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NEW CONGRUENCES INVOLVING PRODUCTS OF TWO BINOMIAL COEFFICIENTS

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Abstract. Let $p>3$ be a prime and let a be a positive integer. We show that if $p\equiv 1 \pmod 4$ or $a>1$ then

$$\sum_{k=0}^{\lfloor \frac{3}{4}p^a\rfloor}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p^a}\right) \pmod{p^3}$$

with $(-)$ the Jacobi symbol, which confirms a conjecture of Z.-W. Sun. We also establish the following new congruences:

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\equiv\left(\frac{p}{3}\right)\frac{2^p+1}{3} \pmod{p^2},$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{6k}{3k}\binom{3k}{k}}{(2k+1)432^k}\equiv\left(\frac{p}{3}\right)\frac{3^p+1}{4} \pmod{p^2},$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{4k}{2k}\binom{2k}{k}}{(2k+1)64^k}\equiv\left(\frac{-1}{p}\right)2^{p-1} \pmod{p^2}.$$

1. Introduction

Let $p>3$ be a prime. In 1862 J. Wolstenholme [W] established the well-known congruence

$$\frac{1}{2}\binom{2p}{p}=\binom{2p-1}{p}\equiv 1 \pmod{p^3}.$$

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In 2006, as a corollary of the combinatorial identity

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{n}\binom{2k}{k-2l+m} = \sum_{k=0}^{l}\binom{l}{k}\binom{2k}{n}\binom{n-l}{m+n-3k-l}$$

with $l,m,n\in\mathbb{N}=\{0,1,2,\ldots\}$, H. Pan and Z.-W. Sun [PS06] obtained that

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\pmod p\quad\text{for all } d=0,\ldots,p,$$

where $(-)$ is the Jacobi symbol. Later Sun and R. Tauraso [ST] showed further that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\equiv\left(\frac{p}{3}\right)\pmod{p^2}.$$

For a general integer $m\not\equiv 0\pmod p$, Sun [Su10] and [Su13a] determined

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}\quad\text{and}\quad\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{m^k}$$

modulo p^2 via Lucas sequences.

Now we turn to congruences for combinatorial sums with summands involving products of two binomial coefficients. For any $k\in\mathbb{N}$, we clearly have

$$\binom{-1/2}{k}^2=\frac{\binom{2k}{k}^2}{16^k},\quad\binom{-1/3}{k}\binom{-2/3}{k}=\frac{\binom{2k}{k}\binom{3k}{k}}{27^k},$$
$$\binom{-1/4}{k}\binom{-3/4}{k}=\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k},\quad\binom{-1/6}{k}\binom{-5/6}{k}=\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}.$$

Let $p>3$ be a prime. In 2003, via the Gross-Koblitz formula and the p-adic Γ-function, E. Mortenson [M1, M2] proved the following congruences

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2},\ \sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}.$$

These supercongruences were conjectured in [RV] motivated by the p-adic analogues of Gaussian hypergeometric series and the Calabi-Yau manifolds. In 2011 Sun [Su11] showed further that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)-p^2E_{p-3}\pmod{p^3}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)+p^2E_{p-3}\pmod{p^3},\tag{1.1}$$

where $E_0,E_1,E_2,\ldots$ are the Euler numbers defined by

$$\frac{2}{e^x+e^{-x}}=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}\quad\left(|x|<\frac{\pi}{2}\right).$$

It is well known that

$$E_0=1,\ \text{and}\ E_n=-\sum_{k=1}^{\lfloor n/2\rfloor}\binom{n}{2k}E_{n-2k}\ \text{for}\ n=1,2,3,\ldots.$$

The series $\sum_{k=0}^{\infty}\binom{2k}{k}^2/((2k+1)16^k)$ can be evaluated via `Mathematica`, namely we have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2}{(2k+1)16^k}=\frac{4G}{\pi},$$

where G is the Catalan constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$. Motivated by this, Sun [Su14] determined

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)16^k}\quad\text{and}\quad\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^2}{(2k+1)16^k}$$

modulo p^3 for any prime $p>3$. Sun [Su11, Conjecture 5.12] also conjectured for any prime $p>3$ that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(2k+1)27^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(2k+1)64^k}\equiv\left(\frac{-1}{p}\right)-3p^2E_{p-3}\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(2k+1)432^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2},$$

which were confirmed by Z.-H. Sun [S16] via the WZ method. Note that $\binom{4k}{2k}/(2k+1)$ is the $2k$-th Catalan number C_{2k} and $\binom{3k}{k}/(2k+1)$ is the k-th second-order Catalan number $C_k^{(2)}$. The Catalan numbers play important roles in enumerative combinatorics and they have lots of combinatorial interpretations.

Let $p = 2n+1$ be an odd prime, and let $\lambda \in \{0, 1, \ldots, p-1\}$. It is easy to see that

$$\begin{aligned}&\sum_{x=0}^{p-1}\left(\frac{x(x-1)(x-\lambda)}{p}\right)\\ \equiv&\sum_{x=0}^{p-1}x^n(x-1)^n(x-\lambda)^n\equiv(-1)^{n-1}\sum_{k=0}^{n}\binom{n}{k}^2\lambda^k\\ \equiv&(-1)^{n-1}\sum_{k=0}^{n}\binom{-1/2}{k}^2\lambda^k=-\left(\frac{-1}{p}\right)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\lambda^k \pmod p.\end{aligned}$$

(See [A, Theorem 2] for this basic fact and [Su13, Theorem 1.1] for an extension of this.) As $p \mid \binom{2k}{k}$ for all $k = (p+1)/2, \ldots, p-1$, we also have

$$\sum_{k=0}^{\lfloor\frac34 p\rfloor}\frac{\binom{2k}{k}^2}{16^k}\lambda^k\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\lambda^k\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\lambda^k \pmod{p^2}.$$

