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RESTRICTED SUMS OF FOUR SQUARES

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Abstract. We refine Lagrange's four-square theorem in new ways by imposing some restrictions involving powers of two (including 1). For example, we show that each $n = 1, 2, 3, \ldots$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N} = \{0, 1, 2, \ldots\})$ with $|x + y - z| \in \{4^k : k \in \mathbb{N}\}$ (or $|2x - y| \in \{4^k : k \in \mathbb{N}\}$, or $x + y - z \in \{\pm 8^k : k \in \mathbb{N}\} \cup \{0\} \subseteq \{t^3 : t \in \mathbb{Z}\}$), and that we can write any positive integer as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y + 2z$ (or $x + 2y + 2z$) a power of four. We also prove that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y + z + w$ a square (or a cube). In addition, we pose some open conjectures for further research; for example, we conjecture that any integer $n > 1$ can be written as $a^2 + b^2 + 3^c + 5^d$ with $a, b, c, d \in \mathbb{N}$.

1. Introduction

The celebrated four-square theorem (cf. [N, pp. 5-7]) proved by J. L. Lagrange in 1770 states that any $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$. Recently, the author [S17b] found that this can be refined in various ways by requiring additionally that $P(x, y, z, w)$ is a square, where $P(x, y, z, w)$ is a suitable polynomial with integer coefficients. (For example, we may take $P(x, y, z, w) = x^2y^2 + y^2z^2 + z^2x^2$.) Here is a challenging conjecture posed by the author.

1-3-5 Conjecture ([S17b, Conjecture 4.3(i)]). *Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 3y + 5z$ is a square.*

In this paper we aim to refine Lagrange's four-square theorem in a new direction by imposing restrictions involving power of two (including $2^0 = 1$). [S17b, Theorem 1.1] asserts that for any $a \in \{1, 4\}$ and $m \in \{4, 5, 6\}$ we can write each $n \in \mathbb{N}$ as $ax^m + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$. Actually the proof in [S17b] works

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for a stronger result which requires additionally that $x \in \{2^k : k \in \mathbb{N}\} \cup \{0\}$. Similarly, by modifying the proof of [S17b, Theorem 1.2(i)] slightly we get that for any $a \in \{1, 2\}$ each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $a(x - y) \in \{4^k : k \in \mathbb{N}\} \cup \{0\}$.

Now we state our first theorem.

Theorem 1.1. (i) *Any $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ can be written as $x^2+y^2+z^2+8^k$ with $x, y, z \in \mathbb{N}$ and $k \in \{0, 1, 2\}$. Also, for each $r \in \{0, 1\}$ and integer $n > r$, we can write n^2 as $x^2 + y^2 + z^2 + 4^{2k+r}$ with $k, x, y, z \in \mathbb{N}$.*

(ii) *Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x - y = 2^{\lfloor \mathrm{ord}_2(n)/2 \rfloor}$, where $\mathrm{ord}_2(n)$ is the 2-adic order of n.*

(iii) *Each $n \in \mathbb{N}$ not of the form $2^{6k+3} \times 7$ $(k \in \mathbb{N})$ can be written as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{N}$ and $x - y \in \{8^k : k \in \mathbb{N}\} \cup \{0\}$. Consequently, we can write any $n \in \mathbb{N}$ as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + y \in \{8^k : k \in \mathbb{N}\} \cup \{0\}$.*

(iv) *Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + y + z + w = 2^{\lfloor (\mathrm{ord}_2(n)+1)/2 \rfloor}$.*

Remark 1.1. For integers y and z of the same parity, clearly

$$y^2 + z^2 = 2\left(\frac{y+z}{2}\right)^2 + 2\left(\frac{y-z}{2}\right)^2.$$

So, the first assertion in Theorem 1.1(i) implies that any $n \in \mathbb{N}$ can be written as $x^2 + 2y^2 + 2z^2 + 8^k$ with $x, y, z \in \mathbb{N}$ and $k \in \{0, 1, 2\}$. If $a \in \mathbb{Z}^+$, $4 \nmid a$, and $2^2 = x^2 + y^2 + z^2 + w^2$ for some $x, y, z, w \in \mathbb{N}$ with $ax \in \{4^k : k \in \mathbb{N}\}$, then $x \in \{1, 2\}$ and hence $a \in \{1, 2\}$. As $\{8^k : k \in \mathbb{N}\} \cup \{0\} \subseteq \{t^3 : t \in \mathbb{N}\}$, Theorem 1.1(iii) implies a conjecture stated in [S17b, Remark 1.2]. Theorem 1.1(iv) with $\mathrm{ord}_2(n) = 0$ was first realized by Euler in a letter to Goldbach dated June 9, 1750 (cf. Question 37278 in `MathOverFlow`). See [S, A281494] for the number of ways to represent $n \in \mathbb{Z}^+$ as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{Z}$ and $|x| \leqslant |y| \leqslant |z| \leqslant |w|$ such that $x + y + z + w = 2^{\lfloor (\mathrm{ord}_2(n)+1)/2 \rfloor}$. For example,

$$14 = 0^2 + 1^2 + (-2)^2 + 3^2 \quad \text{with } 0 + 1 + (-2) + 3 = 2 = 2^{\lfloor (\mathrm{ord}_2(14)+1)/2 \rfloor}$$

and

$$107 = (-1)^2+(-3)^2+(-4)^2+9^2 \text{ with } (-1)+(-3)+(-4)+9 = 1 = 2^{\lfloor (\mathrm{ord}_2(107)+1)/2 \rfloor}.$$

It was proved in [SS] that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y + z + w$ a square.

The author (cf. [S17b, Conjecture 4.1]) conjectured that for each $\varepsilon \in \{\pm 1\}$ any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $2x + \varepsilon y$ a square. Y.-C. Sun and the author [SS] confirmed this for $\varepsilon = 1$, but the case $\varepsilon = -1$ remains unsolved. The author (cf. [S17b, Conjecture 4.1]) also conjectured that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 3y$ a square. Our next theorem provides an advance in this direction.

