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CONSECUTIVE PRIMES AND LEGENDRE SYMBOLS

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Abstract. Let m be any positive integer and let $\delta_1, \delta_2 \in \{1, -1\}$. We show that for some constant $C_m > 0$ there are infinitely many integers $n > 1$ with $p_{n+m} - p_n \leqslant C_m$ such that

$$\left(\frac{p_{n+i}}{p_{n+j}}\right) = \delta_1 \quad \text{and} \quad \left(\frac{p_{n+j}}{p_{n+i}}\right) = \delta_2$$

for all $0 \leqslant i < j \leqslant m$, where p_k denotes the k-th prime, and $(\frac{\cdot}{p})$ denotes the Legendre symbol for any odd prime p. We also prove that under the Generalized Riemann Hypothesis there are infinitely many positive integers n such that p_{n+i} is a primitive root modulo p_{n+j} for any distinct i and j among $0, 1, \ldots, m$.

1. Introduction

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ let p_n denote the n-th prime. The famous twin prime conjecture asserts that $p_{n+1} - p_n = 2$ for infinitely many $n \in \mathbb{Z}^+$. Although this remains open, recently Y. Zhang [Z] was able to prove that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leqslant 7 \times 10^7.$$

The upper bound 7×10^7 was later reduced to 4680 by the `Polymath` team [Po] led by T. Tao, and 600 by J. Maynard [M], and 246 by the `Polymath` team [Po]. Moreover, J. Maynard [M], as well as T. Tao, established the following deep result.

Theorem 1.1 (Maynard-Tao). *For any positive integer m, we have*

$$\liminf_{n\to\infty}(p_{n+m} - p_n) \leqslant Cm^3 e^{4m},$$

where $C > 0$ is an absolutely constant.

Earlier than this work, in 2000 D.K.L. Shiu [S] proved the following nice theorem.

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Theorem 1.2 (Shiu)**.** *Let $a \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$ be relatively prime. Then, for any $m \in \mathbb{Z}^+$ there is a positive integer n such that*

$$p_n \equiv p_{n+1} \equiv \cdots \equiv p_{n+m} \equiv a \pmod{q}.$$

This was recently re-deduced in [BFTB] via the Maynard-Tao method.

In this paper we mainly establish the following new result on consecutive primes and Legendre symbols.

Theorem 1.3. *Let m be any positive integer and let $\delta_1, \delta_2 \in \{1, -1\}$. For some constant $C_m > 0$ depending only on m, there are infinitely many integers $n > 1$ with $p_{n+m} - p_n \leqslant C_m$ such that for any $0 \leqslant i < j \leqslant m$ we have*

$$\left(\frac{p_{n+i}}{p_{n+j}}\right) = \delta_1 \quad \text{and} \quad \left(\frac{p_{n+j}}{p_{n+i}}\right) = \delta_2. \tag{1.1}$$

Remark 1.4. (a) Instead of (1.1) in Theorem 1.3, actually we may require both (1.1) and the following property:

$$p_{ij} \| (p_{n+i} - p_{n+j}) \quad \text{for some prime } p_{ij} > 2m + 1. \tag{1.2}$$

(As usual, for a prime p and an integer a, by $p\|a$ we mean $p \mid a$ but $p^2 \nmid a$.)

(b) We conjecture the following extension of Theorem 1.3: For any $m \in \mathbb{Z}^+$, $\delta \in \{1, -1\}$ and $\delta_{ij} \in \{1, -1\}$ with $0 \leqslant i < j \leqslant m$, there are infinitely many integers $n > 1$ such that

$$\left(\frac{p_{n+i}}{p_{n+j}}\right) = \delta_{ij} = \delta \left(\frac{p_{n+j}}{p_{n+i}}\right)$$

for all $0 \leqslant i < j \leqslant m$.

Example 1.5. The smallest integer $n > 1$ with

$$\left(\frac{p_{n+i}}{p_{n+j}}\right) = 1 \quad \text{for all } i, j = 0, \ldots, 6 \text{ with } i \neq j$$

is 178633, and a list of the first 200 such values of n is available from [Su, A243901]. The seven consecutive primes p_{178633}, p_{178634}, $\ldots$, p_{178639} have the concrete values

2434589, 2434609, 2434613, 2434657, 2434669, 2434673, 2434681

respectively.

Example 1.6. The smallest integer $n>1$ with

$$\left(\frac{p_{n+i}}{p_{n+j}}\right)=-1\quad\text{for all } i,j=0,\ldots,5 \text{ with } i\neq j$$

is 2066981, and the six consecutive primes $p_{2066981},\ p_{2066982},\ \ldots,\ p_{2066986}$ have the concrete values

$$33611561,\ 33611573,\ 33611603,\ 33611621,\ 33611629,\ 33611653$$

respectively.