Thus, such sums are related to the number of the points on the cubic curve $\mathbb{E}_p(\lambda):\ y^2 = x(x-1)(x-\lambda)$ over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. In view of (1.1), it is natural to determine $\sum_{k=0}^{\lfloor 3p/4\rfloor}\binom{2k}{k}^2/16^k \bmod p^3$. There are some earlier congruences involving sums of the type $\sum_{k=1}^{\lfloor 3p/4\rfloor} a_k$; for example, Sun [Su95] showed that

$$\sum_{k=1}^{\lfloor\frac34 p\rfloor}\frac{(-1)^{k-1}}{k}\equiv\sum_{k=1}^{(p-1)/2}\frac{1}{k2^k} \pmod p$$

for any odd prime p, and Pan and Sun [PS14] proved the supercongruence

$$\sum_{k=0}^{\lfloor\frac34 p\rfloor}\frac{\binom{2k}{k}}{4^k}\equiv\left(\frac{2}{p}\right) \pmod{p^2}$$

for any prime $p \equiv 1 \pmod 4$.

With the above backgrounds, we first establish the following result.

Theorem 1.1. *Let p be any odd prime.*

(i) *We have*

$$\sum_{k=0}^{\lfloor 3p/4\rfloor}\frac{\binom{2k}{k}^2}{16^k}\equiv\begin{cases}1 \pmod{p^3} & \text{if } p\equiv 1 \pmod 4,\\ -1+p^2/(2\binom{(p-3)/2}{(p-3)/4}^2) \pmod{p^3} & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{1.2}$$

(ii) *For each $a=2,3,4,\ldots$, we have*

$$\sum_{k=0}^{\lfloor \frac34 p^a\rfloor}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p^a}\right) \pmod{p^3}. \tag{1.3}$$

Remark 1.1. Part (i) in the case $p\equiv 1 \pmod 4$ and part (ii) were conjectured by Sun [Su11]. A more challenging conjecture of Sun [Su11, Conjecture 1.3] involving products of three binomial coefficients states that for any prime $p>3$ and positive integer a with $p^a\equiv 1 \pmod 3$ we have

$$\sum_{k=0}^{\lfloor \frac23 p^a\rfloor}(21k+8)\binom{2k}{k}^3\equiv 8p^a \pmod{p^{a+4}}.$$

Our second theorem is as follows.

Theorem 1.2. *Let $p>3$ be a prime. Then we have*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\equiv\left(\frac{p}{3}\right)\frac{2^p+1}{3} \pmod{p^2}, \tag{1.4}$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{6k}{3k}\binom{3k}{k}}{(2k+1)432^k}\equiv\left(\frac{p}{3}\right)\frac{3^p+1}{4} \pmod{p^2}, \tag{1.5}$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{4k}{2k}\binom{2k}{k}}{(2k+1)64^k}\equiv\left(\frac{-1}{p}\right)2^{p-1} \pmod{p^2}. \tag{1.6}$$

Remark 1.2. We are also able to show the congruence

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}}{(2k+1)27^k}\equiv\left(\frac{p}{3}\right)(3^p+2-2^{p+1}) \pmod{p^2}$$

for any prime $p>3$.

Our proof of Theorem 1.1 given in the next section is somewhat sophisticated. It utilizes Kummer's classical theorem on the p-adic valuation of a binomial coefficient, a curious identity for $\sum_{k=0}^n\binom{n}{k}^{-2}$ given in [SWZ], and a congruence of Sun [Su11] on $\sum_{k=1}^{(p-1)/2}4^k/(k^2\binom{2k}{k})$ modulo p. Our proof of Theorem 1.2 presented in Section 3 employs two identities (3.3) and (3.4) recently observed in [S16].

2. Proof of Theorem 1.1

Lemma 2.1. (Sun [Su11, (1.4)]) *For any prime $p>3$ we have*

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{k^2\binom{2k}{k}}\equiv(-1)^{(p-1)/2}4E_{p-3}\pmod{p}. \tag{2.1}$$

Proof of Theorem 1.1(i). In view of (1.1), (1.2) has the following equivalent form:

$$\sum_{k=(p+1)/2}^{\lfloor 3p/4\rfloor}\frac{\binom{2k}{k}^2}{16^k}\equiv -p^2E_{p-3}+\frac{1-(-1)^{(p-1)/2}}{2}\cdot\frac{p^2}{2\binom{(p-3)/2}{\lfloor p/4\rfloor}^2}\pmod{p^3}. \tag{2.2}$$

By [Su11, Lemma 2.1],

$$k\binom{2k}{k}\binom{2(p-k)}{p-k}\equiv(-1)^{\lfloor 2k/p\rfloor-1}2p\pmod{p^2}\ \text{for all}\ k=1,\ldots,p-1.$$

Thus

$$\sum_{k=(p+1)/2}^{\lfloor 3p/4\rfloor}\frac{\binom{2k}{k}^2}{16^k}\equiv\sum_{k=(p+1)/2}^{\lfloor 3p/4\rfloor}\frac{4p^2}{k^2\binom{2(p-k)}{p-k}^2 16^k}=\sum_{j=\lfloor p/4\rfloor+1}^{(p-1)/2}\frac{4p^2}{(p-j)^2\binom{2j}{j}^2 16^{p-j}}$$
$$\equiv\frac{p^2}{4}\sum_{j=\lfloor p/4\rfloor+1}^{(p-1)/2}\frac{16^j}{j^2\binom{2j}{j}^2}\pmod{p^3}$$

and hence we have reduced (2.2) to the following simpler form

$$\sum_{k=\lfloor n/2\rfloor+1}^{n}\frac{16^k}{k^2\binom{2k}{k}^2}\equiv -4E_{p-3}+\frac{1-(-1)^n}{\binom{n-1}{\lfloor n/2\rfloor}^2}\pmod{p}, \tag{2.3}$$

where $n=(p-1)/2$.

