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# ON $q$-ANALOGUES OF SOME SERIES FOR $\pi$ AND $\pi^2$

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ABSTRACT. We obtain a new $q$-analogue of the classical Leibniz series $\sum_{k=0}^\infty (-1)^k/(2k+1)=\pi/4$, namely

$$\sum_{k=0}^\infty \frac{(-1)^k q^{k(k+3)/2}}{1-q^{2k+1}} = \frac{(q^2;q^2)_\infty (q^8;q^8)_\infty}{(q;q^2)_\infty (q^4;q^8)_\infty},$$

where $q$ is a complex number with $|q|<1$. We also show that the Zeilberger-type series $\sum_{k=1}^\infty (3k-1)16^k/(k\binom{2k}{k})^3=\pi^2/2$ has two $q$-analogues with $|q|<1$, one of which is

$$\sum_{n=0}^\infty q^{n(n+1)/2}\frac{1-q^{3n+2}}{1-q}\cdot\frac{(q;q)_n^3(-q;q)_n}{(q^3;q^2)_n^3} = (1-q)^2\frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4}.$$

## 1. Introduction

Let $q$ be a complex number with $|q|<1$. As usual, for $n\in\mathbb{N}=\{0,1,2,\ldots\}$ and a complex number $a$, we define the $q$-shifted factorial by

$$(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k).$$

(An empty product is considered to take the value 1, and thus $(a;q)_0=1$.) We also adopt the standard notion

$$(a;q)_\infty=\lim_{n\to\infty}(a;q)_n=\prod_{k=0}^\infty(1-aq^k).$$

By the definition of the $q$-Gamma function [6, p. 20], we have

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}=\Gamma_{q^2}\left(\frac{1}{2}\right)(1-q^2)^{-1/2}.$$

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Therefore,

$$\lim_{q\to 1}(1-q)\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}=\Gamma\left(\frac{1}{2}\right)^2\lim_{q\to 1}\frac{1-q}{1-q^2}=\frac{\pi}{2} \tag{1.1}$$

and

$$\lim_{q\to 1}(1-q^2)\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2}=\frac{\pi}{2}.$$

In view of this, Ramanujan's formula

$$\sum_{k=0}^{\infty}\frac{(-q)^k}{1-q^{2k+1}}=\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2}\quad (|q|<1) \tag{1.2}$$

(equivalent to Example (iv) in [2, p. 139]) can be viewed as a $q$-analogue of Leibniz's identity

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}=\frac{\pi}{4}. \tag{1.3}$$

Guo and Liu [8] used the WZ method to deduce the identities

$$\sum_{n=0}^{\infty}q^{n^2}\frac{1-q^{6n+1}}{1-q}\cdot\frac{(q;q^2)_n^2(q^2;q^4)_n}{(q^4;q^4)_n^3}=\frac{(1+q)(q^2;q^4)_\infty(q^6;q^4)_\infty}{(q^4;q^4)_\infty^2} \tag{1.4}$$

and

$$\sum_{n=0}^{\infty}(-1)^n q^{3n^2}\frac{1-q^{6n+1}}{1-q}\cdot\frac{(q;q^2)_n^3}{(q^4;q^4)_n^3}=\frac{(q^3;q^4)_\infty(q^5;q^4)_\infty}{(q^4;q^4)_\infty^2} \tag{1.5}$$

with $|q|<1$, which are $q$-analogues of Ramanujan's formulas [3, p. 352]

$$\sum_{n=0}^{\infty}(6n+1)\frac{(1/2)_n^3}{n!^3 4^n}=\sum_{n=0}^{\infty}(6n+1)\frac{\binom{2n}{n}^3}{256^n}=\frac{4}{\pi}$$

and

$$\sum_{n=0}^{\infty}(6n+1)(-1)^n\frac{(1/2)_n^3}{n!^3 8^n}=\sum_{n=0}^{\infty}(6n+1)\frac{\binom{2n}{n}^3}{(-512)^n}=\frac{2\sqrt{2}}{\pi},$$

where $(a)_n=\prod_{k=0}^{n-1}(a+k)$ is the Pochhammer symbol. Note that

$$\frac{(1/2)_n}{n!}=(-1)^n\binom{-1/2}{n}=\frac{\binom{2n}{n}}{4^n}\quad \text{for all } n\in\mathbb{N}.$$