Example 1.7. The smallest integer $n>1$ with

$$-\left(\frac{p_{n+i}}{p_{n+j}}\right)=1=\left(\frac{p_{n+j}}{p_{n+i}}\right)\quad\text{for all } 0\leqslant i<j\leqslant 6$$

is 7455790, and the seven consecutive primes $p_{7455790},\ p_{7455791},\ \ldots,\ p_{7455796}$ have the concrete values

$$131449631,\ 131449639,\ 131449679,\ 131449691,\ 131449727,\ 131449739,\ 131449751$$

respectively.

Example 1.8. The smallest integer $n>1$ with

$$\left(\frac{p_{n+i}}{p_{n+j}}\right)=1=-\left(\frac{p_{n+j}}{p_{n+i}}\right)\quad\text{for all } 0\leqslant i<j\leqslant 5$$

is 59753753, and the six consecutive primes $p_{59753753},\ p_{59753754},\ \ldots,\ p_{59753758}$ have the concrete values

$$1185350899,\ 1185350939,\ 1185350983,\ 1185351031,\ 1185351059,\ 1185351091$$

respectively.

Actually Theorem 1.3 is motivated by the following conjecture of the second author.

Conjecture 1.9 (Sun [Su, A243837])**.** *For any positive integer m, there are infinitely many $n\in\mathbb{Z}^+$ such that for any distinct i and j among $0,1,\ldots,m$ the prime p_{n+i} is a primitive root modulo p_{n+j}.*

Example 1.10. The least $n\in\mathbb{Z}^+$ with p_{n+i} a primitive root modulo p_{n+j} for any distinct i and j among $0,1,2,3$ is 8560, and a list of the first 50 such values of n is available from [Su, A243839]. Note that

$$p_{8560}=88259,\ p_{8561}=88261 \text{ and } p_{8562}=88289.$$

Our second result confirms Conjecture 1.9 under the Generalized Riemann Hypothesis.

Theorem 1.11. *Let m be any positive integer. Assuming the GRH (Generalized Riemann Hypothesis). for some constant $C_m > 0$ depending only on m, there are infinitely many integers $n > 1$ with $p_{n+m} - p_n \leqslant C_m$, such that the prime p_{n+i} is a primitive root modulo p_{n+j} for any distinct $i, j \in \{0, 1, \ldots, m\}$.*

We will prove Theorem 1.3 in the next section with the help of the Maynard-Tao work, and show Theorem 1.11 in Section 3 by combining our method with a recent result of P. Pollack [P] motivated by the Maynard-Tao work on bounded gaps of primes and Artin's conjecture on primitive roots modulo primes.

Throughout this paper, p always represents a prime. For two integers a and b, their greatest common divisor is denoted by $\gcd(a, b)$.

2. Proof of Theorem 1.3

Let $h_1, h_2, \ldots, h_k$ be distinct positive integers. If $\bigcup_{j=1}^{k} h_i (\mathrm{mod}\ p) \neq \mathbb{Z}$ for any prime p (where $a(\mathrm{mod}\ p)$ denotes the residue class $a + p\mathbb{Z}$), then we call $\{h_i :\ i = 1, \ldots, k\}$ an *admissible set.* Hardy and Littlewood conjectured that if $\mathcal{H} = \{h_i :\ i = 1, \ldots, k\}$ is admissible then there are infinitely many $n \in \mathbb{Z}^+$ such that $n + h_1, n + h_2, \ldots, n + h_k$ are all prime. We need the following result in this direction.

Lemma 2.1 (Maynard-Tao)**.** *Let m be any positive integer. Then there is an integer $k > m$ depending only on m such that if $\mathcal{H} = \{h_i :\ i = 1, \ldots, k\}$ is an admissible set of cardinality k and $W = q_0 \prod_{p \leqslant w} p$ (with $q_0 \in \mathbb{Z}^+$) is relatively prime to $\prod_{i=1}^{k} h_i$ with $w = \log\log\log x$ large enough, then for some integer $n \in [x, 2x]$ with $W \mid n$ there are more than m primes among $n+h_1, n+h_2, \ldots, n+h_k$.*

Lemma 2.2. *Let $k > 1$ be an integer. Then there is an admissible set $\mathcal{H} = \{h_1, \ldots, h_k\}$ with $h_1 = 0 < h_2 < \ldots < h_k$ which has the following properties:*

(i) *All those $h_1, h_2, \ldots, h_k$ are multiples of $K = 4 \prod_{p < 2k} p$ with p prime.*

(ii) *Each $h_i - h_j$ with $1 \leqslant i < j \leqslant k$ has a prime divisor $p > 2k$ with $h_i \not\equiv h_j$* (mod p^2).

