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TWO q-ANALOGUES OF EULER'S FORMULA $\zeta(2)=\pi^2/6$

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Abstract. It is well known that $\zeta(2)=\pi^2/6$ as discovered by Euler. In this paper we present the following two q-analogues of this celebrated formula:
$$\sum_{k=0}^\infty\frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2}=\prod_{n=1}^\infty\frac{(1-q^{2n})^4}{(1-q^{2n-1})^4}$$
and
$$\sum_{k=0}^\infty\frac{q^{2k-\lfloor(-1)^kk/2\rfloor}}{(1-q^{2k+1})^2}=\prod_{n=1}^\infty\frac{(1-q^{2n})^2(1-q^{4n})^2}{(1-q^{2n-1})^2(1-q^{4n-2})^2},$$
where q is any complex number with $|q|<1$. We also give a q-analogue of the identity $\zeta(4)=\pi^4/90$, and pose a problem on q-analogues of Euler's formula for $\zeta(2m)$ $(m=3,4,\ldots)$.

1. Introduction

For $n\in\mathbb{N}=\{0,1,2,\ldots\}$, the q-analogue of n is defined as
$$[n]_q:=\frac{1-q^n}{1-q}=\sum_{0\leqslant k<n}q^k\in\mathbb{Z}[q].$$
Note that $\lim_{q\to1}[n]_q=n$. For $|q|<1$, the q-*Gamma function* introduced by F. H. Jackson [J] in 1905 is given by
$$\Gamma_q(x):=(1-q)^{1-x}\prod_{n=1}^\infty\frac{1-q^n}{1-q^{n+x-1}}.$$
In view of the basic properties of the q-Gamma function (cf. [AAR, pp. 493–496]), we have

$$\lim_{\substack{q\to1\\|q|<1}}(1-q^2)\prod_{n=1}^\infty\frac{(1-q^{2n})^2}{(1-q^{2n-1})^2}=\lim_{\substack{q\to1\\|q|<1}}\Gamma_{q^2}\left(\frac12\right)^2=\Gamma\left(\frac12\right)^2=\pi, \tag{1.1}$$

which is essentially equivalent to Wallis' formula
$$\prod_{n=1}^\infty\frac{4n^2}{4n^2-1}=\frac{\pi}{2}$$
since
$$\prod_{n=1}^\infty\frac{(1-q^{2n})^2}{(1-q^{2n-1})(1-q^{2n+1})}=(1-q)\prod_{n=1}^\infty\frac{(1-q^{2n})^2}{(1-q^{2n-1})^2}\quad\text{for }|q|<1.$$

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In light of (1.1),

$$\lim_{\substack{q\to 1\\ |q|<1}} (1-q)\prod_{n=1}^{\infty}\frac{(1-q^{4n})^2}{(1-q^{4n-2})^2} = \lim_{\substack{q\to 1\\ |q|<1}} \frac{1-q}{1-q^4}\times\pi = \frac{\pi}{4}$$

and hence we may view Ramanujan's formula

$$\sum_{k=0}^{\infty}\frac{(-q)^k}{1-q^{2k+1}} = \prod_{n=1}^{\infty}\frac{(1-q^{4n})^2}{(1-q^{4n-2})^2} \quad (|q)<1)$$

(equivalent to Example (iv) in B. C. Berndt [B91, p. 139]) as a q-analogue of Leibniz's classical identity

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

Recently, V.J.W. Guo and J.-C. Liu [GL] gave some q-analogues of two Ramanujan-type series for $1/\pi$.

Let $\mathbb{C}$ be the field of complex numbers. The *Riemann zeta function* is given by

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s} \quad \text{for } s\in\mathbb{C} \text{ with } \Re(s)>1.$$

In 1734 L. Euler obtained the elegant formula

$$\zeta(2) = \frac{\pi^2}{6}. \tag{1.2}$$

In 2011 Kh. Hessami Pilehrood and T. Hessami Pilehrood [HP] gave an interesting q-analogue of the known identity $3\sum_{n=1}^{\infty}1/(n^2\binom{2n}{n}) = \zeta(2)$, which states that

$$\sum_{n=1}^{\infty}\frac{q^{n^2}(1+2q^n)}{[n]_q^2\begin{bmatrix}2n\\ n\end{bmatrix}_q} = \sum_{n=1}^{\infty}\frac{q^n}{[n]_q^2} \quad \text{for } |q|<1,$$

where

$$\begin{bmatrix}2n\\ n\end{bmatrix}_q = \frac{\prod_{k=1}^{2n}[k]_q}{\prod_{j=1}^{n}[j]_q^2}$$

is the q-analogue of the central binomial coefficient $\binom{2n}{n}$.