Theorem 1.2. (i) *Any* $n \in \mathbb{Z}^+$ *can be written as* $x^2+y^2+z^2+w^2$ *with* $x, y, z, w \in \mathbb{N}$ *and* $|2x-y| \in \{4^k : k \in \mathbb{N}\}$.

(ii) *Let* $a \in \{1,2\}$. *Then any* $n \in \mathbb{N}$ *can be written as* $x^2+y^2+z^2+w^2$ *with* $x, y, z, w \in \mathbb{N}$ *and* $2x-y \in \{\pm a8^k : k \in \mathbb{N}\} \cup \{0\} \subseteq \{at^3 : t \in \mathbb{Z}\}$.

(iii) *If any positive integer* $n \equiv 9 \pmod{20}$ *can be written as* $5x^2+5y^2+z^2$ *with* $x, y, z \in \mathbb{Z}$ *and* $2 \nmid z$, *then any* $n \in \mathbb{Z}^+$ *can be written as* $x^2+y^2+z^2+w^2$ $(x, y, z, w \in \mathbb{Z})$ *with* $x+3y \in \{4^k : k \in \mathbb{N}\}$.

Remark 1.2. The author [S15, Remark 1.8] conjectured that for each $n \in \mathbb{N}$ we can write $20n+9$ as $5x^2+5y^2+z^2$ with $x, y, z \in \mathbb{Z}$ and $2 \nmid z$.

The author (cf. [S17b, Conjecture 4.3(iii)-(iv)]) conjectured that each $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ $(x, y, z, w \in \mathbb{N})$ with $x+y-z$ (or $x-y-z$) a square. In contrast, we have the following result.

Theorem 1.3. (i) *Any* $n \in \mathbb{Z}^+$ *can be written as* $x^2+y^2+z^2+w^2$ *with* $x, y, z, w \in \mathbb{N}$ *and* $|x+y-z| \in \{4^k : k \in \mathbb{N}\}$.

(ii) *Let* $a \in \{1,2\}$. *Then any* $n \in \mathbb{N}$ *can be written as* $x^2+y^2+z^2+w^2$ *with* $x, y, z, w \in \mathbb{N}$ *and* $x+y-z \in \{\pm a8^k : k \in \mathbb{N}\} \cup \{0\} \subseteq \{at^3 : t \in \mathbb{Z}\}$.

Remark 1.3. We conjecture that each $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{N}$, $x \equiv y \pmod 2$ and $|x+y-z| \in \{4^k : k \in \mathbb{N}\}$ (cf. [S, A299825]) but we are unable to prove this which is stronger than Theorem 1.3(i). In contrast with Theorem 1.3(ii), we conjecture that any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x+y-z$ an integer cube, where $x, y, z, w \in \mathbb{N}$, $x \geqslant y \leqslant z$ and $x \equiv y \pmod 2$ (cf. [S, A282091]).

The author [S17b] conjectured that any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z)$ is a square, where $P(x, y, z)$ may be any of the polynomials

$$x+y-2z,\ 2x+y-z,\ 2x-y-z,\ x+2y-2z,\ 2x-y-2z.$$

Here we give the following result.

Theorem 1.4. (i) *Let* $c \in \{1,2\}$. *Then each* $n \in \mathbb{Z}^+$ *can be written as* $x^2+y^2+z^2+w^2$ *with* $x, y, z, w \in \mathbb{Z}$ *and* $x+y+2z \in \{c4^k : k \in \mathbb{N}\}$.

(ii) *Any* $n \in \mathbb{Z}^+$ *can be written as* $x^2+y^2+z^2+w^2$ *with* $x, y, z, w \in \mathbb{Z}$ *and* $x+2y+2z \in \{4^k : k \in \mathbb{N}\}$. *Also, for each* $n \in \mathbb{Z}^+$ *we can write* $n^2 = x^2+y^2+z^2+w^2$ *with* $x, y, z, w \in \mathbb{Z}$ *and* $x+2y+2z \in \{8^k : k \in \mathbb{N}\}$.

(iii) *If* $n \in \mathbb{Z}^+$ *does not belong to* $\bigcup_{k \in \mathbb{N}} \{2^{4k}, 2^{4k+3}\}$, *then we can write* n *as* $x^2+y^2+z^2+w^2$ *with* $x, y, z, w \in \mathbb{Z}$ *and* $x+2y+2z \in \{3 \times 4^k : k \in \mathbb{N}\}$. *Consequently, any integer* $n > 1$ *can be written as* $x^2+y^2+z^2+w^2$ *with* $x, y, z, w \in \mathbb{Z}$ *and* $x+2y+2z \in \{3 \times 2^k : k \in \mathbb{N}\}$.

Remark 1.4. Y.-C. Sun and the author [SS, Theorem 1.2(ii)] showed that for each $d = 1, 2, 3$ and $m = 2, 3$, we can write any $n \in \mathbb{N}$ as $x^2+y^2+z^2+w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x+2y+2z \in \{dt^m : t \in \mathbb{N}\}$.

In contrast with the 1-3-5 Conjecture, we have the following curious conjecture motivated by Theorem 1.4.

Conjecture 1.1. (i) *Any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ such that $x+2(y-z)$ is a power of four (including $4^0=1$). Also, for each $n \in \mathbb{Z}^+$ we can write n^2 as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ and $x+2(y-z) \in \{8^k:\ k \in \mathbb{N}\}$.*

(ii) *Let $c \in \{1,2,4\}$. Then each $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ and $y \leqslant z$ such that $c(2x+y-z) \in \{8^k:\ k \in \mathbb{N}\} \cup \{0\} \subseteq \{t^3:\ t \in \mathbb{N}\}$.*

Remark 1.5. (i) We have verified the first assertion in part (i) for all $n=1,\ldots,2\times 10^7$, and Qing-Hu Hou extended the verification for n up to 10^9. See [S, A279612 and A279616] for related data. For example,

$$111 = 9^2+1^2+5^2+2^2 \ \text{ with } 9+2\times 1-2\times 5 = 4^0.$$

(ii) We have verified part (ii) of Conjecture 1.1 for all $n=0,\ldots,2\times 10^6$. See [S, A284343] for related data. For example,

$$2976 = 20^2+16^2+48^2+4^2 \ \text{ with } 16<48 \text{ and } 2\times 20+16-48=8.$$

In 1917 Ramanujan [R] listed 55 possible quadruples (a,b,c,d) of positive integers with $a \leqslant b \leqslant c \leqslant d$ such that any $n \in \mathbb{N}$ can be written as $ax^2+by^2+cz^2+dw^2$ with $x,y,z,w \in \mathbb{Z}$, and 54 of them were later confirmed by Dickson [D27] with the remaining one wrong (see also [W]).