(iii) *If $1 \leqslant i < j \leqslant k$, $1 \leqslant s < t \leqslant k$ and $\{i, j\} \neq \{s, t\}$, then no prime $p > 2k$ divides both $h_i - h_j$ and $h_s - h_t$.*

Proof. Set $h_1 = 0$ and let $1 \leqslant r < k$. Suppose that we have found nonnegative integers $h_1 < \ldots < h_r$ divisible by K such that each $h_i - h_j$ with $1 \leqslant i < j \leqslant r$ has a prime divisor $p > 2k$ with $h_i \not\equiv h_j \pmod{p^2}$, and that no prime $p > 2k$ divides both $h_i - h_j$ and $h_s - h_t$ if $1 \leqslant i < j \leqslant r$, $1 \leqslant s < t \leqslant r$ and $\{i, j\} \neq \{s, t\}$. Let

$$X_r = \{p > 2k :\ p \text{ is prime and } p \mid h_s - h_t \quad \text{for some } 1 \leqslant s < t \leqslant r\}.$$

As K is relatively prime to $\prod_{p \in X_r} p$, for each $i = 1, \ldots, r$ there is an integer b_i with $Kb_i \equiv h_i \pmod{\prod_{p \in X_r} p}$. For each $p \in X_r$, as $r < k < p$ there is an integer $a_p \not\equiv b_i \pmod{p}$ for all $i = 1, \ldots, r$. Choose distinct primes $q_1, \ldots, q_r$ which are greater than $2k$ but not in the set X_r. For any $i = 1, \ldots, r$, there is an integer c_i with $Kc_i \equiv h_i \pmod{q_i^2}$ since K is relatively prime to q_i^2. By the Chinese Remainder Theorem, there is an integer $b > h_r/K$ such that $b \equiv a_p \pmod{p}$ for all $p \in X_r$, and $b \equiv c_i + q_i \pmod{q_i^2}$ for all $i = 1, \ldots, r$.

Set $h_{r+1} = Kb > h_r$. If $1 \leqslant s \leqslant r$, then

$$h_{r+1} - h_s \equiv Kb - Kc_s = K(b - c_s) \equiv Kq_s \pmod{q_s^2},$$

hence $q_s > 2k$ is a prime divisor of $h_{r+1} - h_s$ but $h_{r+1} \not\equiv h_s \pmod{q_s^2}$.

For $s, t \in \{1, \ldots, r\}$ with $s \neq t$, clearly

$$\gcd(h_{r+1} - h_s, h_{r+1} - h_t) = \gcd(h_{r+1} - h_s, h_s - h_t).$$

Let $1 \leqslant i < j \leqslant r$ and $1 \leqslant s \leqslant r$. If a prime $p > 2k$ divides $h_i - h_j$, then $p \in X_r$ and hence

$$h_{r+1} - h_s \equiv Ka_p - Kb_s = K(a_p - b_s) \not\equiv 0 \pmod{p}.$$

So $\gcd(h_{r+1} - h_s, h_i - h_j)$ has no prime divisor greater than $2k$.

In view of the above, we have constructed nonnegative integers $h_1 < \ldots < h_k$ satisfying (i)-(iii) in Lemma 2.2. Note that $\bigcup_{i=1}^k h_i (\mathrm{mod}\ p) \neq \mathbb{Z}$ if $p > k$. For each $p \leqslant k$, clearly $h_i \equiv 0 \not\equiv 1 \pmod{p}$ for any $i = 1, \ldots, k$. Therefore the set $\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$ is admissible. This concludes the proof. $\square$

Proof of Theorem 1.3. By Lemma 2.1, there is an integer $k = k_m > m$ depending on m such that for any admissible set $\mathcal{H} = \{h_1, \ldots, h_k\}$ of cardinality k if x is sufficiently large and $\prod_{i=1}^k h_i$ is relatively prime to $W = 4\prod_{p \leqslant w} p$ then for some integer $n \in [x/W, 2x/W]$ there are more than m primes among $Wn + h_1, Wn + h_2, \ldots, Wn + h_k$, where $w = \log\log\log x$.

Let $\mathcal{H} = \{h_1, \ldots, h_k\}$ with $h_1 = 0 < h_2 < \ldots < h_k$ be an admissible set satisfying the conditions (i)-(iii) in Lemma 2.2. Clearly $K = 4\prod_{p < 2k} p \equiv 0 \pmod{8}$. Let x be sufficiently large with the interval $(h_k, w]$ containing more than $h_k - k$ primes. Note that $8 \mid W$ since $w \geqslant 2$. Our goal is to construct a new admissible set $\mathcal{H}'$ to which we will apply Lemma 2.1 in order to complete the proof.

