^2}\binom{n-1}{l}\binom{n}{l}\binom{2l}{l}(98l^2+77l)\\ \equiv& -\frac{n}{3}\sum_{l=0}^{n-1}\frac{2^l}{(l+1)^2}\binom{n-1}{l}\binom{n}{l}\binom{2l}{l}(2l^2+2l)\\ =& -\frac{2n^2}{3}\sum_{l=0}^{n-1}\frac{2^l}{l+1}\binom{n-1}{l}\binom{n-1}{n-l}\binom{2l}{l}\\ \equiv& 0 \pmod n\end{aligned}$$

since the Catalan number $C_l = \binom{2l}{l}/(l+1)$ is always integral. Replacing n by $n+1$, we get

$$2\sum_{k=0}^{n} k^2(4k+3)g_k(2) \equiv 0 \pmod{n+1}.$$

Therefore

$$a_n := \frac{2}{n(n+1)}\sum_{k=0}^{n} k^2(4k+3)g_k(2) \in \mathbb{Z}.$$

By (4.2), we have

$$\begin{aligned} a_n =& -\frac{1}{3n}\sum_{l=0}^{n}\frac{2^l}{l+1}\binom{n}{l}\binom{n+1}{l}C_lQ(n+1,l) \\ =& -\frac{Q(n+1,0)}{3n}-\frac{n+1}{3}Q(n+1,1) \\ &-\frac{2}{3}\sum_{1<l\leqslant n}\frac{2^{l-1}}{l(l+1)}\binom{n-1}{l-1}\binom{n+1}{l}C_lQ(n+1,l). \end{aligned}$$

If $l>1$, then $2^{l-1}/(l(l+1))=2^{l-1}/l-2^{l-1}/(l+1)$ is a 2-adic integer. Also, $Q(n+1,0)\equiv n \pmod{2n}$ and $Q(n+1,1)\equiv 0 \pmod 2$. So we have $a_n\equiv 1 \pmod 2$.

Finally we prove (1.15) for any prime $p>3$. Clearly,

$$\frac{p}{(l+1)^2}\binom{p}{l}=\frac{p^2}{l(l+1)^2}\binom{p-1}{l-1}\equiv 0 \pmod{p^2}\quad\text{for all } l=1,\ldots,p-2.$$

Thus, by applying (4.2) with $n=p$ we get

$$\begin{aligned} &\sum_{k=0}^{p-1}k^2(4k+3)g_k(2) \\ \equiv& -\frac{p}{6}\left(Q(p,0)+\frac{2^{p-1}}{p^2}\binom{p}{p-1}\binom{2(p-1)}{p-1}Q(p,p-1)\right) \\ \equiv& -\frac{p}{6}\left(21+\frac{2^{p-1}}{2p-1}\binom{2p-1}{p}(98(p-1)^2+77(p-1)+21)\right)\equiv\frac{7}{2}p \pmod{p^2}, \end{aligned}$$

where we have used that $2^{p-1}\equiv 1 \pmod p$ (by Fermat's little theorem) and $\binom{2p-1}{p}\equiv 1 \pmod p$.

In view of the above, we have completed the proof of Theorem 1.3. □

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References

[1] P. Barrucand, *A combinatorial identity, Problem 75–4*, SIAM Rev. **17** (1975), 168.

[2] D. Callan, *A combinatorial interpretation for an identity of Barrucand*, J. Integer Seq. **11** (2008), Article 08.3.4, 3pp (electronic).

[3] W.Y.C. Chen, Q.-H. Hou, and Y.-P. Mu, *A telescoping method for double summations*, J. Comput. Appl. Math. **196** (2006) 553–566.

[4] H.W. Gould, *Combinatorial Identities*, Morgantown Printing and Binding Co., 1972.

[5] V.J.W. Guo, *Proof of two conjectures of Sun on congruences for Franel numbers*, Integral Transforms Spec. Funct. **24** (2013), 532–539.

[6] V.J.W. Guo and J.-C. Liu, *Proof of some conjectures of Z.-W. Sun on the divisibility of certain double-sums*, Int. J. Number Theory **12**(2016), 615–623.
[7] V.J.W. Guo, G.-S. Mao and H. Pan, *Proof of a conjecture involving Sun polynomials*, J. Diference Equ. Appl. **22**(2016), 1184–1197.
[8] Y.-P. Mu and Z.-W. Sun, *Telescoping method and congruences for double sums*, Int. J. Number Theory **14** (2018), 143–165.
[9] M. Petkovsěk, H. S. Wilf and D. Zeilberger, *A=B*, A K Peters, Wellesley, 1996.
[10] N.J.A. Sloane, Sequence A000172 in OEIS (On-Line Encyclopedia of Integer Sequences), `http://oeis.org/A000172`.
[11] V. Strehl, *Binomial identities–combinatorial and algorithmic aspects*, Discrete Math. **136**(1994), 309–346.
[12] Z.-W. Sun, *Congruences for Franel numbers*, Adv. in Appl. Math.**51**(2013), 524–535.
[13] Z.-W. Sun, *Connections between* $p = x^2 + 3y^2$ *and Franel numbers*, J. Number Theory **133**(2013), 2914-2928.
[14] Z.-W. Sun, *Conjectures and results on* x^2 *mod* p^2 *with* $4p = x^2 + dy^2$, in: Number Theory and Related Area (eds., Y. Ouyang, C. Xing, F. Xu and P. Zhang), Adv. Lect. Math. 27, Higher Education Press and International Press, Beijing-Boston, 2013, pp. 149-197.
[15] Z.-W. Sun, *Two new kinds of numbers and related divisibility results*, Colloq. Math., to appear. Preprint version: arXiv:1408.5381.
[16] Z.-W. Sun, *Congruences involving* $g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k$, Ramanujan J. **40**(2016), 511–533.
[17] Z.-W. Sun, *Supercongruences involving Lucas sequences*, preprint, 2016, arXiv:1610.03384.

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