Theorem 1.5. (i) *Any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+2w^2$ with $x,y,z,w \in \mathbb{N}$ and $x-y=1$. Also, any $n \in \mathbb{Z}^+$ can be written as $x^2+2y^2+2z^2+2w^2$ with $x,y,z,w \in \mathbb{Z}$ and $x+y+z=1$.*

(ii) *Any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+2w^2$ $(x,y,z,w \in \mathbb{Z})$ with $x+y+2z=1$. Also, any $n \in \mathbb{Z}^+$ can be written as $x^2+4y^2+z^2+2w^2$ $(x,y,z,w \in \mathbb{Z})$ with $x+2y+2z=1$, and any $n \in \mathbb{Z}^+$ can be written as $x^2+2y^2+2z^2+2w^2$ $(x,y,z,w \in \mathbb{Z})$ with $x+y+3z=1$.*

(iii) *Any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+2w^2$ $(x,y,z,w \in \mathbb{Z})$ with $y+z+2w=1$. Also, any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+4z^2+2w^2$ $(x,y,z,w \in \mathbb{Z})$ with $y+2z+2w=1$, and any $n \in \mathbb{Z}^+$ can be written as $x^2+2y^2+2z^2+2w^2$ $(x,y,z,w \in \mathbb{Z})$ with $x+y+z+2w=1$.*

(iv) *Any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+2w^2$ $(x,y,z,w \in \mathbb{Z})$ with $y+z+w=1$.*

(v) *Any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+2z^2+5w^2$ with $x,y,z,w \in \mathbb{Z}$ and $y+w=1$. For each $\delta=1,2$, any positive integer $n \not\equiv 2 \pmod 3$ can be written as $x^2+y^2+2z^2+(6/\delta)w^2$ with $y+\delta w=1$.*

(vi) *Any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+3w^2$ $(x,y,z,w \in \mathbb{Z})$ with $x+y+2z=2$.*

(vii) *Any integer $n>4$ can be written as $x^2+y^2+z^2+2w^2$ with $x,y,z,w \in \mathbb{Z}$ such that $x+y+z=t^2$ for some $t=1,2$.*

(viii) *Any integer* $n > 7$ *can be written as* $x^2 + y^2 + z^2 + 2w^2$ $(x, y, z, w \in \mathbb{Z})$ *with* $x + y + 2z = 2t^2$ *for some* $t = 1, 2$.

Remark 1.6. We can prove several other results similar to the first assertion in Theorem 1.5(v). As a supplement to the second assertion in Theorem 1.5(v), we conjecture that any positive integer $n \equiv 2 \pmod 3$ also can be written as $x^2 + y^2 + 2z^2 + 6w^2$ with $x, y, z, w \in \mathbb{Z}$ and $y + w = 1$, and that 11 is the only positive integer which cannot be written as $x^2 + y^2 + 2z^2 + 3w^2$ with $x, y, z, w \in \mathbb{Z}$ and $y + 2w = 1$.

Theorem 1.6. *Let* $m \in \{2, 3\}$.

(i) *Any* $n \in \mathbb{N}$ *can be written as* $x^2 + y^2 + z^2 + 2w^2$ $(x, y, z, w \in \mathbb{Z})$ *with* $x + y + z + w$ *an* m*-th power.*

(ii) *Any* $n \in \mathbb{N}$ *can be written as* $x^2 + y^2 + z^2 + 2w^2$ $(x, y, z, w \in \mathbb{Z})$ *with* $x + 2y + 2z$ *an* m*-th power.*

Remark 1.7. Our proof of Theorem 1.6 depends heavily on a new identity similar to Euler's four-square identity. We even conjecture that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ $(x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z})$ with $x + y - z + w \in \{0, 1\}$.

We will prove Theorems 1.1-1.4 and 1.5-1.6 in Sections 2 and 3 respectively, and pose more related conjectures in Sec. 4.

2. Proofs of Theorems 1.1-1.4

It is known that the set

$$E(a, b, c) := \{n \in \mathbb{N} : \ n \neq ax^2 + by^2 + cz^2 \text{ for all } x, y, z \in \mathbb{Z}\} \tag{2.1}$$

is infinite for any $a, b, c \in \mathbb{Z}^+$.

Lemma 2.1 (The Gauss-Legendre Theorem). *We have*

$$E(1, 1, 1) = E_0 := \{4^k(8l + 7) : \ k, l \in \mathbb{N}\}. \tag{2.2}$$

Remark 2.1. This is a well-known result on sums of three squares, see, e.g., [N, p. 23] or [MW, p. 42]).

Lemma 2.2. *Let* $m, n \in \mathbb{N}$ *with* $16 \mid m$ *and* $16 \nmid n$. *Then* $\{n - 1, n - m\} \not\subseteq E_0$, *where* E_0 *is the set defined in* (2.2).

Proof. Clearly one of $n - 1$ or $n - m$ is odd. If $n - 1 \equiv 7 \pmod 8$, then $n - m \equiv n \equiv 8 \pmod{16}$. If $n - m \equiv 7 \pmod 8$, then $n - 1 \equiv n - m - 1 \equiv 6 \pmod 8$. Thus $\{n - 1, n - m\} \not\subseteq E_0$ as desired. □

Remark 2.2. It follows from Lemmas 2.1 and 2.2 that any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with x a power of two and $y, z, w \in \mathbb{N}$. A stronger result given in [S17b, Theorem 1.2(v)] states that any positive integer can be written as $4^k(1 + 4x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$.

Proof of Theorem 1.1. (i) (a) If $n-1 \notin E(1,1,1)$, then $n = 8^0 + x^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{N}$.