Quite recently, Sun [12] provided $q$-analogues of Euler's classical formulas $\zeta(2)=\pi^2/6$ and $\zeta(4)=\pi^4/90$.

In 1993, Zeilberger [13] used the WZ method to show that

$$\sum_{n=0}^{\infty}(21n+13)\frac{n!^6}{8(2n+1)!^3}=\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{6}.$$

A complicated $q$-analogue of the identity $\sum_{k=1}^{\infty}(21k-8)/(k^3\binom{2k}{k})^3=\zeta(2)$ was given by Hessami Pilehrood and Hessami Pilehrood [9] in 2011. Following Zeilberger's work, in 2008 Guillera [7, Identity 1] employed the WZ method to obtain the Zeilberger-type series

$$\sum_{k=1}^{\infty}\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{2}. \tag{1.6}$$

In this paper we study $q$-analogues of the identities (1.3) and (1.6).

Now we state our main results.

**Theorem 1.1.** *For $|q|<1$ we have*

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k+3)/2}}{1-q^{2k+1}}=\frac{(q^2;q^2)_\infty(q^8;q^8)_\infty}{(q;q^2)_\infty(q^4;q^8)_\infty}. \tag{1.7}$$

The above identity gives a new $q$-analogue of Leibniz's identity (1.3) since

$$\lim_{q\to1}(1-q^2)\frac{(q^2;q^2)_\infty(q^8;q^8)_\infty}{(q;q^2)_\infty(q^4;q^8)_\infty}=\lim_{q\to1}\Gamma_{q^2}\left(\frac{1}{2}\right)\Gamma_{q^8}\left(\frac{1}{2}\right)=\Gamma\left(\frac{1}{2}\right)^2=\pi.$$

**Theorem 1.2.** *For $|q|<1$ we have*

$$\sum_{n=0}^{\infty}q^{2n(n+1)}(1+q^{2n+2}-2q^{4n+3})\frac{(q^2;q^2)_n^3}{(q;q^2)_{n+1}^3(-1;q)_{2n+3}}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{2n}}{(1-q^{2n+1})^2} \tag{1.8}$$

*and*

$$\sum_{n=0}^{\infty}q^{n(n+1)/2}\frac{1-q^{3n+2}}{1-q}\cdot\frac{(q;q)_n^3(-q;q)_n}{(q^3;q^2)_n^3}=(1-q)^2\frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4}. \tag{1.9}$$

Multiplying both sides of (1.8) by $(1-q)^2$ and then letting $q\to1$, we obtain

$$\frac{1}{4}\sum_{n=0}^{\infty}(3n+2)\frac{2^{4n}n!^6}{(2n+1)!^3}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{1}{2}\left(1-\frac{1}{4}\right)\zeta(2)=\frac{\pi^2}{16},$$

which is equivalent to (1.6). In view of (1.1) and the fact that

$$\lim_{q\to1}q^{n(n+1)/2}\frac{1-q^{3n+2}}{1-q}\cdot\frac{(q;q)_n^3(-q;q)_n}{(q^3;q^2)_n^3}=\frac{(3n+2)16^{n+1}}{2(n+1)^3\binom{2n+2}{n+1}^3},$$

the identity (1.9) is also a $q$-analogue of (1.6). The expansions of both sides of (1.9) are

$$1+2q-q^2+3q^4-6q^5+3q^6+8q^7-16q^8+8q^9+10q^{10}+\cdots.$$

In [11, Conjecture 1.4], Sun presented several conjectural identities similar to Zeilberger-type series (one of which is $\sum_{k=1}^\infty (10k-3)8^k/(k^3\binom{2k}{k}^2\binom{3k}{k}) = \pi^2/2$), but we could not find $q$-analogues of them.