For each $k=0,\ldots,n$, clearly

$$\binom{n}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\pmod{p}.$$

Thus

$$\sum_{k=\lfloor n/2\rfloor+1}^{n}\frac{16^k}{k^2\binom{2k}{k}^2}\equiv\sum_{k=\lfloor n/2\rfloor+1}^{n}\frac{1}{k^2\binom{n}{k}^2}\equiv 4\sum_{k=\lfloor n/2\rfloor+1}^{n}\frac{1}{\binom{n-1}{k-1}^2}\pmod{p}.$$

Note that

$$\sum_{k=\lfloor n/2\rfloor+1}^{n}\frac{1}{\binom{n-1}{k-1}^2}=\sum_{k=\lfloor n/2\rfloor}^{n-1}\frac{1}{\binom{n-1}{k}^2}=\frac12\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}^2}+\frac{1-(-1)^n}{4\binom{n-1}{\lfloor n/2\rfloor}^2}$$

and

$$\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}^2}=\frac{2n^2}{n+1}\sum_{k=1}^{n}\frac{1}{k\binom{2n+1-k}{n-k}} \tag{2.4}$$

(cf. [SWZ]). So we have

$$\begin{aligned}&\sum_{k=\lfloor n/2\rfloor+1}^{n}\frac{16^k}{k^2\binom{2k}{k}^2}-\frac{1-(-1)^n}{\binom{n-1}{\lfloor n/2\rfloor}^2}\\ \equiv&\frac{4n^2}{n+1}\sum_{k=1}^{n}\frac{1}{k\binom{2n+1-k}{n-k}}\equiv 2\sum_{k=1}^{n}\frac{1}{k\binom{-k}{n-k}} \pmod p.\end{aligned}$$

Observe that

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k\binom{-k}{n-k}}=&\sum_{k=1}^{n}\frac{(-1)^{n-k}}{k\binom{n-1}{k-1}}=n\sum_{k=1}^{n}\frac{(-1)^{n-k}}{k^2\binom{n}{k}}\\ \equiv&\frac{(-1)^{n-1}}{2}\sum_{k=1}^{n}\frac{4^k}{k^2\binom{2k}{k}} \pmod p.\end{aligned}$$

Therefore, with the help of Lemma 2.1, we finally obtain

$$\sum_{k=\lfloor n/2\rfloor+1}^{n}\frac{16^k}{k^2\binom{2k}{k}^2}-\frac{1-(-1)^n}{\binom{n-1}{\lfloor n/2\rfloor}^2}\equiv(-1)^{n-1}\sum_{k=1}^{n}\frac{4^k}{k^2\binom{2k}{k}}\equiv -4E_{p-3} \pmod p.$$

This proves (2.3) and hence (1.2) follows. □

Now we give a lemma which is a natural extension of (1.1).

Lemma 2.2. *Let $p>3$ be a prime and let a be any positive integer. Then*

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p^a}\right)+\left(\frac{-1}{p^{a-1}}\right)p^2E_{p-3} \pmod{p^3}. \tag{2.5}$$

Proof. Theorem 1.2 of Sun [Su13] states that for any $d=0,\ldots,(p-1)/2$ we have

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{2k}{k+d}}{16^k}\equiv\left(\frac{-1}{p}\right)+\frac{(-1)^d}{4}p^2E_{p-3}\left(d+\frac12\right) \pmod{p^3},$$

where $E_n(x)$ denotes the Euler polynomial of degree n given by

$$E_n(x)=\sum_{k=0}^{n}\frac{E_k}{2^k}\left(x-\frac12\right)^{n-k}.$$

In the case $d=0$ this yields (1.1). Modifying this proof of (1.1) slightly we immediately get (2.5). □

In 1852, Kummer (cf. [R, pp. 22-24]) proved that for any $m,n\in\mathbb{N}$ the p-adic valuation of the binomial coefficient $\binom{m+n}{m}$ is equal to the number of *carry-overs* when performing the addition of m and n written in base p.

Lemma 2.3. *Let p be an odd prime and let $a\in\{1,2,3,\ldots\}$. For any $k=1,2,\ldots,(p^a-1)/2$, we have*

$$\mathrm{ord}_p\binom{p^a-k}{\frac{p^a-1}{2}-k}\le a-1,$$

where $\mathrm{ord}_p(x)$ denotes the p-adic valuation of a p-adic integer x.

Proof. It is well known that

$$\mathrm{ord}_p(n!)=\sum_{j=1}^{\infty}\left\lfloor\frac{n}{p^j}\right\rfloor.$$

Thus

$$\mathrm{ord}_p\binom{p^a-k}{\frac{p^a-1}{2}-k}=\sum_{j=1}^{a-1}\left(\left\lfloor\frac{p^a-k}{p^j}\right\rfloor-\left\lfloor\frac{(p^a+1)/2}{p^j}\right\rfloor-\left\lfloor\frac{(p^a-1)/2-k}{p^j}\right\rfloor\right)$$

does not exceed $a-1$ as each summand in the sum is at most one. This concludes the proof. $\square$

Proof of Theorem 1.1(ii). In view of Lemma 2.2, we just need to verify that

$$\sum_{k=(p^a+1)/2}^{\lfloor 3p^a/4\rfloor}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p^{a-1}}\right)p^2E_{p-3}\pmod{p^3}.\tag{2.6}$$