Let $\delta := \delta_1\delta_2$. For any integer $b \equiv \delta \pmod{K}$ and each prime $p < 2k$, clearly $b + h_i \equiv \delta + 0 \pmod{p}$ and hence $\gcd(b + h_i, p) = 1$ for all $i = 1, \ldots, k$.

By the property (ii) in Lemma 2.2, for any $1 \leqslant i < j \leqslant k$, the number $h_i - h_j$ has a prime divisor $p_{ij} > 2k$ with $h_i \not\equiv h_j \pmod{p_{ij}^2}$. Let $p > 2k$ be an arbitrary prime dividing $\prod_{1 \leqslant i < j \leqslant k}(h_i - h_j)$. Then there is a unique pair $\{i, j\}$ with $1 \leqslant i < j \leqslant k$ such that $h_i \equiv h_j \pmod{p}$. Note that $p \leqslant h_k$ and p may be different from p_{ij}. All the $k - 2 < (p-3)/2$ numbers $h_i - h_s$ with $1 \leqslant s \leqslant k$ and $s \neq i, j$ are relatively prime to p, so there is an integer $r_p \not\equiv h_i - h_s \pmod{p}$ for all $s = 1, \ldots, k$ such that

$$\left(\frac{r_p\,\delta}{p}\right) = \begin{cases} \delta_2 & \text{if } p = p_{ij}, \\ 1 & \text{otherwise.} \end{cases}$$

So, for any integer $b \equiv r_p - h_i \pmod p$, we have $b + h_s \not\equiv 0 \pmod p$ for all $s = 1, \dots, k$.

Assume that $S = \{h_1, h_1 + 1, \dots, h_k\} \setminus \mathcal{H}$ is a set $\{a_i : \ i = 1, \dots, t\}$ of cardinality $t > 0$. Clearly $t \leqslant h_k - k + 1$ and hence we may choose t distinct primes $q_1, \dots, q_t \in (h_k, w]$. If $b \equiv -a_i \pmod{q_i}$, then $b + h_s \equiv h_s - a_i \not\equiv 0 \pmod{q_i}$ for all $s = 1, \dots, k$ since $0 < |h_s - a_i| < h_k < q_i$.

Let

$$Q = \left\{p \in (2k, w] : \ p \text{ is prime and } p \nmid \prod_{1 \leqslant i < j \leqslant k} (h_i - h_j)\right\} \setminus \{q_i : \ i = 1, \dots, t\}.$$

For any prime $q \in Q$, there is an integer $r_q \not\equiv -h_i \pmod q$ for all $i = 1, \dots, k$ since $\mathcal{H}$ is admissible.

By the Chinese Remainder Theorem, there is an integer b satisfying the following (1)-(4).

(1) $b \equiv \delta = \delta_1 \delta_2 \pmod K$.

(2) $b \equiv r_p - h_i \equiv r_p - h_j \pmod p$ if $p > 2k$ is a prime dividing $h_i - h_j$ with $1 \leqslant i < j \leqslant k$.

(3) $b \equiv -a_i \pmod{q_i}$ for all $i = 1, \dots, t$.

(4) $b \equiv r_q \pmod q$ for all $q \in Q$.

By the above analysis, as we have ensured that $b + h_s \not\equiv 0 \pmod p$ for each prime $p \leqslant w$, the product $\prod_{s=1}^{k}(b + h_s)$ is relatively prime to W. As $\mathcal{H}' = \{b + h_s : \ s = 1, \dots, k\}$ is also an admissible set of cardinality k, for large x there is an integer $n \in [x/W, 2x/W]$ such that there are more than m primes among $Wn + b + h_s$ $(s = 1, \dots, k)$. For $a_i \in S$, we have

$$Wn + b + a_i \equiv 0 - a_i + a_i = 0 \pmod{q_i}$$

and hence $Wn + b + a_i$ is composite since $W > q_i$. Therefore, there are consecutive primes $p_N, p_{N+1}, \dots, p_{N+m}$ with $p_{N+i} = Wn + b + h_{s(i)}$ for all $i = 0, \dots, m$, where $1 \leqslant s(0) < s(1) < \ldots < s(m) \leqslant k$. Note that

$$p_{N+m} - p_N = (Wn + b + h_{s(m)}) - (Wn + b + h_{s(0)}) = h_{s(m)} - h_{s(0)} \leqslant h_k.$$