Below we assume that $n-1 \in E(1,1,1)$. Then $n-1 = 4^k(8l+7)$ for some $k, l \in \mathbb{N}$.

If $k > 0$, then $n-8 \equiv n \equiv 1 \pmod 4$ and hence $n-8 = x^2+y^2+z^2$ for some $x, y, z \in \mathbb{N}$.

Now we consider the case $k = 0$. In this case $n = 8l+8$. If l is odd, then by Lemma 2.1 we can write $n-8 = 2^3 l$ as $x^2+y^2+z^2$ with $x, y, z \in \mathbb{N}$. Clearly,

$$8 = 0^2+0^2+0^2+8,\ 3\times 8 = 4^2+0^2+0^2+8,\ 5\times 8 = 4^2+4^2+0^2+8,\ 7\times 8 = 4^2+4^2+4^2+8.$$

If l is even and at least 8, then $n-64 = 8(l+1)-64 = 2^3(l-7) \notin E(1,1,1)$ and hence $n = 8^2+x^2+y^2+z^2$ for some $x, y, z \in \mathbb{N}$. This proves the first assertion in Theorem 1.1(i).

(b) Let $n = 4^k m$ with $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $4 \nmid m$. If $m = 1$, then $n^2 = 0^2+0^2+0^2+4^{2k}$. If $m = 1$ and $n > 1$, then

$$n^2 = (2^{2k-1})^2 + (2^{2k-1})^2 + (2^{2k-1})^2 + 4^{2(k-1)+1}.$$

Now let $m > 1$. If $m < 4$, then $m^2 - 16^\delta \notin E_0$ with $\delta = 0$. If $m > 4$, then by Lemma 2.2, for some $\delta \in \{0,1\}$ we have $m^2 - 16^\delta \notin E_0$. So $m^2 - 16^\delta = x^2+y^2+z^2$ for some $x, y, z \in \mathbb{N}$ and hence

$$n^2 = 16^k m^2 = (4^k x)^2 + (4^k y)^2 + (4^k z)^2 + 4^{2(k+\delta)}.$$

This proves the second assertion in Theorem 1.1(i) for $r = 0$.

Finally, we handle the case $r = 1$. If $2 \nmid m$, then $m^2 - 4 \notin E_0$. If $2 \mid m$ and $m < 8$, then $(m/2)^2 - 16^\delta \notin E_0$ with $\delta = 0$. If $2 \mid m$ and $m > 8$, then $(m/2)^2 - 16^\delta \notin E_0$ for some $\delta \in \{0,1\}$ (by Lemma 2.2). Anyway, for some $\delta \in \{0,1\}$ we can write $m^2 - 4 \times 16^\delta$ as $x^2+y^2+z^2$ with $x, y, z \in \mathbb{N}$, and thus

$$n^2 = 16^k m^2 = (4^k x)^2 + (4^k y)^2 + (4^k z)^2 + 4^{2(k+\delta)+1}$$

as desired.

(ii) Write $n = 2^a m$ with $a \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $2 \nmid m$. Let $n_0 = n/4^{\lfloor a/2 \rfloor} = 2^{a_0} m$, where a_0 is 0 or 1 according as a is even or odd. As $4 \nmid n_0$, we have $2n_0 - 1 \not\equiv 7 \pmod 8$. By Lemma 2.1, $2n_0 - 1 = u^2 + v^2 + (2y+1)^2$ for some $u, v, y \in \mathbb{N}$ with $u \equiv v \pmod 2$. Let $z = (u+v)/2$ and $w = |u-v|/2$. Then $2n_0 - 2 = 4y^2 + 4y + (z+w)^2 + (z-w)^2$ and hence $n_0 = x^2+y^2+z^2+w^2$ with $x = y+1$. It follows that

$$n = \left(2^{\lfloor a/2 \rfloor} x\right)^2 + \left(2^{\lfloor a/2 \rfloor} y\right)^2 + \left(2^{\lfloor a/2 \rfloor} z\right)^2 + \left(2^{\lfloor a/2 \rfloor} w\right)^2$$

with

$$2^{\lfloor a/2 \rfloor} x - 2^{\lfloor a/2 \rfloor} y = 2^{\lfloor a/2 \rfloor}.$$

(iii) Obviously, for any $k \in \mathbb{N}$ we have

$$2^{6k+3} \times 7 = (8^k \times 6)^2 + (8^k \times 4)^2 + (8^k \times 2)^2 + 0^2 \quad \text{with } 8^k \times 6 + 8^k \times 2 = 8^{k+1}.$$

So it suffices to prove the first assertion in Theorem 1.1(iii) by induction.

For each $n \in \{0, 1, \dots, 63\} \setminus \{56\}$, we can verify via a computer that n can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x - y \in \{8^k : k \in \mathbb{N}\} \cup \{0\}$.

Now let $n \geqslant 64$ be an integer not of the form $2^{6k+3} \times 7 = 64^k \times 56$ ($k \in \mathbb{N}$), and assume that any $m \in \{0, 1, \dots, n-1\}$ not of the form $2^{6k+3} \times 7$ ($k \in \mathbb{N}$) can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x - y \in \{8^k : k \in \mathbb{N}\} \cup \{0\}$.

If $64 \mid n$, then by the induction hypothesis we can write $n/64$ as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{N}$ and $x - y \in \{8^k : k \in \mathbb{N}\} \cup \{0\}$, and hence

$$n = (8x)^2 + (8y)^2 + (8z)^2 + (8w)^2 \quad \text{with } 8x - 8y = 8(x - y) \in \{8^k : k \in \mathbb{N}\} \cup \{0\}.$$

Below we suppose that $64 \nmid n$.

Case 1. $n \notin 4^k(16l + 14)$ for any $k, l \in \mathbb{N}$.

In this case, we have $2n \notin E(1, 1, 1)$ by (2.2), and hence $2n = (2y)^2 + z^2 + w^2$ for some $y, z, w \in \mathbb{N}$ with $z \equiv w \pmod 2$. Thus

$$n = y^2 + y^2 + \left(\frac{z+w}{2}\right)^2 + \left(\frac{z-w}{2}\right)^2 \quad \text{with } y - y = 0.$$

Case 2. $n = 16l + 14$ for some $l \in \mathbb{N}$.