Let k and l be positive integers with $k+l=p^a$ and $0<l<p^a/2$. Then

$$\frac{\binom{2k}{k}^2}{\binom{2p^a-2}{p^a-1}^2}=\frac{(2p^a-2l)!^2}{(2p^a-2)!^2}\left(\frac{(p^a-1)!}{(p^a-l)!}\right)^4=\frac{\prod_{0<i<l}(p^a-i)^4}{\prod_{1<j<2l}(2p^a-j)^2}$$

and hence

$$\frac{\binom{2k}{k}^2}{\binom{2p^a-2}{p^a-1}^2}\cdot\frac{(2l-1)!^2}{(l-1)!^4}=\frac{\prod_{0<i<l}(1-p^a/i)^4}{\prod_{1<j<2l}(1-2p^a/j)^2}\equiv 1\pmod{p}.$$

Note that

$$\binom{2p^a-2}{p^a-1}^2=p^{2a}\prod_{j=2}^{p^a-1}\left(\frac{2p^a-j}{j}\right)^2\equiv p^{2a}\pmod{p^{2a+1}}$$

and

$$\binom{2k}{k}^2 = \binom{p^a + (2k - p^a)}{0p^a + k}^2 \equiv \binom{2k - p^a}{k}^2 = 0 \pmod{p^2}$$

by Lucas' theorem. So we have

$$\frac{l^2}{4}\binom{2l}{l}^2 = \frac{(2l-1)!^2}{(l-1)!^4} \not\equiv 0 \pmod{p^{2a}}$$

and

$$\binom{2k}{k}^2 \equiv p^{2a}\frac{(l-1)!^4}{(2l-1)!^2} = \frac{4p^{2a}}{l^2\binom{2l}{l}^2} \pmod{p^3}.$$

Therefore

$$\begin{aligned}\sum_{k=(p^a+1)/2}^{\lfloor 3p^a/4\rfloor}\frac{\binom{2k}{k}^2}{16^k} \equiv& \sum_{k=(p^a+1)/2}^{\lfloor 3p^a/4\rfloor}\frac{4p^{2a}}{16^k(p^a-k)^2\binom{2(p^a-k)}{p^a-k}^2}\\ \equiv&\frac{p^{2a}}{4}\sum_{l=\lfloor p^a/4\rfloor+1}^{(p^a-1)/2}\frac{16^l}{l^2\binom{2l}{l}^2} \pmod{p^3}.\end{aligned}$$

For $k = 1, \ldots, (p^a - 1)/2$, clearly

$$\begin{aligned}\frac{\binom{(p^a-1)/2}{k}}{\binom{2k}{k}/(-4)^k} =& \frac{\binom{(p^a-1)/2}{k}}{\binom{-1/2}{k}} = \prod_{j=0}^{k-1}\frac{(p^a-1)/2-j}{-1/2-j}\\ =&\prod_{j=0}^{k-1}\left(1-\frac{p^a}{2j+1}\right) \equiv 1 \pmod{p}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=(p^a+1)/2}^{\lfloor 3p^a/4\rfloor}\frac{\binom{2k}{k}^2}{16^k} \equiv&\frac{p^{2a}}{4}\sum_{k=\lfloor p^a/4\rfloor+1}^{(p^a-1)/2}\frac{1}{k^2\binom{(p^a-1)/2}{k}^2}\\ \equiv&p^{2a}\sum_{k=\lfloor p^a/4\rfloor+1}^{(p^a-1)/2}\frac{1}{\binom{(p^a-3)/2}{k-1}^2} \pmod{p^3}.\end{aligned}$$

So (2.6) is reduced to

$$p^{2a-2}\sum_{k=\lfloor p^a/4\rfloor}^{(p^a-3)/2}\frac{1}{\binom{(p^a-3)/2}{k}^2} \equiv -\left(\frac{-1}{p^{a-1}}\right)E_{p-3} \pmod{p}. \tag{2.7}$$

If $p^a\equiv 1\pmod 4$, then $(p^a-3)/2$ is odd and hence

$$\sum_{k=\lfloor p^a/4\rfloor}^{(p^a-3)/2}\frac{1}{\binom{(p^a-3)/2}{k}^2}=\frac12\sum_{k=0}^{(p^a-3)/2}\frac{1}{\binom{(p^a-3)/2}{k}^2}.$$

If $p^a\equiv 3\pmod 4$, then $a\in\{3,5,\ldots\}$ and

$$\sum_{k=\lfloor p^a/4\rfloor}^{(p^a-3)/2}\frac{1}{\binom{(p^a-3)/2}{k}^2}=\frac12\sum_{k=0}^{(p^a-3)/2}\frac{1}{\binom{(p^a-3)/2}{k}^2}+\frac12\cdot\frac{1}{\binom{(p^a-3)/2}{(p^a-3)/4}^2}.$$

In the case $p^a\equiv 3\pmod 4$, as the fractional parts of $(p^a-3)/(2p)$ and $(p^a-3)/(4p)$ are $(p-3)/(2p)$ and $(p-3)/(4p)$ respectively, we have

$$\left\lfloor\frac{(p^a-3)/2}{p}\right\rfloor=2\left\lfloor\frac{(p^a-3)/4}{p}\right\rfloor$$

and hence

$$\mathrm{ord}_p\binom{(p^a-3)/2}{(p^a-3)/4}^2=2\sum_{j=1}^{a-1}\left(\left\lfloor\frac{(p^a-3)/2}{p^j}\right\rfloor-2\left\lfloor\frac{(p^a-3)/4}{p^j}\right\rfloor\right)<2a-2.$$

No matter $p^a\equiv 1\pmod 4$ or not, we always have

$$p^{2a-2}\sum_{k=\lfloor p^a/4\rfloor}^{(p^a-3)/2}\frac{1}{\binom{(p^a-3)/2}{k}^2}\equiv\frac{p^{2a-2}}{2}\sum_{k=0}^{(p^a-3)/2}\frac{1}{\binom{(p^a-3)/2}{k}^2}\pmod p.$$