We are going to show Theorems 1.1 and 1.2 in Sections 2 and 3 respectively. Finally, in Section 4, we give alternative proofs for (1.4) and (1.5).

## 2. Proof of Theorem 1.1

As usual, for $x \in \mathbb{Z}$ we let $T_x$ denote the triangular number $x(x+1)/2$.

**Lemma 2.1.** *Let $n \in \mathbb{N}$, and define*

$$t_2(n) := |\{(x,y) \in \mathbb{N}^2 :\ T_x + 4T_y = n\}|.$$

*Then*

$$t_2(n) = \sum_{\substack{d \mid 8n+5 \\ d < \sqrt{8n+5}}} (-1)^{(d-1)/2}. \tag{2.1}$$

*Proof.* By Theorem 3.2.1 of [4, p. 56], for any positive integer $m$ we have

$$r_2(m) = 4 \sum_{2 \nmid d \mid m} (-1)^{(d-1)/2},$$

where $r_2(m) := |\{(x,y) \in \mathbb{Z}^2 :\ x^2 + y^2 = m\}|$. Observe that

$$\begin{aligned} t_2(n) =& |\{(x,y) \in \mathbb{N}^2 :\ (2x+1)^2 + 4(2y+1)^2 = 8n+5\}| \\ =& \frac{1}{4}|\{(x,y) \in \mathbb{Z}^2 :\ x^2 + (2y)^2 = 8n+5\}| = \frac{1}{8} r_2(8n+5) \\ =& \frac{1}{2}\sum_{d \mid 8n+5} (-1)^{(d-1)/2} = \sum_{\substack{d \mid 8n+5 \\ d < \sqrt{8n+5}}} \frac{(-1)^{(d-1)/2} + (-1)^{((8n+5)/d-1)/2}}{2} \\ =& \sum_{\substack{d \mid 8n+5 \\ d < \sqrt{8n+5}}} (-1)^{(d-1)/2}. \end{aligned}$$

This proves (2.1). ∎

As usual, for $|q| < 1$ we define

$$\psi(q) := \sum_{n=0}^\infty q^{T_n}.$$

By a known formula of Gauß (cf. (1.3.14) of [4, p. 11]),

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{2.2}$$

*First proof of* (1.7). Let $L$ and $R$ denote the left-hand side and the right-hand side of (1.7), respectively. In view of Gauß' identity (2.2) and (2.1), we have

$$R = \psi(q)\psi(q^4) = \sum_{n=0}^{\infty} t_2(n)q^n = \sum_{n=0}^{\infty}\Bigg(\sum_{\substack{d\mid 8n+5\\ d<\sqrt{8n+5}}}(-1)^{(d-1)/2}\Bigg)q^n.$$

On the other hand,

$$\begin{aligned}L &= \sum_{k=0}^{\infty}(-1)^k\sum_{m=0}^{\infty}q^{k(k+3)/2+(2k+1)m}\\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^k q^{((2k+1)(2k+1+4(2m+1))-5)/8} = \sum_{n=0}^{\infty}\sum_{\substack{d\mid 8n+5\\ d<\sqrt{8n+5}}}(-1)^{(d-1)/2}q^n.\end{aligned}$$

Therefore (1.7) is valid. ∎

*Second proof of* (1.7). Recall the standard basic hypergeometric notation

$${}_r\phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right] = \sum_{\ell=0}^{\infty}\frac{(a_1;q)_\ell\cdots(a_r;q)_\ell}{(q;q)_\ell(b_1;q)_\ell\cdots(b_s;q)_\ell}\left((-1)^\ell q^{\binom{\ell}{2}}\right)^{s-r+1}z^\ell$$