Euler's celebrated formula (1.2) plays very important roles in modern mathematics. Though $\sum_{n=1}^{\infty}q^n/[n]_q^2$ (with $q|<1$) is a natural q-analogue of $\zeta(2)$, it seems hopeless to use it to give a q-analogue of (1.2). As nobody has given a q-analogue of Euler's formula (1.2) before, we aim to present two q-analogues of (1.2) in this paper.

Our main result is as follows.

Theorem 1.1. *For any $q\in\mathbb{C}$ with $|q|<1$, we have*

$$\sum_{k=0}^{\infty}\frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2} = \prod_{n=1}^{\infty}\frac{(1-q^{2n})^4}{(1-q^{2n-1})^4}, \tag{1.3}$$

$$\sum_{k=0}^{\infty}\frac{q^{2k-\lfloor(-1)^k k/2\rfloor}}{(1-q^{2k+1})^2}=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2(1-q^{4n})^2}{(1-q^{2n-1})^2(1-q^{4n-2})^2}. \tag{1.4}$$

Clearly,

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{k=1}^{\infty}\frac{1}{(2k)^2}=\frac{3}{4}\zeta(2)$$

and hence (1.2) has the equivalent form

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{\pi^2}{8}. \tag{1.5}$$

Now we explain why (1.5) follows from (1.3) or (1.4). In view of (1.1),

$$\lim_{\substack{q\to1\\|q|<1}}(1-q^2)^2\prod_{n=1}^{\infty}\frac{(1-q^{2n})^4}{(1-q^{2n-1})^4}=\pi^2,$$

$$\lim_{\substack{q\to1\\|q|<1}}(1-q^2)(1-q^4)\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2(1-q^{4n})^2}{(1-q^{2n-1})^2(1-q^{4n-2})^2}=\pi^2.$$

Thus

$$\lim_{\substack{q\to1\\|q|<1}}(1-q)^2\prod_{n=1}^{\infty}\frac{(1-q^{2n})^4}{(1-q^{2n-1})^4}=\frac{\pi^2}{4} \tag{1.6}$$

and

$$\lim_{\substack{q\to1\\|q|<1}}(1-q)^2\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2(1-q^{4n})^2}{(1-q^{2n-1})^2(1-q^{4n-2})^2}=\frac{\pi^2}{8}.$$

On the other hand,

$$\lim_{\substack{q\to1\\|q|<1}}(1-q)^2\sum_{k=0}^{\infty}\frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2}=\lim_{\substack{q\to1\\|q|<1}}\sum_{k=0}^{\infty}\frac{q^k(1+q^{2k+1})}{[2k+1]_q^2}=\sum_{k=0}^{\infty}\frac{2}{(2k+1)^2},$$

$$\lim_{\substack{q\to1\\|q|<1}}(1-q)^2\sum_{k=0}^{\infty}\frac{q^{2k-\lfloor(-1)^k k/2\rfloor}}{(1-q^{2k+1})^2}=\lim_{\substack{q\to1\\|q|<1}}\sum_{k=0}^{\infty}\frac{q^{2k-\lfloor(-1)^k k/2\rfloor}}{[2k+1]_q^2}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}.$$

Therefore (1.3) and (1.4) indeed give q-analogues of (1.2).

We also deduce a q-analogue of the known formula $\zeta(4)=\pi^4/90$, which has the equivalent form

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}=\frac{\pi^4}{96}. \tag{1.7}$$

Theorem 1.2. *For any $q\in\mathbb{C}$ with $|q|<1$, we have*

$$\sum_{k=0}^{\infty}\frac{q^{2k}(1+4q^{2k+1}+q^{4k+2})}{(1-q^{2k+1})^4}=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^8}{(1-q^{2n-1})^8}. \tag{1.8}$$

As is clearly seen, the left-hand side of (1.8) times $(1-q)^4$ tends to $6\sum_{k=0}^\infty 1/(2k+1)^4$ as $q\to 1$. In view of (1.6), the right-hand side of (1.8) times $(1-q)^4$ has the limit $\pi^4/16$ as $q\to 1$. So (1.8) implies (1.7) and hence it gives a q-analogue of the formula $\zeta(4)=\pi^4/90$.