So (2.7) has the following equivalent form:

$$p^{2a-2}\sum_{k=0}^{(p^a-3)/2}\frac{1}{\binom{(p^a-3)/2}{k}^2}\equiv-2\left(\frac{-1}{p^{a-1}}\right)E_{p-3}\pmod p.\tag{2.8}$$

The identity (2.4) with $n=(p^a-1)/2$ yields that

$$\sum_{k=0}^{(p^a-3)/2}\frac{1}{\binom{(p^a-3)/2}{k}^2}=\frac{2((p^a-1)/2)^2}{(p^a+1)/2}\sum_{k=1}^{(p^a-1)/2}\frac{1}{k\binom{p^a-k}{(p^a-1)/2-k}}.$$

So (2.8) is reduced to

$$p^{2a-2}\sum_{k=1}^{(p^a-1)/2}\frac{1}{k\binom{p^a-k}{(p^a+1)/2}}\equiv-2\left(\frac{-1}{p^{a-1}}\right)E_{p-3}\pmod p.\tag{2.9}$$

In view of Lemma 2.3, if $1\leqslant k\leqslant(p^a-1)/2$ and $p^{a-1}\nmid k$, then

$$\frac{p^{2a-2}}{k\binom{p^a-k}{(p^a+1)/2}}\equiv 0\pmod p.$$

Thus

$$\begin{aligned}p^{2a-2}\sum_{k=1}^{(p^a-1)/2}\frac{1}{k\binom{p^a-k}{(p^a+1)/2}}\equiv&p^{2a-2}\sum_{j=1}^{(p-1)/2}\frac{1}{p^{a-1}j\binom{p^a-p^{a-1}j}{(p^a+1)/2}}\\=&\frac{p^a+1}{2}\sum_{j=1}^{(p-1)/2}\frac{1}{j(p-j)\binom{p^a-p^{a-1}j-1}{(p^a-1)/2}}\\\equiv&-\frac{1}{2}\sum_{j=1}^{(p-1)/2}\frac{1}{j^2\binom{p^a-p^{a-1}j-1}{(p^a-1)/2}}\pmod p.\end{aligned}$$

For each $j=1,\ldots,(p-1)/2$, by Lucas' theorem we have

$$\begin{aligned}\binom{p^{a-1}(p-j)-1}{(p^a-1)/2}=&\binom{p^{a-1}(p-1-j)+p^{a-1}-1}{p^{a-1}(p-1)/2+(p^{a-1}-1)/2}\\\equiv&\binom{p-1-j}{(p-1)/2}\binom{p^{a-1}-1}{(p^{a-1}-1)/2}\\\equiv&(-1)^{(p^{a-1}-1)/2}\binom{p-j-1}{(p-1)/2}\pmod p,\end{aligned}$$

also

$$\begin{aligned}\binom{p-j-1}{(p-1)/2}=&\binom{p-1-j}{(p-1)/2-j}=(-1)^{(p-1)/2-j}\binom{-p+(p-1)/2}{(p-1)/2-j}\\\equiv&(-1)^{(p-1)/2-j}\binom{(p-1)/2}{j}\equiv(-1)^{(p-1)/2-j}\binom{-1/2}{j}\\=&(-1)^{(p-1)/2}\frac{\binom{2j}{j}}{4^j}\pmod p.\end{aligned}$$

Therefore

$$\begin{aligned}p^{2a-2}\sum_{k=1}^{(p^a-1)/2}\frac{1}{k\binom{p^a-k}{(p^a+1)/2}}\equiv&\frac{(-1)^{(p^{a-1}+1)/2}}{2}\sum_{j=1}^{(p-1)/2}\frac{1}{j^2\binom{p-j-1}{(p-1)/2}}\\\equiv&\frac{(-1)^{(p^{a-1}+1)/2}}{2}(-1)^{(p-1)/2}\sum_{j=1}^{(p-1)/2}\frac{4^j}{j^2\binom{2j}{j}}\pmod p.\end{aligned}$$

This, together with (2.1), yields the desired (2.9).

The proof of Theorem 1.1(ii) is now complete. □

3. Proof of Theorem 1.2

Lemma 3.1. *Let $p>3$ be a prime, and let $m\in\{1,2,\ldots,(p-1)/2\}$. For any p-adic integer t, we have*

$$\binom{m+pt-1}{(p-1)/2}\binom{-1-pt-m}{(p-1)/2}\equiv\frac{pt}{m}\pmod{p^2}. \tag{3.1}$$

Proof. Since

$$\begin{aligned}\binom{m+pt-1}{(p-1)/2}=&\frac{\prod_{r=0}^{m-1}(pt+r)\times\prod_{s=1}^{(p-1)/2-m}(pt-s)}{((p-1)/2)!}\\ \equiv&\frac{(m-1)!pt(-1)^{(p-1)/2-m}((p-1)/2-m)!}{((p-1)/2)!}\pmod{p^2},\end{aligned}$$

and

$$\begin{aligned}\binom{-m-pt-1}{(p-1)/2}=&\frac{\prod_{j=1}^{(p-1)/2}(-m-pt-j)}{((p-1)/2)!}\\ \equiv&\frac{(-1)^{(p-1)/2}(m+1)(m+2)\cdots(m+(p-1)/2)}{((p-1)/2)!}\pmod{p},\end{aligned}$$

we have

$$\begin{aligned}&\binom{m+pt-1}{(p-1)/2}\binom{-m-pt-1}{(p-1)/2}\\ \equiv&\frac{pt(m-1)!(-1)^m((p-1)/2-m)!(m+1)(m+2)\cdots(m+(p-1)/2)}{((p-1)/2)!^2}\\ =&\frac{pt}{m}\frac{(-1)^m((p-1)/2-m)!(m+(p-1)/2)!}{((p-1)/2)!^2}=\frac{pt}{m}(-1)^m\frac{\binom{p-1}{(p-1)/2}}{\binom{p-1}{(p-1)/2+m}}\\ \equiv&\frac{pt}{m}(-1)^m(-1)^{(p-1)/2}(-1)^{(p-1)/2+m}=\frac{pt}{m}\pmod{p^2}.\end{aligned}$$