(with $(a_1;q)_\ell\cdots(a_r;q)_\ell$ often abbreviated as $(a_1,\dots,a_r;q)_\ell$). We start with the transformation formula (cf. [6, Eq. (3.10.4)])

$$\begin{aligned}&{}_{10}\phi_9\left[\begin{matrix}a,\sqrt{a}q,-\sqrt{a}q,b,x,-x,y,-y,\\ \sqrt{a},-\sqrt{a},aq/b,aq/x,-aq/x,aq/y,-aq/y,\end{matrix}\right.\\ &\qquad\qquad\left.\begin{matrix}-q^{-n},q^{-n}\\ -aq^{n+1},aq^{n+1}\end{matrix};q,-\frac{a^3q^{2n+3}}{bx^2y^2}\right]\\ &= \frac{(a^2q^2,a^2q^2/x^2y^2;q^2)_n}{(a^2q^2/x^2,a^2q^2/y^2;q^2)_n}\,{}_5\phi_4\left[\begin{matrix}q^{-2n},x^2,y^2,-aq/b,-aq^2/b\\ x^2y^2/a^2q^{2n},a^2q^2/b^2,-aq,-aq^2\end{matrix};q^2,q^2\right],\end{aligned}$$

where $n$ is a nonnegative integer. In this identity, set $a=q$, $x=y=\sqrt{q}$, and let $n\to\infty$. In this way, we obtain

$$\begin{aligned}&{}_4\phi_5\left[\begin{matrix}q,b,\sqrt{q},-\sqrt{q}\\ q^2/b,q^{3/2},-q^{3/2},0,0\end{matrix};q,\frac{q^4}{b}\right]\\ &\qquad= \frac{(q^2;q^2)_\infty\,(q^4;q^2)_\infty}{{(q^3;q^2)_\infty}^2}\,{}_4\phi_3\left[\begin{matrix}q,-q^2/b,-q^3/b,q\\ -q^3,-q^2,q^4/b^2\end{matrix};q^2,q^2\right].\end{aligned}$$

By performing the limit as $b \to 0$, the above transformation formula is reduced to

$$ {}_3\phi_3\left[\begin{matrix} q, \sqrt{q}, -\sqrt{q} \\ -q^{3/2}, q^{3/2}, 0 \end{matrix}; q, q^2\right] = \frac{(q^2;q^2)_\infty\,(q^4;q^2)_\infty}{(q^3;q^2)_\infty^{\ 2}}\, {}_2\phi_2\left[\begin{matrix} q, q \\ -q^2, -q^3 \end{matrix}; q^2, q^3\right]. $$

If the left-hand side is written out explicitly, we see that it agrees with the left-hand side in (1.7) up to a multiplicative factor of $1-q$. On the other hand, the ${}_2\phi_2$-series on the right-hand side can be evaluated by means of the summation formula (cf. [6, Ex. 1.19(i); Appendix (II.10)])

$$ {}_2\phi_2\left[\begin{matrix} a, q/a \\ -q, b \end{matrix}; q, -b\right] = \frac{(ab, bq/a; q^2)_\infty}{(b;q)_\infty}. $$

Thus, we arrive at

$$ \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+3)/2}\,(1-q)}{1-q^{2k+1}} = \frac{(q^2;q^2)_\infty\,(q^4;q^2)_\infty\,(-q^4;q^4)_\infty^2}{(-q^3;q^2)_\infty\,(q^3;q^2)_\infty^2}, $$

which is indeed equivalent to (1.7). ∎

## 3. Proof of Theorem 1.2

*Proof of* (1.8). We construct a $q$-analogue of the WZ pair given by Guillera [7, Identity 1].