In this case, $2n - 1 \equiv 3 \pmod 8$ and hence by (2.2) we can write $2n - 1$ as $(2y + 1)^2 + z^2 + w^2$ with $y, z, w \in \mathbb{N}$ and $z \equiv w \pmod 2$. It follows that

$$n = (y + 1)^2 + y^2 + \left(\frac{z+w}{2}\right)^2 + \left(\frac{z-w}{2}\right)^2 \quad \text{with } (y + 1) - y = 1.$$

Case 3. $n = 4^k(16l + 14)$ for some $k \in \{1, 2\}$ and $l \in \mathbb{N}$.

In this case, $2n - 64 = 4^{k+1}(8l + 3k)$. In light of (2.2), $8l + 3k = x^2 + y^2 + z^2$ for some integers $x \geqslant y \geqslant z \geqslant 0$. If $k = 1$, then $8l + 3k = (2n - 64)/16 \geqslant 64/16 = 4$. If $k = 2$, then $8l + 3k \geqslant 3k = 6$. So, $x \geqslant 2$ and hence $2^{k+1}x = 2v + 8$ for some $v \in \mathbb{N}$. Therefore

$$2n-64 = (2^{k+1}x)^2+(2^{k+1}y)^2+(2^{k+1}z)^2 = (2v+8)^2+2(2^k y+2^k z)^2+2(2^k y-2^k z)^2$$

and hence

$$n = (v + 8)^2 + v^2 + (2^k(y + z))^2 + (2^k(y - z))^2 \quad \text{with } (v + 8) - v = 8^1.$$

The induction proof of Theorem 1.1(iii) is now complete.

(iv) We distinguish two cases.

Case 1. $4 \nmid n$.

Let
$$\delta_n = 1 - \mathrm{ord}_2(n) = \begin{cases} 1 & \text{if } 2 \nmid n, \\ 0 & \text{if } 2 \| n. \end{cases}$$
Then
$$4^{\delta_n} n - 1 \equiv \begin{cases} 3 \pmod 8 & \text{if } 2 \nmid n, \\ 1 \pmod 4 & \text{if } 2 \| n. \end{cases}$$
By Lemma 2.1, there are $u, v, w \in \mathbb{Z}$ such that
$$4^{\delta_n} n - 1 = \left(2^{\delta_n} u - 1\right)^2 + \left(2^{\delta_n} v - 1\right)^2 + \left(2^{\delta_n} w - 1\right)^2.$$
If $2 \| n$, then $\delta_n = 0$ and $u + v + w \equiv n \equiv 0 \pmod 2$. In the case $2 \nmid n$, since $(2u-1)^2 = (2(1-u)-1)^2$, without loss of generality we may also assume that $u + v + w \equiv 0 \pmod 2$. Set
$$x = \frac{u+v-w}{2},\ y = \frac{u-v+w}{2},\ z = \frac{-u+v+w}{2}.$$
Then $u = x + y$, $v = x + z$ and $w = y + z$. Therefore
$$\begin{aligned} 4^{\delta_n} n - 1 &= \left(2^{\delta_n}(x+y) - 1\right)^2 + \left(2^{\delta_n}(x+z) - 1\right)^2 + \left(2^{\delta_n}(y+z) - 1\right)^2 \\ &= \left(2^{\delta_n}(x+y+z) - 2\right)^2 + \left(2^{\delta_n} x\right)^2 + \left(2^{\delta_n} y\right)^2 + \left(2^{\delta_n} z\right)^2 - 1 \end{aligned}$$
and hence
$$n = \left((x+y+z) - 2^{1-\delta_n}\right)^2 + x^2 + y^2 + z^2 = x^2 + y^2 + z^2 + \left(2^{1-\delta_n} - x - y - z\right)^2$$
with $x + y + z + (2^{1-\delta_n} - x - y - z) = 2^{\lfloor (\mathrm{ord}_2(n)+1)/2 \rfloor}$.

Case 2. $4 \mid n$.

Write $n = 4^k n_0$ with $k, n_0 \in \mathbb{Z}^+$ and $4 \nmid n_0$. By the above, there are $x_0, y_0, z_0, w_0 \in \mathbb{Z}$ such that
$$n_0 = x_0^2 + y_0^2 + z_0^2 + w_0^2 \ \text{ with } \ x_0 + y_0 + z_0 + w_0 = 2^{\lfloor (\mathrm{ord}_2(n_0)+1)/2 \rfloor}.$$
It follows that
$$n = \left(2^k x_0\right)^2 + \left(2^k y_0\right)^2 + \left(2^k z_0\right)^2 + \left(2^k w_0\right)^2$$
with
$$2^k x_0 + 2^k y_0 + 2^k z_0 + 2^k w_0 = 2^{k + \lfloor (\mathrm{ord}_2(n_0)+1)/2 \rfloor} = 2^{\lfloor (\mathrm{ord}_2(n)+1)/2 \rfloor}.$$

Combining the above, we have proved Theorem 1.1. $\square$

Lemma 2.3. (i) (Dickson [D39, pp. 112-113]) *We have*

$$E(1,5,5)=\{n\in\mathbb{N}:\ n\equiv 2,3 \pmod 5\}\cup E_0. \tag{2.3}$$

(ii) ([S17a, Lemma 2.1]) *Let u and v be integers with u^2+v^2 a positive multiple of 5. Then $u^2+v^2=x^2+y^2$ for some $x,y\in\mathbb{Z}$ with $5\nmid xy$.*

Lemma 2.4. *Let $n\geqslant 4$ be an integer not divisible by 64.*
(i) *We have $\{n,n-4\}\not\subseteq E_0$.*
(ii) *If $16\nmid n$, then $\{n,n-1\}\not\subseteq E_0$. If $16\mid n$, then $\{n,n-1,n-64\}\not\subseteq E_0$.*

Proof. (i) If $n=4^k(8l+7)$ for some $k\in\mathbb{N}$ and $l\in\mathbb{N}$, then $k<3$ as $64\nmid n$, and hence $n-4\notin E_0$ since

$$n-4=\begin{cases}8l+7-4=8l+3 & \text{if } k=0,\\ 4(8l+7)-4=4(8l+6) & \text{if } k=1,\\ 4^2(8l+7)-4=4(8(4l+3)+3) & \text{if } k=2.\end{cases}$$

(ii) If $n=8l+7$ for some $l\in\mathbb{N}$, then $n-1=8l+6\notin E_0$. If $n=4(8l+7)$ for some $l\in\mathbb{N}$, then $n-1=32l+27=8(4l+3)+3\notin E_0$. So $\{n,n-1\}\not\subseteq E_0$ if $16\nmid n$.