We will show Theorems 1.1 and 1.2 in the next section.

The Bernoulli numbers $B_0, B_1, \ldots$ are given by $B_0=1$ and

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k=0\quad (n=1,2,3,\ldots).$$

Euler proved (cf. [IR, pp. 231–232]) that for each $m=1,2,3,\ldots$ we have

$$\zeta(2m)=(-1)^{m-1}\frac{2^{2m-1}\pi^{2m}}{(2m)!}B_{2m}. \tag{1.9}$$

To seek for q-analogues of (1.9) is our novel idea in this paper. We don't know whether one can find a q-analogue of (1.9) similar to (1.3), (1.4) and (1.8) for $m=3,4,5,\ldots$, and this problem might stimulate further research.

2. Proofs of Theorems 1.1 and 1.2

Recall that the *triangular numbers* are the integers

$$T_n:=\frac{n(n+1)}{2}\ \ (n=0,1,2,\ldots).$$

As usual, we set

$$\psi(q):=\sum_{n=0}^\infty q^{T_n}\quad \text{for } |q|<1. \tag{2.1}$$

Lemma 2.1. *For $|q|<1$ we have*

$$\psi(q)=\prod_{n=1}^\infty\frac{1-q^{2n}}{1-q^{2n-1}}. \tag{2.2}$$

Remark 2.2. This is a well-known result due to Gauss (cf. Berndt [B06, (1.3.14), p. 11]).

Lemma 2.3. *Let $n\in\mathbb{N}$ and*

$$t_4(n):=|\{(w,x,y,z)\in\mathbb{N}^4:\ T_w+T_x+T_y+T_z=n\}|.$$

Then

$$t_4(n)=\sigma(2n+1), \tag{2.3}$$

where $\sigma(m)$ denotes the sum of all positive divisors of a positive integer m.

Remark 2.4. This is also a known result, see [B06, (3.6.6.), p.72]. In contrast with (2.3), for any positive integer n Jacobi showed that

$$r_4(n) := |\{(w,x,y,z) \in \mathbb{Z}^4 :\ w^2+x^2+y^2+z^2 = n\}| = 8\sum_{4\nmid d \mid n} d$$

(cf. [B06, (3.3.1), p. 59]).

Lemma 2.5. *For $|q|<1$ we have*

$$\sum_{k=0}^{\infty} \frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2} = \sum_{n=0}^{\infty} \sigma(2n+1)q^n. \tag{2.4}$$

Proof. For each $k \in \mathbb{N}$, clearly

$$\begin{aligned}\frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2} =& 2q^k(1-q^{2k+1})^{-2} - q^k(1-q^{2k+1})^{-1} \\ =& 2q^k\sum_{j=0}^{\infty}\binom{-2}{j}(-q^{2k+1})^j - q^k\sum_{j=0}^{\infty} q^{(2k+1)j} \\ =& \sum_{j=0}^{\infty}(2j+1)q^{(2j+1)(k+1/2)-1/2}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2} &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(2j+1)q^{((2j+1)(2k+1)-1)/2} \\ &= \sum_{n=0}^{\infty}\Bigg(\sum_{d\mid 2n+1} d\Bigg)q^{(2n+1-1)/2} = \sum_{n=0}^{\infty}\sigma(2n+1)q^n.\end{aligned}$$

This concludes the proof. □

Lemma 2.6. *For each $n \in \mathbb{N}$, we have*

$$|\{(u,v,x,y) \in \mathbb{N}^4 :\ T_u+T_v+2T_x+2T_y = n\}| = \sum_{d\mid 4n+3}\frac{d-(-1)^{(d-1)/2}}{4}. \tag{2.5}$$

Remark 2.7. This is a known result due to K. S. Williams [W].

Proof of Theorem 1.1. In view of Lemmas 2.1 and 2.3,

$$\prod_{n=1}^{\infty}\frac{(1-q^{2n})^4}{(1-q^{2n-1})^4} = \psi(q)^4 = \sum_{n=0}^{\infty} t_4(n)q^n = \sum_{n=0}^{\infty}\sigma(2n+1)q^n. \tag{2.6}$$

Combining this with (2.4) we immediately obtain (1.3).