Recall that a pair of bivariate functions $(F(n,k), G(n,k))$ is called a *WZ pair* [10, Chapter 7] if

$$ F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). $$

It was shown (cf. [1]) that

$$ \sum_{n=0}^{\infty} G(n,0) = \lim_{k\to\infty} \sum_{n=0}^{\infty} G(n,k) + \sum_{k=0}^{\infty} F(0,k). \tag{3.1} $$

We make the following construction. Let

$$ F_q(n,k) = 4 \cdot \frac{1-q^{2n}}{1-q} \cdot B_q(n,k) $$

and

$$ G_q(n,k) = \frac{4(1+q^{2n+1} - 2q^{4n+2k+1})}{(1-q)(1+q^{2n})(1+q^{2n+1})} B_q(n,k), $$

where

$$ B_q(n,k) = \frac{(q;q^2)_k^2 (q;q^2)_n^3}{(q^{2n+2};q^2)_k^2 (q^2;q^2)_n^3 (-1;q)_{2n}} q^{2n^2+4nk}. $$

We can extend the definition of $B_q(n,k)$ from nonnegative integers $n,k$ to any real numbers $n,k$ by defining

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Let $a$ be a positive real number. It is straightforward to check that $(F_q(n+a,k), G_q(n+a,k))$ is a WZ pair. Observing that $B_q(n,k)$ contains the factor $q^{4nk}$, we get

$$\lim_{k\to\infty}\sum_{n=0}^{\infty} G_q(n+a,k) = 0.$$

Thus,

$$\sum_{n=0}^{\infty} G_q(n+a,0) = \sum_{n=0}^{\infty} F_q(a,k). \tag{3.2}$$

Setting $a=1/2$ and noting that

$$(q;q^2)_{n+1/2} = \frac{(q;q^2)_\infty}{(q\cdot q^{2n+1};q^2)_\infty} = \frac{(q;q^2)_\infty(q^2;q^2)_n}{(q^2;q^2)_\infty}$$

and

$$(q^2;q^2)_{n+1/2} = \frac{(q^2;q^2)_\infty}{(q^2\cdot q^{2n+1};q^2)_\infty} = \frac{(q^2;q^2)_\infty(q;q^2)_{n+1}}{(q;q^2)_\infty}$$

for any $n\in\mathbb{N}$, we infer that

$$F_q\left(\frac{1}{2},k\right) = \frac{2q^{1/2}}{1-q}\cdot\frac{q^{2k}}{(1-q^{2k+1})^2}\cdot\frac{(q;q^2)_\infty^6}{(q^2;q^2)_\infty^6},$$

and

$$\begin{aligned} G_q\left(n+\frac{1}{2},0\right) =& \frac{4q^{2n^2+2n+1/2}(1+q^{2n+2}-2q^{4n+3})}{(1-q)} \\ &\times \frac{(q;q^2)_\infty^3(q^{2n+3};q^2)_\infty^3}{(-1;q)_{2n+3}(q^{2n+2};q^2)_\infty^3(q^2;q^2)_\infty^3}. \end{aligned}$$

After cancelling the common factors, we arrive at (1.8). ∎

*Remark* 3.1. Guillera [7] obtained the identity (1.6) via the WZ pair

$$F(n,k) = 8nB(n,k), \quad G(n,k) = (6n+4k+1)B(n,k),$$

where

$$B(n,k) = \frac{(2k)!^2(2n)!^3}{2^{8n+4k}(n+k)!^2k!^2n!^4}.$$

Since

$$\lim_{q\to1}F_q(n,k) = F(n,k) \quad \text{and} \quad \lim_{q\to1}G_q(n,k) = G(n,k),$$

the pair $(F_q(n,k), G_q(n,k))$ is indeed a $q$-analogue of the pair $(F(n,k), G(n,k))$.