Now we consider the case $16\mid n$. As $64\nmid n$, if $n\notin E_0$, then $n=4^2(8l+7)$ for some $l\in\mathbb{N}$, and hence $n-64=4^2(8l+3)\notin E_0$.

The proof of Lemma 2.4 is now complete. □

Proof of Theorem 1.2. (i) For $n=1,\ldots,15$ we can easily verify the desired result.

Now fix an integer $n\geqslant 16$ and assume that each $m=1,\ldots,n-1$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ with $|2x-y|\in\{4^k:\ k\in\mathbb{N}\}$.

Let's first consider the case $16\mid n$. By the the induction hypothesis, there are $x,y,z,w\in\mathbb{N}$ for which $n/16=x^2+y^2+z^2+w^2$ with $|2x-y|\in\{4^k:\ k\in\mathbb{N}\}$, and hence $|2(4x)-4y|=4|2x-y|\in\{4^k:\ k\in\mathbb{N}\}$.

Now we suppose that $16\nmid n$. Then $\{5n-1,5n-16\}\not\subseteq E_0$ by Lemma 2.2. Let $\delta=0$ if $5n-1\notin E_0$, and $\delta=1$ otherwise. In view of Lemma 2.1, we can write $5n-16^\delta$ as $x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$. Since a square is congruent to one of $0,1,-1$ modulo 5. one of x^2,y^2,z^2 must be congruent to -1 modulo 5. Without loss of generality, we may assume that $x^2+1\equiv y^2+z^2\equiv 0 \pmod 5$. If $y^2+z^2\neq 0$, then by Lemma 2.3(ii) we can write $y^2+z^2=y_1^2+z_1^2$ with $y_1,z_1\in\mathbb{Z}$ and $5\nmid y_1z_1$. Without loss of generality, we simply assume that $x\equiv -2\times 4^\delta \pmod 5$ (otherwise we use $-x$ instead of x) and that either $y=z=0$ or $y\equiv 2z\equiv -2\times 4^\delta \pmod 5$.

Clearly, $r=(x+2\times 4^\delta)/5$, $s=(2x-4^\delta)/5$, $u=(2y+z)/5$ and $v=(2z-y)/5$ are all integers. Observe that

$$r^2+s^2+u^2+v^2=\frac{(4^\delta)^2+x^2}{5}+\frac{y^2+z^2}{5}=n\ \text{ with }\ 2r-s=4^\delta.$$

If $x > -2$, then $x \geqslant 2$ since $x \equiv (-1)^{\delta-1}2 \pmod 5$, and hence $r, s \in \mathbb{N}$. When $x \leqslant -2$, clearly $s < 0$, and

$$r > 0 \iff (\delta = 1 \text{ and } x = -3).$$

If $r \leqslant 0$ and $s \leqslant 0$, then

$$|2|r| - |s|| = |2(-r) - (-s)| = |2r - s| = 4^\delta.$$

Now it remains to consider the case $\delta = 1$ and $x = -3$. Note that

$$\frac{y^2+z^2}{5} = u^2 + v^2 = n - r^2 - s^2 = n - 1^2 - (-2)^2 > 0$$

and hence $y \equiv 2z \equiv -2 \times 4^\delta \pmod 5$. When $y \neq -3$, in the spirit of the above arguments, all the four numbers

$$\bar{r} = \frac{y + 2 \times 4^\delta}{5},\ \bar{s} = \frac{2y - 4^\delta}{5},\ \bar{u} = \frac{2x+z}{5},\ \bar{v} = \frac{2z-x}{5}$$

are integral, and

$$\bar{r}^2 + \bar{s}^2 + \bar{u}^2 + \bar{v}^2 = \frac{(4^\delta)^2 + y^2}{5} + \frac{x^2+z^2}{5} = n$$

with $|2|\bar{r}| - |\bar{s}|| = |2\bar{r} - \bar{s}| = 4^\delta$. If $y = -3$, then

$$5n - 16 = x^2 + y^2 + z^2 = (-3)^2 + (-3)^2 + z^2 \equiv z^2 + 2 \not\equiv 0, 1 \pmod 4,$$

thus $5n - 1 \not\equiv 0, 3 \pmod 4$ and hence $5n - 1 \notin E_0$ which contradicts $\delta = 1$. This concludes our induction proof of Theorem 1.2(i).

(ii) For every $n = 0, 1, \ldots, 63$, we can verify the desired result directly.

Now fix an integer $n \geqslant 64$ and assume that the desired result holds for all smaller values of n.

If $64 \mid n$, then by the induction hypothesis we can write $n/64$ as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{N}$ and $2x - y \in \{\pm a8^k :\ k \in \mathbb{N}\} \cup \{0\}$, and hence

$$n = (8x)^2 + (8y)^2 + (8z)^2 + (8w)^2 \text{ with } 2(8x) - (8y) \in \{\pm a8^k :\ k \in \mathbb{N}\} \cup \{0\}.$$

Below we suppose that $64 \nmid n$.