For each $s = 1, \dots, k$, clearly $Wn + b + h_s \equiv 0 + \delta + 0 = \delta \pmod 8$ and hence

$$\left(\frac{-1}{Wn + b + h_s}\right) = \delta \quad \text{and} \quad \left(\frac{2}{Wn + b + h_s}\right) = 1,$$

where $(-)$ denotes the Jacobi symbol. As $p_{N+i} = Wn + b + h_{s(i)} \equiv \delta \pmod 8$ for all $i = 0, \dots, m$, by the law of Quadratic Reciprocity we have

$$\left(\frac{p_{n+j}}{p_{N+i}}\right) = \delta \left(\frac{p_{n+i}}{p_{N+j}}\right) \quad \text{for all } 0 \leqslant i < j \leqslant m.$$

Let $0 \leqslant i < j \leqslant m$. Then

$$\begin{aligned}\left(\frac{p_{N+i}}{p_{N+j}}\right) &= \left(\frac{Wn+b+h_{s(i)}}{Wn+b+h_{s(j)}}\right) = \left(\frac{h_{s(i)}-h_{s(j)}}{Wn+b+h_{s(j)}}\right) \\ &= \left(\frac{-1}{Wn+b+h_{s(j)}}\right)\left(\frac{h_{s(j)}-h_{s(i)}}{Wn+b+h_{s(j)}}\right) = \delta\left(\frac{h^*_{ij}}{Wn+b+h_{s(j)}}\right),\end{aligned}$$

where h^*_{ij} is the odd part (i.e., the largest odd divisor) of $h_{s(j)} - h_{s(i)}$. For any prime divisor p of h^*_{ij}, clearly $p \leqslant h_k \leqslant w$ and

$$\left(\frac{p}{Wn+b+h_{s(j)}}\right) = \delta^{(p-1)/2}\left(\frac{Wn+b+h_{s(j)}}{p}\right) = \delta^{(p-1)/2}\left(\frac{b+h_{s(j)}}{p}\right).$$

If $p < 2k$, then $p \mid K$, hence $b + h_j \equiv \delta + 0 \pmod{p}$ and thus

$$\left(\frac{p}{Wn+b+h_{s(j)}}\right) = \delta^{(p-1)/2}\left(\frac{b+h_{s(j)}}{p}\right) = \delta^{(p-1)/2}\left(\frac{\delta}{p}\right) = 1.$$

If $p > 2k$, then by the choice of b we have

$$\begin{aligned}\left(\frac{p}{Wn+b+h_{s(j)}}\right) =&\delta^{(p-1)/2}\left(\frac{b+h_{s(j)}}{p}\right) = \delta^{(p-1)/2}\left(\frac{r_p}{p}\right) \\ =&\left(\frac{r_p\,\delta}{p}\right) = \begin{cases}\delta_2 & \text{if } p = p_{s(i),s(j)}, \\ 1 & \text{otherwise.}\end{cases}\end{aligned}$$

Recall that $p_{s(i),s(j)} \| h^*_{ij}$. Therefore,

$$\left(\frac{p_{N+i}}{p_{N+j}}\right) = \delta\left(\frac{h^*_{ij}}{Wn+b+h_{s(j)}}\right) = \delta\delta_2 = \delta_1$$

and

$$\left(\frac{p_{N+j}}{p_{N+i}}\right) = \delta\left(\frac{p_{N+i}}{p_{N+j}}\right) = \delta_2.$$

This concludes the proof of Theorem 1.3. □

3. Proof of Theorem 1.11

The following lemma is a slight modification of Lemma 2.2 which can be proved in a similar way.

Lemma 3.1. *Let $k > 1$ be an integer. Then there is an admissible set $\mathcal{H} = \{h_1, \dots, h_k\}$ with $h_1 = 0 < h_2 < \ldots < h_k$ such that:*

(i) All those $h_1, h_2, \dots, h_k$ are multiples of $K = 4\prod_{p<4k} p$.

(ii) Each $h_i - h_j$ with $1 \leqslant i < j \leqslant k$ has a prime divisor $p > 4k$ with $h_i \not\equiv h_j \pmod{p^2}$.

(iii) If $1 \leqslant i < j \leqslant k$, $1 \leqslant s < t \leqslant k$ and $\{i, j\} \neq \{s, t\}$, then no prime $p > 4k$ divides both $h_i - h_j$ and $h_s - h_t$.