*Proof of* (1.9). We start from the quadratic transformation formula (cf. [6, Eq. (3.8.13)])

$$\sum_{n=0}^{\infty} \frac{(1-aq^{3n})\,(a,d,aq/d;q^2)_n\,(b,c,aq/bc;q)_n}{(1-a)\,(q,aq/d,d;q)_n\,(aq^2/b,aq^2/c,bcq;q^2)_n} q^n \\ = \frac{(aq^2,bq,cq,aq^2/bc;q^2)_\infty}{(q,aq^2/b,aq^2/c,bcq;q^2)_\infty} {}_3\phi_2\left[\begin{matrix} b,c,aq/bc \\ dq,aq^2/d \end{matrix}; q^2, q^2\right]. \tag{3.3}$$

We set $a = q^2$ and $b = c = q$. This yields

$$\sum_{n=0}^{\infty} \frac{(1-q^{3n+2})\,(q^2,d,q^3/d;q^2)_n\,(q,q,q;q)_n}{(1-q^2)\,(q,q^3/d,d;q)_n\,(q^3,q^3,q^3;q^2)_n} q^n \\ = \frac{(q^4,q^2,q^2,q^2;q^2)_\infty}{(q,q^3,q^3,q^3;q^2)_\infty} {}_3\phi_2\left[\begin{matrix} q,q,q \\ dq,q^4/d \end{matrix}; q^2, q^2\right]. \tag{3.4}$$

At this point, we observe that the limit $d \to 0$ applied to the left-hand side of (3.4) produces exactly the left-hand side of (1.9). Thus, it remains to show the limit $d \to 0$ applied to the right-hand side of (3.4) yields the right-hand side of (1.9).

In order to see this, we take recourse to the transformation formula (cf. [6, Eq. (3.3.1) or Appendix (III.34)])

$${}_3\phi_2\left[\begin{matrix} A,B,C \\ D,E \end{matrix}; q, q\right] = \frac{(BCq/E,q/E;q)_\infty}{(Cq/E,Bq/E;q)_\infty} {}_3\phi_2\left[\begin{matrix} D/A,B,C \\ D,BCq/E \end{matrix}; q, \frac{Aq}{E}\right] \\ - \frac{(q/E,A,B,C,Dq/E;q)_\infty}{(Cq/E,Bq/E,D,E/q,Aq/E;q)_\infty} {}_3\phi_2\left[\begin{matrix} Cq/E,Bq/E,Aq/E \\ Dq/E,q^2/E \end{matrix}; q, q\right].$$

Here we replace $q$ by $q^2$ and set $A = B = C = q$, $D = dq$, and $E = q^4/d$, to obtain

$${}_3\phi_2\left[\begin{matrix} q,q,q \\ dq,q^4/d \end{matrix}; q^2, q^2\right] = \frac{(d,d/q^2;q^2)_\infty}{(d/q,d/q;q^2)_\infty} {}_2\phi_1\left[\begin{matrix} q,q \\ dq \end{matrix}; q^2, \frac{d}{q}\right] \\ - \frac{(d/q^2,q,q,q,d^2/q;q^2)_\infty}{(d/q,d/q,dq,q^2/d,d/q;q^2)_\infty} {}_3\phi_2\left[\begin{matrix} d/q,d/q,d/q \\ d^2/q,d \end{matrix}; q^2, q^2\right].$$

From here it is evident that

$$\lim_{d\to 0} {}_3\phi_2\left[\begin{matrix} q,q,q \\ dq,q^4/d \end{matrix}; q^2, q^2\right] = 1. \tag{3.5}$$

Indeed, the first term on the right-hand side of (3.5) converges trivially to 1, while in the second term everywhere the substitution $d = 0$ is fine — and thus produces a well-defined *finite* value —, except for the factor $(q^2/d,q^2)_\infty$ in the denominator. However, as $d \to 0$ (to be precise, we take $d = q^2/D$

with $D \to -\infty$), this term becomes unbounded, whence the whole term tends to 0. Thus, performing this limit, we get

$$\sum_{n=0}^{\infty} \frac{(1-q^{3n+2})\,(q^2;q^2)_n\,(q,q,q;q)_n}{(1-q^2)\,(q;q)_n\,(q^3,q^3,q^3;q^2)_n} q^{n+\binom{n}{2}} = \frac{(q^4,q^2,q^2,q^2;q^2)_\infty}{(q,q^3,q^3,q^3;q^2)_\infty} = \frac{(1-q)^3\,(q^2,q^2,q^2,q^2;q^2)_\infty}{(1-q^2)\,(q,q,q,q;q^2)_\infty},$$

as desired. ∎

## 4. Alternative proofs of the identities of Guo and Liu

In this last section, we provide alternative proofs of (1.4) and (1.5), showing that they can be obtained as special/limiting cases of a quadratic summation formula due to Gasper and Rahman [5].