By Lemma 2.4, $5n - (a\delta)^2 \notin E_0$ for some $\delta \in \{0, 1, 8\}$ satisfying

$$\delta = 8 \implies (a = 1 \text{ and } 16 \mid n). \tag{2.4}$$

In view of (2.2), $5n - (a\delta)^2 = x^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{Z}$. Since any square is congruent to one of $0, \pm 1$ modulo 5, one of x^2, y^2, z^2, say x^2, is congruent to

$-(a\delta)^2$ modulo 5. Without loss of generality, we simply suppose that $x \equiv -2a\delta \pmod 5$. As $y^2 \equiv (2z)^2 \pmod 5$, we may also suppose that $y \equiv 2z \pmod 5$. Thus all the numbers

$$r = \frac{x + 2a\delta}{5}, \ s = \frac{2x - a\delta}{5}, \ u = \frac{2y + z}{5}, \ v = \frac{y - 2z}{5}$$

are integral. Note that

$$n = \frac{(a\delta)^2 + x^2}{5} + \frac{y^2 + z^2}{5} = r^2 + s^2 + u^2 + v^2$$

with

$$2r - s = a\delta \in \{a8^k : \ k \in \mathbb{N}\} \cup \{0\}.$$

If $x \geqslant a\delta/2$, then $r \geqslant 0$ and $s \geqslant 0$. If $x \leqslant -2a\delta$, then $r \leqslant 0$ and $s \leqslant -a\delta \leqslant 0$.

Now we handle the remaining case $-2a\delta < x < a\delta/2$. Clearly, $r > 0 > s$, $a\delta = 2r - s > 2$, hence $\delta = 8$, and $a = 1$ and $16 \mid n$ by (2.4). As $2r - s = 8$, we must have

$$(r, s) \in \{(1, -6), \ (2, -4), \ (3, -2)\}.$$

Note that $2 \times 2 - 4 = 0$ and $2 \times 2 - 3 = 1$. If $(r, s) = (1, -6)$, then

$$n = 1^2 + (-6)^2 + u^2 + v^2 \not\equiv 0 \pmod 4,$$

which contradicts $16 \mid n$. This ends our proof of Theorem 1.2(ii).

(iii) Suppose that any positive integer $n \equiv 9 \pmod{20}$ can be written as $5x^2 + 5y^2 + z^2$ with $x, y, z \in \mathbb{Z}$ and $2 \nmid z$. Below we prove by induction that any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 3y \in \{4^k : \ k \in \mathbb{N}\}$.

It is easy to verify that each $n = 1, \ldots, 15$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 3y \in \{4^k : \ k \in \mathbb{N}\}$.

Now let $n \in \mathbb{Z}^+$ with $n \geqslant 16$, and assume that each $m = 1, \ldots, n - 1$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 3y \in \{4^k : \ k \in \mathbb{N}\}$. If $16 \mid n$, then by the induction hypothesis there are $x, y, z, w \in \mathbb{Z}$ such that $n/16 = x^2 + y^2 + z^2 + w^2$ and $x + 3y \in \{4^k : \ k \in \mathbb{N}\}$, and hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $4x + 3(4y) = 4(x + 3y) \in \{4^k : \ k \in \mathbb{N}\}$.

Below we suppose that $16 \nmid n$.

Case 1. $2 \nmid n$.

In this case, $10n - 1 \equiv 9 \pmod{20}$ and hence $10n - 1 = 5u^2 + 5v^2 + x^2$ for some $u, v, x \in \mathbb{Z}$ with $2 \nmid x$. As $u^2 + v^2$ is even, both $y = (u + v)/2$ and $z = (u - v)/2$ are integers. Note that $10n - 1 = x^2 + 10y^2 + 10z^2$.

Case 2. $n = 2m$ for some $m \in \mathbb{Z}^+$.

Note that $8 \nmid m$ since $16 \nmid n$. For $m = 9, 10, 11, 12$ we can easily verify the desired result. Assume that $m \geqslant 13$. Then $5m - 64 \geqslant 5 \times 13 - 64 > 0$. If m is odd, then either $5m - 4$ or $5m - 64$ is not congruent to 7 mod 8. If $2 \| m$, then $5m - 4 \equiv 5m - 64 \equiv 2 \pmod 4$. If $4 \| m$ and $(5m - 64)/4 = 5m/4 - 16 \equiv 7$

(mod 8), then $(5m-4)/4 = 5m/4 - 1 \equiv 6 \pmod 8$. So, for some $\delta \in \{2,4\}$ we have $5m - (\delta^2/2)^2 \notin E_0$, and hence by (2.3) there are $u,v,w \in \mathbb{Z}$ such that $5m - \delta^4/4 = 5u^2 + 5v^2 + w^2$. Clearly,

$$10n-\delta^4 = 4\left(5m - \frac{\delta^4}{4}\right) = 10(2u^2+2v^2)+(2w)^2 = (2w)^2+10(u+v)^2+10(u-v)^2.$$

In view of the above, for some $\delta \in \{1,2,4\}$ we have $10n-\delta^4 = x^2+10y^2+10z^2$ for some $x,y,z \in \mathbb{Z}$. As $x \equiv \pm 3\delta^2 \pmod{10}$, we may simply assume that $x = 10w + 3\delta^2$ for some $w \in \mathbb{Z}$. Thus

$$10n - \delta^4 = (10w+3\delta^2)^2 + 10y^2 + 10z^2$$

and hence

$$n = 10w^2 + 6\delta^2 w + \delta^4 + y^2 + z^2 = (3w+\delta^2)^2 + (-w)^2 + y^2 + z^2$$

with $(3w+\delta^2) + 3(-w) = \delta^2 \in \{4^k : k \in \mathbb{N}\}$. This concludes the induction proof of Theorem 1.1(iii). $\square$

Lemma 2.5. *We have*

$$E(1,3,6) = \{3q+2 : q \in \mathbb{N}\} \cup \{4^k(16l+14) : k,l \in \mathbb{N}\} \tag{2.5}$$

and

$$E(2,3,6) = \{3q+1 : q \in \mathbb{N}\} \cup E_0. \tag{2.6}$$

Remark 2.3. (2.5) and (2.6) are known results, see, e.g., L. Dickson [D39, pp. 112-113].

Proof of Theorem 1.3. (i) We can easily verify the desired result for $n = 1,\ldots,15$.

Now let $n \geqslant 16$ and assume that each $m = 1,2,\ldots,n-1$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{N})$ with $|x+y-z| \in \{4^k : k \in \mathbb{N}\}$.