Lemma 3.2. *Let $k > 1$ be an integer, and let $\mathcal{H} = \{h_1, \dots, h_k\}$ with $h_1 = 0 < h_2 < \dots < h_k$ be an admissible set satisfying* (i)-(iii) *in Lemma* 3.1*. Then there is a positive integer b with all of the following properties:*

(i) *$\prod_{i=1}^k (b + h_i)$ is relatively prime to the least common multiple W of those $h_j - h_i$ with $1 \leqslant i < j \leqslant k$ and $\prod_{2<p\leqslant w} p$ if w is large enough.*

(ii) *$\prod_{i=1}^k (b + h_i - 1)$ is relatively prime to $\prod_{2<p\leqslant w} p$ if w is large enough.*

(iii) *For any $i, j \in \{1, \dots, k\}$ with $i \neq j$, we have*

$$\left(\frac{h_i - h_j}{b + h_j}\right) = -1.$$

(iv) *If $n > b$, $n \equiv b \pmod W$ and $a \in \{h_1, h_1 + 1, \dots, h_k\} \setminus \mathcal{H}$, then $n + a$ is not prime.*

Proof. For any $1 \leqslant i < j \leqslant k$, the number $h_i - h_j$ has a prime divisor $p_{ij} > 4k$ with $h_i \not\equiv h_j \pmod{p_{ij}^2}$. Suppose that $p > 4k$ is a prime dividing $\prod_{1\leqslant i<j\leqslant k}(h_i - h_j)$. Then there is a unique pair $\{i, j\}$ with $1 \leqslant i < j \leqslant k$ such that $h_i \equiv h_j \pmod p$. Note that $p \leqslant h_k$. As $h_i - h_j \equiv h_i - h_i \pmod p$, we have

$$\begin{aligned}
&|\{0 \leqslant r < p:\ r \not\equiv h_i - h_s, h_i - h_s + 1 \pmod p \text{ for all } s = 1, \dots, k\}| \\
=&|\{0 \leqslant r < p:\ r \not\equiv h_i - h_s, h_i - h_s + 1 \pmod p \text{ for all } s \in \{1, \dots, k\} \setminus \{i\}\}| \\
\leqslant&2(k-1) < \frac{p-1}{2}.
\end{aligned}$$

Recall that $\{1, \dots, p-1\}$ contains exactly $(p-1)/2$ quadratic residues modulo p and $(p-1)/2$ quadratic nonresidues modulo p. So there is an integer $r_p \not\equiv h_i - h_s,\ h_i - h_s + 1 \pmod p$ for all $s = 1, \dots, k$ such that

$$\left(\frac{r_p}{p}\right) = \begin{cases} -(-1)^{\mathrm{ord}_3(h_j - h_i)} & \text{if } p = p_{ij}, \\ 1 & \text{otherwise,} \end{cases}$$

where $\mathrm{ord}_3(h_i - h_j)$ is the largest nonnegative integer a such that $3^a | (h_i - h_j)$. So, for any integer $b \equiv r_p - h_i \pmod p$, we have $b + h_s \not\equiv 0, 1 \pmod p$.

Assume that

$$S = \{h_1 < a < h_k : a \neq h_s, h_s - 1 \text{ for all } s = 1, \dots, k\} = \{a_i :\ i = 1, \dots, t\}.$$

Clearly $t \leqslant h_k - k$ and hence we may choose t distinct primes $q_1, \dots, q_t \in (h_k, w]$ if w is large enough. If $b \equiv -a_i \pmod{q_i}$, then $b + h_s \equiv h_s - a_i \not\equiv 0, 1 \pmod{q_i}$ for all $s = 1, \dots, k$ since $|h_s - a_i| < h_k < q_i$.

Let

$$Q = \left\{p \in (4k, w] :\ p \text{ is prime and } p \nmid \prod_{1\leqslant i<j\leqslant k} (h_i - h_j)\right\} \setminus \{q_i :\ i = 1, \dots, t\}.$$

For any prime $q \in Q$, there is an integer $r_q \not\equiv -h_i, -h_i + 1 \pmod q$ for all $i = 1, \dots, k$ since $q > 2k$.

By the Chinese Remainder Theorem, there is a positive integer b satisfying the following (1)-(4).

(1) $b \equiv 17 \pmod{24}$, and $b \equiv 4 \pmod p$ for all primes $p \in [5, 4k]$.

(2) $b \equiv r_p - h_i \equiv r_p - h_j \pmod p$ if $p > 4k$ is a prime dividing $h_i - h_j$ with $1 \leqslant i < j \leqslant k$.

(3) $b \equiv -a_i \pmod{q_i}$ for all $i = 1, \dots, t$.

(4) $b \equiv r_q \pmod q$ for all $q \in Q$.

By the above analysis, $\prod_{s=1}^{k}(b + h_s)(b + h_s - 1)$ is relatively prime to $\prod_{2<p\leqslant w} p$. Note that $b + h_i \equiv 17 + 0 \pmod{24}$ for all $i = 1, \dots, k$. If $w \geqslant h_k$, then any prime divisor of W does not exceed w. So parts (i) and (ii) of Lemma 3.2 are valid.