This concludes the proof. □

Remark 3.1. Let $p>3$ be a prime and let $m\in\{(p+1)/2,\ldots,p-1\}$. For any p-adic integer t, by Lemma 3.1 we have

$$\begin{aligned}&\binom{m+pt-1}{(p-1)/2}\binom{-1-pt-m}{(p-1)/2}\\ =&\binom{(m-p)+p(t+1)-1}{(p-1)/2}\binom{-1-p(t+1)-(m-p)}{(p-1)/2}\\ \equiv&\frac{p(t+1)}{m-p}\equiv\frac{p(t+1)}{m}\pmod{p^2}.\end{aligned}$$

Lemma 3.2. *Let $p > 3$ be a prime. For $k \in \{1, 2, \ldots, p-1\}$ and p-adic integer t, we have*

$$\binom{pt}{k}\binom{-1-pt}{k} \equiv -\frac{p^2t^2}{k^2} - \frac{pt}{k} \pmod{p^3}. \tag{3.2}$$

Proof. This is almost trivial. In fact,

$$\begin{aligned}\binom{pt}{k}\binom{-1-pt}{k} =& \frac{pt}{pt-k}\binom{-1+pt}{k}\binom{-1-pt}{k} \\ \equiv& \frac{pt}{pt-k}\binom{-1}{k}^2 = \frac{pt(p^2t^2+ptk+k^2)}{(pt)^3-k^3} \\ \equiv& -\frac{p^2t^2}{k^2} - \frac{pt}{k} \pmod{p^3}.\end{aligned}$$

This proves (3.2). □

Recall that those $H_n = \sum_{0<k\leqslant n} 1/k$ with $n \in \mathbb{N}$ are called harmonic numbers. If a prime p does not divide an integer a, then we let $q_p(a)$ denote the Fermat quotient $(a^{p-1}-1)/p$.

Lemma 3.3. (Lemma [L]). *For any prime $p > 3$, we have*

$$H_{\lfloor p/2\rfloor} \equiv -2q_p(2) \pmod{p},\ H_{\lfloor p/4\rfloor} \equiv -3q_p(2) \pmod{p},$$
$$H_{\lfloor p/3\rfloor} \equiv -\frac{3}{2}q_p(3) \pmod{p} \text{ and } H_{\lfloor p/6\rfloor} \equiv -2q_p(2) - \frac{3}{2}q_p(3) \pmod{p}.$$

For $n \in \mathbb{N}$, define

$$S_n(x) = \sum_{k=0}^{n}\binom{x}{k}\binom{-1-x}{k} \quad\text{and}\quad T_n(x) = \sum_{k=0}^{n}\binom{x}{k}\binom{-1-x}{k}\frac{1+2x}{1+2k}.$$

By [S16, (2.2)] with $a = x+1$ and $b = 0$, we have

$$S_n(x) + S_n(x+1) = 2\binom{x}{n}\binom{-2-x}{n}. \tag{3.3}$$

By [S16, (2.2)] with $b = 2$, we get

$$T_n(x) - T_n(x-1) = 2\binom{x-1}{n}\binom{-x-1}{n}. \tag{3.4}$$

Proof of Theorem 1.2. For any p-adic integer a, we let $\langle a\rangle_p$ denote the least nonnegative integer r with $a \equiv r \pmod{p}$. For convenience, we also set $n = (p-1)/2$.

We first prove (1.4). For any p-adic integer $a \not\equiv 0 \pmod p$, by using (3.3) we get

$$\begin{aligned} &S_n(a) - (-1)^{\langle a\rangle_p} S_n(a - \langle a\rangle_p) \\ =& \sum_{k=0}^{\langle a\rangle_p - 1} (-1)^k (S_n(a-k) + S_n(a-k-1)) \\ =& \sum_{k=0}^{\langle a\rangle_p - 1} (-1)^k 2\binom{a-k-1}{n}\binom{k-a-1}{n} \end{aligned}$$

and hence

$$\begin{aligned} &S_n(a) - (-1)^{\langle a\rangle_p} S_n(pt) \\ =&2\sum_{k=0}^{\langle a\rangle_p - 1} (-1)^k \binom{\langle a\rangle_p + pt - k - 1}{n}\binom{-1-pt-(\langle a\rangle_p - k)}{n}, \end{aligned}$$

where $t := (a - \langle a\rangle_p)/p$. By Lemma 3.2,

$$S_n(pt) = \sum_{k=0}^{n} \binom{pt}{k}\binom{-1-pt}{k} \equiv 1 - \sum_{k=1}^{n} \frac{pt}{k} = 1 - ptH_n \pmod{p^2}.$$

So, with helps of Lemma 3.1 and Remark 3.1, we have

$$S_n(a) - (-1)^{\langle a\rangle_p}(1 - ptH_n) \equiv 2\sum_{k=0}^{\langle a\rangle_p - 1} (-1)^k \frac{p(t+\delta_k)}{\langle a\rangle_p - k} \pmod{p^2}, \quad (3.5)$$

where δ_k takes 1 or 0 according as $\langle a\rangle_p - k > p/2$ or not.