By Lemmas 2.1 and 2.6, we have

$$\begin{aligned}&\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2(1-q^{4n})^2}{(1-q^{2n-1})^2(1-q^{4n-2})^2}\\ =&\psi(q)^2\psi(q^2)^2=\sum_{n=0}^{\infty}|\{(u,v,x,y)\in\mathbb{N}^4:\ T_u+T_v+2T_x+2T_y=n\}|q^n\\ =&\sum_{n=0}^{\infty}\sum_{d\mid 4n+3}\frac{d-(-1)^{(d-1)/2}}{4}q^n\\ =&\sum_{j=0}^{\infty}\frac{(4j+1)-1}{4}\sum_{k=0}^{\infty}q^{((4j+1)(4k+3)-3)/4}\\ &+\sum_{j=0}^{\infty}\frac{(4j+3)+1}{4}\sum_{k=0}^{\infty}q^{((4j+3)(4k+1)-3)/4}\\ =&\sum_{k=0}^{\infty}\left(q^k\sum_{j=0}^{\infty}jq^{j(4k+3)}+q^{3k}\sum_{j=0}^{\infty}(j+1)q^{j(4k+1)}\right).\end{aligned}$$

For $|z|<1$, clearly

$$\sum_{j=0}^{\infty}(j+1)z^j=\sum_{j=1}^{\infty}jz^{j-1}=\sum_{j=0}^{\infty}\frac{d}{dz}z^j=\frac{d}{dz}(1-z)^{-1}=\frac{1}{(1-z)^2}.$$

Therefore

$$\begin{aligned}&\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2(1-q^{4n})^2}{(1-q^{2n-1})^2(1-q^{4n-2})^2}\\ =&\sum_{k=0}^{\infty}\left(q^k\frac{q^{4k+3}}{(1-q^{4k+3})^2}+\frac{q^{3k}}{(1-q^{4k+1})^2}\right)=\sum_{m=0}^{\infty}\frac{q^{2m-\lfloor(-1)^m m/2\rfloor}}{(1-q^{2m+1})^2}.\end{aligned}$$

This proves (1.4). The proof of Theorem 1.1 is now complete. $\square$

Remark 2.8. In light of (1.6) and (2.6), we have the curious formula

$$\lim_{\substack{q\to 1\\ |q|<1}}(1-q)^2\sum_{n=0}^{\infty}\sigma(2n+1)q^n=\frac{\pi^2}{4}. \tag{2.7}$$

Lemma 2.9. *Let $n\in\mathbb{N}$ and*

$$t_8(n)=|\{(x_1,\ldots,x_8)\in\mathbb{N}^8:\ T_{x_1}+T_{x_2}+\cdots+T_{x_8}=n\}|.$$

Then

$$t_8(n)=\sum_{2\nmid d\mid n+1}\frac{(n+1)^3}{d^3}. \tag{2.8}$$

Remark 2.10. This is a result due to A. M. Legendre, see [B06, p. 139] or [ORW, Theorem 5].

Proof of Theorem 1.2. For $z\in\mathbb{C}$ with $|z|<1$, we have

$$\frac{z}{(1-z)^4}=z\sum_{k=0}^{\infty}\binom{-4}{k}(-z)^k=\sum_{k=0}^{\infty}\binom{k+3}{3}z^{k+1}=\sum_{k=1}^{\infty}\frac{k(k+1)(k+2)}{6}z^k$$

and hence

$$\begin{aligned}&\frac{z(1+4z+z^2)}{(1-z)^4}\\ =&(1+4z+z^2)\sum_{k=1}^{\infty}k(k+1)(k+2)\frac{z^k}{6}\\ =&\sum_{k=1}^{\infty}(k(k+1)(k+2)+4(k-1)k(k+1)+(k-2)(k-1)k)\frac{z^k}{6}\\ =&\sum_{k=1}^{\infty}k^3z^k.\end{aligned}$$

Thus

$$\sum_{k=0}^{\infty}\frac{q^{2k+1}(1+4q^{2k+1}+q^{4k+2})}{(1-q^{2k+1})^4}=\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}m^3q^{(2k+1)m}=\sum_{n=1}^{\infty}\Bigg(\sum_{2\nmid d\mid n}\frac{n^3}{d^3}\Bigg)q^n.$$

Combining this with Lemma 2.9 we obtain

$$\sum_{k=0}^{\infty}\frac{q^{2k}(1+4q^{2k+1}+q^{4k+2})}{(1-q^{2k+1})^4}=\sum_{n=1}^{\infty}t_8(n-1)q^{n-1}=\psi(q)^8.$$

So, with the help of Lemma 2.1 we get the desired identity (1.8). This completes the proof. □

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