For each $s = 1, \dots, k$, clearly $b + h_s \equiv 17 + 0 \equiv 1 \pmod 8$ and hence

$$\left(\frac{-1}{b + h_s}\right) = \left(\frac{2}{b + h_s}\right) = 1;$$

also $b + h_s \equiv 17 \equiv 5 \pmod{12}$ and hence

$$\left(\frac{3}{b + h_s}\right) = \left(\frac{b + h_s}{3}\right) = \left(\frac{5}{3}\right) = -1.$$

Let $i, j \in \{0, \dots, m\}$ with $i \neq j$. Then

$$\left(\frac{h_i - h_j}{b + h_j}\right) = \left(\frac{h_{ij}}{b + h_j}\right),$$

where h_{ij} is the odd part of $|h_i - h_j|$. For any prime divisor p of h_{ij}, clearly $p \leqslant h_k \leqslant w$ and

$$\left(\frac{p}{b + h_j}\right) = \left(\frac{b + h_j}{p}\right).$$

If $3 < p < 4k$, then $p \mid K$, hence $b + h_j \equiv 4 + 0 \pmod p$ and thus

$$\left(\frac{p}{b + h_j}\right) = \left(\frac{b + h_j}{p}\right) = \left(\frac{4}{p}\right) = 1.$$

If $p > 4k$, then by the choice of b we have

$$\begin{aligned}\left(\frac{p}{b + h_j}\right) = \left(\frac{b + h_j}{p}\right) = \left(\frac{r_p}{p}\right)\\ = \begin{cases} -(-1)^{\mathrm{ord}_3|h_j - h_i|} & \text{if } p = p_{\min\{i,j\},\max\{i,j\}}, \\ 1 & \text{otherwise.}\end{cases}\end{aligned}$$

Recall that $p_{\min i,j,\max i,j} \| h_{ij}$. Therefore,

$$\left(\frac{h_i - h_j}{b + h_j}\right) = \left(\frac{h_{ij}}{b + h_j}\right) = \left(\frac{3}{b + h_j}\right)^{\mathrm{ord}_3 |h_i - h_j|} \left(\frac{p_{\min\{i,j\},\max\{i,j\}}}{b + h_j}\right) = -1.$$

So part (iii) of Lemma 3.2 also holds.

Now suppose that $n > b$ is an integer with $n \equiv b \pmod{W}$, and that $a \in \{h_1, h_1 + 1, \dots, h_k\} \setminus \mathcal{H}$. If $a = h_s - 1$ for some $1 \leqslant s \leqslant k$, then $n + a \equiv b + h_s - 1 \equiv 0 \pmod 4$ and hence $n + a$ is not prime. If $a \neq h_s - 1$ for all $s = 1, \dots, k$, then $a = a_i$ for some $1 \leqslant i \leqslant t$, hence $n + a \equiv b + a_i \equiv 0 \pmod{q_i}$ and thus $n + a$ is not prime. (Note that $n + a > W > w \geqslant q_i$.) Thus part (iv) of Lemma 3.2 also holds.

In view of the above, we have completed the proof of Lemma 3.2. $\square$

Proof of Theorem 1.11. Choose an integer $k > m$ as in Pollack [P] in the spirit of Maynard-Tao's work. Let $\mathcal{H} = \{h_1, h_2, \dots, h_k\}$ be an admissible set constructed in Lemma 3.1 and choose an integer b as in Lemma 3.2. Let x be sufficiently large, and let W be the least common multiple of those $h_j - h_i$ $(1 \leqslant i < j \leqslant k)$ and $\prod_{2<p\leqslant \log\log\log x} p$. Then we have an analogue of Pollack [P, Lemma 3.3]. When $n + h_i$ and $n + h_j$ $(i \neq j)$ are both prime with $n \equiv b \pmod{W}$, we see that $n + h_i$ is a primitive root modulo $n + h_j$ if and only if $|h_i - h_j|$ is a primitive root modulo $n + h_j$ since $n + h_j \equiv 1 \pmod 4$.