Observe that

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} = \sum_{k=0}^{n} \binom{-1/3}{k}\binom{-2/3}{k} = S_n(a)$$

with $a = -1/3$. Note that

$$\langle a\rangle_p = \begin{cases} (p-1)/3 & \text{if } p \equiv 1 \pmod 3, \\ (2p-1)/3 & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Hence

$$t := \frac{a - \langle a\rangle_p}{p} = \begin{cases} -1/3 & \text{if } p \equiv 1 \pmod 3, \\ -2/3 & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Case 1. $p \equiv 1 \pmod 3$.

In this case, $\langle a\rangle_p = (p-1)/3$, $t = -1/3$, and $\delta_k = 0$ for all $k = 0, \dots, \langle a\rangle_p - 1$. Thus, in view of (3.5), we have

$$\begin{aligned}
&S_n\left(-\frac{1}{3}\right) - (-1)^{(p-1)/3}(1-ptH_n)\\
\equiv&2pt(-1)^{(p-1)/3}\sum_{j=1}^{(p-1)/3}\frac{(-1)^j}{j} = 2pt\left(H_{(p-1)/6} - H_{(p-1)/3}\right) \pmod{p^2}.
\end{aligned}$$

Combining this with Lemma 3.3 and recalling that $t = -1/3$, we immediately obtain the desired congruence

$$S_n\left(-\frac{1}{3}\right) \equiv 1 + \frac{2}{3}p\,q_p(2) \pmod{p^2}.$$

Case 2. $p \equiv 2 \pmod 3$.

In this case, we have $\langle a\rangle_p = (2p-1)/3$, $t = -2/3$ and

$$\delta_k = \begin{cases} 1 & \text{if } 0 \le k < (p+1)/6,\\ 0 & \text{if } (p+1)/6 \le k \le \langle a\rangle_p - 1.\end{cases}$$

So, by (3.5) we have

$$\begin{aligned}
&S_n\left(-\frac{1}{3}\right) - (-1)^{(2p-1)/3}(1-ptH_n)\\
\equiv&2p(t+1)\sum_{k=0}^{(p-5)/6}\frac{(-1)^k}{\langle a\rangle_p - k} + 2pt\sum_{k=(p+1)/6}^{(2p-4)/3}\frac{(-1)^k}{\langle a\rangle_p - k}\\
=&2p(t+1)(-1)^{(2p-1)/3}\sum_{j=(p+1)/2}^{(2p-1)/3}\frac{(-1)^j}{j} + 2pt(-1)^{(2p-1)/3}\sum_{j=1}^{(p-1)/2}\frac{(-1)^j}{j}\\
=&-2p(t+1)\sum_{j=1}^{(2p-1)/3}\frac{(-1)^j}{j} + 2p\sum_{j=1}^{(p-1)/2}\frac{(-1)^j}{j}\\
=&-2p(t+1)\left(H_{\lfloor p/3\rfloor} - H_{\lfloor 2p/3\rfloor}\right) + 2p\left(H_{\lfloor p/4\rfloor} - H_{\lfloor p/2\rfloor}\right) \pmod{p^2}.
\end{aligned}$$

Note that

$$H_{\lfloor 2p/3\rfloor} = \sum_{k=1}^{(p-1)/2}\left(\frac{1}{k} + \frac{1}{p-k}\right) - \sum_{j=1}^{(p-1)/3}\frac{1}{p-j} \equiv H_{\lfloor p/3\rfloor} \pmod p.$$

Therefore,

$$S_n\left(-\frac{1}{3}\right) + 1 - ptH_{\lfloor p/2\rfloor} \equiv 2p\left(H_{\lfloor p/4\rfloor} - H_{\lfloor p/2\rfloor}\right) \pmod{p^2}.$$

This, together with Lemma 3.3 and the fact that $t=-2/3$, yields the desired congruence

$$S_n\left(-\frac{1}{3}\right)\equiv -1-\frac{2}{3}p\,q_p(2)\ (\mathrm{mod}\ p^2).$$

In view of the above, we have completed the proof of (1.4).

Our next goal is to show (1.5) and (1.6). For any p-adic integer a with $a(2a+1)\not\equiv 0\ (\mathrm{mod}\ p)$, if we set $t=(a-\langle a\rangle_p)/p$ then by (3.4) we have

$$\begin{aligned}T_n(a)-T_n(pt)=&\sum_{k=1}^{\langle a\rangle_p}(T_n(a-k+1)-T_n(a-k))\\=&\sum_{k=1}^{\langle a\rangle_p}2\binom{a-k}{n}\binom{k-a-2}{n}\\=&2\sum_{k=1}^{\langle a\rangle_p}\binom{m_k+pt-1}{n}\binom{-1-pt-m_k}{n},\end{aligned}$$

where $m_k=\langle a\rangle_p-k+1$. In view of Lemmas 3.2 and 3.3,

$$\begin{aligned}T_n(pt)-(1+2pt)=&\sum_{k=0}^{n}\binom{pt}{k}\binom{-1-pt}{k}\frac{1+2pt}{1+2k}-(1+2pt)\\\equiv&\binom{pt}{n}\binom{-1-pt}{n}\frac{1+2pt}{p}-\sum_{k=1}^{n-1}\frac{pt}{k(1+2k)}\\\equiv&\left(-\frac{p^2t^2}{n^2}-\frac{pt}{n}\right)\frac{1+2pt}{p}-\sum_{k=1}^{n-1}\frac{pt}{k(1+2k)}\\\equiv&2t+2pt-pt\sum_{k=1}^{n-1}\frac{1}{k}+2pt\sum_{k=1}^{n-1}\frac{1}{2k+1}\\\equiv&2t-2pt-ptH_n+2pt\left(H_{p-1}-\frac{H_n}{2}\right)\\\equiv&2t-2pt+4ptq_p(2)\ (\mathrm{mod}\ p^2)\end{aligned}$$