*Proof of* (1.4). We start with the quadratic summation formula (cf. [6, Eq. (3.8.12)])

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1-aq^{3k}}{1-a} \cdot \frac{(a,b,q/b;q)_k}{(q^2,aq^2/b,abq;q^2)_k} \cdot \frac{(d,f,a^2q/df;q^2)_k}{(aq/d,aq/f,df/a;q)_k} \cdot q^k \\ &+ \frac{(aq,f/a,b,f/q;q)_\infty}{(a/f,fq/a,aq/d,df/a;q)_\infty} \cdot \frac{(d,a^2q/bd,a^2q/df,fq^2/d,df^2q/a^2;q^2)_\infty}{(aq^2/b,abq,fq/ab,bf/a,aq^2/bf;q^2)_\infty} \\ &\qquad \times {}_3\phi_2\!\left[\begin{matrix} f,bf/a,fq/ab \\ fq^2/d,df^2q/a^2 \end{matrix}; q^2,q^2\right] \\ &= \frac{(aq,f/a;q)_\infty}{(aq/d,df/a;q)_\infty} \cdot \frac{(aq^2/bd,abq/d,bdf/a,dfq/ab;q^2)_\infty}{(aq^2/b,abq,bf/a,fq/ab;q^2)_\infty}. \end{aligned}$$

Now replace $f$ by $a^2q^{2N+1}/d$, with $N$ a positive integer. The effect is that, because of the factor $(a^2q/df;q^2)_\infty$, this kills off the second term on the left-hand side. In other words, now this is indeed a genuine summation formula. Now replace $q$ by $q^2$ and choose $a=b=q$. Then the above identity reduces to

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1-q^{6k+1}}{1-q} \cdot \frac{(d,q^{4N+4}/d,q^{-4N};q^4)_k}{(q^4,q^4,q^4;q^4)_k} \cdot \frac{(q,q,q;q^2)_k}{(q^3/d,dq^{-4N-1},q^{4N+3};q^2)_k} \cdot q^{2k} \\ &= \frac{(q^3,q^{4N+3}/d;q^2)_\infty}{(q^3/d,q^{4N+3};q^2)_\infty} \cdot \frac{(q^4/d,q^4/d,q^{4N+4},q^{4N+4};q^4)_\infty}{(q^4,q^4,q^{4N+4}/d,q^{4N+4}/d;q^4)_\infty}. \end{aligned} \tag{4.1}$$

Finally, we set $d=q^2$ and let $N \to \infty$. Upon little simplification, we arrive at (1.4). ∎

*Proof of* (1.5). We proceed in a similar manner. In (4.1), we set $d = q^{-2N}$ with $N$ a positive integer. This yields the identity

$$\sum_{k=0}^{\infty} \frac{1-q^{6k+1}}{1-q} \cdot \frac{(q^{-2N}, q^{6N+4}, q^{-4N}; q^4)_k}{(q^4, q^4, q^4; q^4)_k} \cdot \frac{(q, q, q; q^2)_k}{(q^{2N+3}, q^{-6N-1}, q^{4N+3}; q^2)_k} \cdot q^{2k} = \frac{(q^3; q^2)_N (q^{2N+4}; q^4)_N^2}{(q^4; q^4)_N^2 (q^{4N+3}; q^2)_N}.$$

Letting $N \to \infty$, we then obtain (1.5). ∎

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