Let P be the set of all primes. For $j = 1 \dots, k$, set

$$P_j := \{p \in P : \ |h_i - h_j| \text{ is a primitive root modulo } p \text{ for every } i \neq j\}.$$

Define the weight function $w(n)$ as in [M, Proposition 4.1] or [P, Proposition 3.1], and let $\chi_A(x)$ be the characteristic function of the set A. We need to show that

$$\sum_{\substack{x\leqslant n\leqslant 2x \\ n\equiv b \pmod{W}}} \left(\sum_{j=1}^{k} \chi_{P_j}(n + h_j)\right) w(n) \sim \sum_{\substack{x\leqslant n\leqslant 2x \\ n\equiv b \pmod{W}}} \left(\sum_{j=1}^{k} \chi_{P}(n + h_j)\right) w(n). \tag{3.1}$$

For a prime q and an integer g, define

$$P_q(g) = \left\{p \in P : \ p \equiv 1 \pmod q \text{ and } g^{(p-1)/q} \equiv 1 \pmod p\right\}$$

and

$$\mathcal{P}_q(g) = P_q(g) \setminus \bigcup_{q'<q} P_{q'}(g).$$

Under the GRH, Pollack [P, the estimation of $\Sigma_1 - \Sigma_4$] used an effective version of the Chebotarey density theorem to show that if $(\frac{g}{b+h_j}) = -1$ then

$$\sum_{q\in P} \sum_{\substack{x\leqslant n\leqslant 2x \\ n\equiv b \pmod{W}}} \chi_{\mathcal{P}_q(g)}(n + h_j) w(n) = o\left(\frac{\varphi(W)^k}{W^{k+1}} x (\log x)^k\right),$$

where the error term is controlled via the GRH. Note that if $n \in P \setminus P_j$ then $n \in \mathcal{P}_q(|h_i - h_j|)$ for some $i \neq j$ and some prime q. Hence

$$\sum_{\substack{x\leqslant n\leqslant 2x \\ n\equiv b \pmod W}} \left(\sum_{j=1}^{k} \chi_P(n+h_j)\right) w(n) - \sum_{\substack{x\leqslant n\leqslant 2x \\ n\equiv b \pmod W}} \left(\sum_{j=1}^{k} \chi_{P_j}(n+h_j)\right) w(n)$$
$$\leqslant \sum_{j=1}^{k} \sum_{\substack{i=1 \\ i\neq j}}^{k} \sum_{q\in P} \sum_{\substack{x\leqslant n\leqslant 2x \\ n\equiv b \pmod W}} \chi_{\mathcal{P}_q(|h_i-h_j|)}(n+h_j)w(n) = o\left(\frac{\varphi(W)^k}{W^{k+1}} x(\log x)^k\right).$$

Maynard and Tao (cf. [M]) have proved

$$\sum_{\substack{x\leqslant n\leqslant 2x \\ n\equiv b \pmod W}} w(n) \sim \frac{\alpha\varphi(W)^k}{W^{k+1}} x(\log x)^k \tag{3.2}$$

and

$$\sum_{\substack{x\leqslant n\leqslant 2x \\ n\equiv b \pmod W}} \left(\sum_{j=1}^{k} \chi_P(n+h_j)\right) w(n) \sim \frac{\beta k\varphi(W)^k}{W^{k+1}} x(\log x)^k, \tag{3.3}$$

where α and β are positive constants only depending on k and w. It follows from (3.1) and (3.3) that

$$\sum_{\substack{x\leqslant n\leqslant 2x \\ n\equiv b \pmod W}} \left(\sum_{j=1}^{k} \chi_{P_j}(n+h_j)\right) w(n) \sim \frac{\beta k\varphi(W)^k}{W^{k+1}} x(\log x)^k.$$

Similarly, for each $j = 1, \dots, k$ we have

$$\sum_{\substack{x\leqslant n\leqslant 2x \\ n\equiv b \pmod W}} \chi_{P_j}(n+h_j)w(n) \sim \frac{\beta\varphi(W)^k}{W^{k+1}} x(\log x)^k,$$

which implies that the set P_j cannot be finite. Moreover, in view of [M], we may choose a sufficiently large integer k and a suitable weight function w such that

$$\beta k > m\alpha,$$

i.e.,

$$\sum_{\substack{x\leqslant n\leqslant 2x \\ n\equiv b \pmod W}} \left(\sum_{j=1}^{k} \chi_{P_j}(n+h_j) - m\alpha\right) w(n) > 0.$$

Since $w(n)$ is non-negative, for some $n \in [x, 2x]$ with $n \equiv b \pmod W$, $\{n + h_1, \ldots, n + h_k\}$ contains at least $m + 1$ primes $n + h_j$ $(j \in J)$ with $|J| > m$ and $n + h_j \in P_j$ for all $j \in J$. According to the construction of b and Lemma 3.2 (iv), for each $j = 1, \ldots, k$, the interval $(n + h_j, n + h_{j+1})$ contains no prime. So those primes in $\{n + h_1, \ldots, n + h_k\}$ are consecutive primes. Note that $n + h_k - (n + h_1) = h_k - h_1$ is a constant C_m only depending on m. For any $i, j \in J$ with $i \neq j$, the number $h_i - h_j$, as well as the prime $n + h_i$, is a primitive root modulo the prime $n + h_j$. This concludes the proof. $\square$

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