and hence

$$T_n(pt)\equiv 1+2t+4ptq_p(2)\ (\mathrm{mod}\ p^2).$$

Therefore, with the helps of Lemma 3.1 and Remark 3.1, we have

$$\begin{aligned}
&T_n(a)-(1+2t+4ptq_p(2))\\
\equiv&2\sum_{k=1}^{\langle a\rangle_p}\binom{m_k+pt-1}{n}\binom{-1-pt-m_k}{n}\\
\equiv&2\sum_{k=1}^{\langle a\rangle_p}\frac{p(t+\delta_k)}{m_k}=2\sum_{j=1}^{\langle a\rangle_p}\frac{pt}{j}+2\sum_{\substack{j=1\\ j>p/2}}^{\langle a\rangle_p}\frac{1}{j}\pmod{p^2},
\end{aligned}$$

where δ_k takes 1 or 0 according as $m_k>p/2$ or not. Below we deal with $a=-1/6,-1/4$.

Clearly,

$$H_{p-k}=H_{p-1}-\sum_{0<j<k}\frac{1}{p-j}\equiv H_{k-1}\pmod{p}$$

for all $k=1,\ldots,p-1$. Thus, with the help of Lemma 3.3 we have

$$H_{\lfloor 3p/4\rfloor}\equiv H_{p-1-\lfloor 3p/4\rfloor}=H_{\lfloor p/4\rfloor}\equiv -3q_p(2)\pmod{p}$$

and

$$H_{\lfloor 5p/6\rfloor}\equiv H_{p-1-\lfloor 5p/6\rfloor}=H_{\lfloor p/6\rfloor}\equiv -2q_p(2)-\frac{3}{2}q_p(3)\pmod{p}.$$

Case I. $\langle a\rangle_p<n$.

If $a=-1/6$, then $p\equiv 1\pmod{6}$, $\langle a\rangle_p=(p-1)/6$ and $t=-1/6$. By the above,

$$\begin{aligned}
T_n\left(-\frac{1}{6}\right)\equiv&\frac{2}{3}-\frac{2}{3}pq_p(2)-\frac{p}{3}H_{\lfloor p/6\rfloor}\\
\equiv&\frac{2}{3}-\frac{2}{3}pq_p(2)-\frac{p}{3}\left(-2q_p(2)-\frac{3}{2}q_p(3)\right)\\
\equiv&\frac{2}{3}+\frac{p}{2}q_p(3)\pmod{p^2}
\end{aligned}$$

and thus

$$\sum_{k=0}^{n}\frac{\binom{6k}{3k}\binom{3k}{k}}{(2k+1)432^k}=\frac{3}{2}T_n\left(-\frac{1}{6}\right)\equiv 1+\frac{3}{4}pq_p(3)=\frac{3^p+1}{4}\pmod{p^2}.$$

If $a=-1/4$, then $p\equiv 1 \pmod 4$, $\langle a\rangle_p=(p-1)/4$ and $t=-1/4$. By the above,

$$\begin{aligned}T_n\left(-\frac{1}{4}\right)\equiv&\frac{1}{2}-pq_p(2)-\frac{p}{2}H_{\lfloor p/4\rfloor}\\ \equiv&\frac{1}{2}-pq_p(2)-\frac{p}{2}(-3q_p(2))\\ \equiv&\frac{1}{2}+\frac{p}{2}q_p(2) \pmod{p^2}\end{aligned}$$

and thus

$$\sum_{k=0}^{n}\frac{\binom{4k}{2k}\binom{2k}{k}}{(2k+1)64^k}=2T_n(-1/4)\equiv 1+pq_p(2)=2^{p-1} \pmod{p^2}.$$

Case II. $\langle a\rangle_p>n$.

If $a=-1/6$, then $p\equiv 5 \pmod 6$, $\langle a\rangle_p=(5p-1)/6$ and $t=-5/6$. By the above,

$$\begin{aligned}T_n\left(-\frac{1}{6}\right)\equiv&-\frac{2}{3}+\frac{2}{3}pq_p(2)+\frac{p}{3}H_{\lfloor 5p/6\rfloor}\\ \equiv&-\frac{2}{3}+\frac{2}{3}pq_p(2)+\frac{p}{3}\left(-2q_p(2)-\frac{3}{2}q_p(3)\right)\\ \equiv&-\frac{2}{3}-\frac{p}{2}q_p(3) \pmod{p^2}\end{aligned}$$

and hence

$$\sum_{k=0}^{n}\frac{\binom{6k}{3k}\binom{3k}{k}}{(2k+1)432^k}=\frac{3}{2}T_n\left(-\frac{1}{6}\right)\equiv -1-\frac{3}{4}pq_p(3)=-\frac{3^p+1}{4} \pmod{p^2}.$$

If $a=-1/4$, then $p\equiv 3 \pmod 4$, $\langle a\rangle_p=(3p-1)/4$ and $t=-3/4$. So

$$\begin{aligned}T_n\left(-\frac{1}{4}\right)\equiv&-\frac{1}{2}+pq_p(2)+\frac{p}{2}H_{\lfloor 3p/4\rfloor}\\ \equiv&-\frac{1}{2}+pq_p(2)+\frac{p}{2}(-3q_p(2))\\ \equiv&-\frac{1}{2}-\frac{p}{2}q_p(2) \pmod{p^2}\end{aligned}$$

and hence

$$\sum_{k=0}^{n}\frac{\binom{4k}{2k}\binom{2k}{k}}{(2k+1)64^k}=2T_n\left(-\frac{1}{4}\right)\equiv -1-pq_p(2)=-2^{p-1} \pmod{p^2}.$$

The proof of Theorem 1.2 is now complete. □

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