We first suppose that $16 \mid n$. By the induction hypothesis, there are $x,y,z,w,k \in \mathbb{N}$ for which $n/16 = x^2+y^2+z^2+w^2$ with $|x+y-z| = 4^k$, and hence $n = (4x)^2+(4y)^2+(4z)^2+(4w)^2$ with $|4x+4y-4z| = 4^{k+1}$.

Now we suppose that $16 \nmid n$. If $16 < n < 86$ then we can verify the desired result via a computer. Thus we simply let $n \geqslant 86$ and hence $3n \geqslant 258 > 16^2$. Let $\delta = 0$ if $3n-1 \notin E_0$. In the case $3n-1 \in E_0$, we let $\delta = 1$ if $n-6$ is not an odd square, and $\delta = 2$ otherwise. By Lemmas 2.2 and 2.5, if $3n-1 \in E_0$ then $3n-16, 3n-16^2 \notin E(2,3,6)$. As $3n > 16^\delta$ and $3n - 16^\delta \notin E(2,3,6)$, there are $x,y \in \mathbb{N}$ and $z \in \mathbb{Z}$ such that

$$3n - 16^\delta = 3x^2 + 6y^2 + 2(3z-4^\delta)^2 = 3(x^2+2y^2+2(3z^2-2\times 4^\delta z)) + 2\times 16^\delta$$

and hence

$$n = x^2 + 2y^2 + 6z^2 - 4^{\delta+1}z + 16^\delta = x^2 + (y+z)^2 + (z-y)^2 + (4^\delta - 2z)^2$$

with $(y+z)+(z-y)+(4^\delta-2z) = 4^\delta$.

When $z \geqslant 2^{2\delta-1}$, we have $2z \geqslant 4^\delta$, hence

$$(y+z) + |z-y| - (2z-4^\delta) = 4^\delta \quad \text{if } y \leqslant z,$$

and

$$||z-y| + (2z-4^\delta) - (y+z)| = 4^\delta \quad \text{if } y > z.$$

When $y \geqslant z$ and $z < 2^{2\delta-1}$,

$$|y+z| + (4^\delta - 2z) - (y-z) = 4^\delta \quad \text{if } y+z \geqslant 0,$$

and

$$|(y-z) + |y+z| - (4^\delta - 2z)| = 4^\delta \quad \text{if } y+z < 0.$$

Below we assume $0 \leqslant y < z < 2^{2\delta-1}$. Clearly $\delta > 0$. If $\delta = 1$, then we must have $y = 0$ and $z = 1$, hence

$$n = x^2 + (y+z)^2 + (z-y)^2 + (4^\delta - 2z)^2 = x^2 + 1^2 + 1^2 + (4-2)^2 = x^2 + 6.$$

If $2 \mid x$, then $3(x^2+6) - 1 \equiv 3 \times 6 - 1 \equiv 1 \pmod 4$ and hence $3(x^2+6)-1 \notin E_0$. Thus $\delta \neq 1$ by the definition of δ. So $\delta = 2$ and $0 \leqslant y < z < 8$. As $n - 6 = x_0^2$ for a positive odd integer, we have

$$n = x^2 + (y+z)^2 + (z-y)^2 + (16-2z)^2 \equiv 7 \pmod 8,$$

hence $2 \nmid x$, $2 \nmid y \pm z$ and $2 \nmid z$.

If $z = 1$, then $y = 0$ and

$$n = x^2 + 1^2 + 1^2 + 14^2 = x^2 + 13^2 + 5^2 + 2^2$$

with $13 + 5 - 2 = 4^2$. If z is 3 or 5, then

$$n = x^2 + (y+z)^2 + (z-y)^2 + (16-2z)^2$$

with $|y + z + (z-y) - (16-2z)| = |4(z-4)| = 4$.

Now we handle the remaining case $z = 7$. Note that $y \in \{0, 2, 4, 6\}$. If $y = 2$, then

$$n = x^2 + (2+7)^2 + (7-2)^2 + (16-14)^2 = x^2 + 6^2 + 5^2 + 7^2$$

with $6 + 5 - 7 = 4$. If $y = 4$, then

$$n = x^2 + (4+7)^2 + (7-4)^2 + (16-14)^2 = x^2 + 9^2 + 2^2 + 7^2$$

with $9+2-7=4$. If $y=6$, then

$$n = x^2 + (6+7)^2 + (7-6)^2 + (16-14)^2 = x^2 + 11^2 + 7^2 + 2^2$$

with $11+7-2=4^2$.

In the case $y=0$ and $z=7$, we have

$$x_0^2 + 6 = n = x^2 + (0+7)^2 + (7-0)^2 + (16-14)^2 = x^2 + 102$$

and hence

$$\frac{x_0-x}{2}\cdot\frac{x_0+x}{2} = \frac{102-6}{4} = 24.$$

As x_0 and x are positive and odd, $(x_0-x)/2 \not\equiv (x_0+x)/2 \pmod 2$, hence either

$$\frac{x_0-x}{2} = 1,\ \frac{x_0+x}{2} = 24,\ \text{and thus } n = x_0^2+6 = 25^2+6,$$

or

$$\frac{x_0-x}{2} = 3,\ \frac{x_0+x}{2} = 8,\ \text{and thus } n = x_0^2+6 = 11^2+6.$$

Note that

$$11^2+6 = 1^2+5^2+10^2+1^2 \text{ with } |1+5-10| = 4$$

and

$$25^2+6 = 1^2+5^2+22^2+11^2 \text{ with } |1+5-22| = 4^2.$$

In view of the above, we have completed the induction proof of Theorem 1.3(i).

(ii) Via a computer we can easily verify the desired result for every $n = 0,1,\ldots,63$.

Now fix an integer $n \geqslant 64$ and assume that the desired result holds for all smaller values of n.

If $64 \mid n$, then by the induction hypothesis we can write $n/64$ as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ and $x+y-z \in \{\pm a8^k:\ k \in \mathbb{N}\} \cup \{0\}$, and hence

$$n = (8x)^2 + (8y)^2 + (8z)^2 + (8w)^2 \quad \text{with } 8x+8y-8z \in \{\pm a8^k:\ k \in \mathbb{N}\} \cup \{0\}.$$

Below we suppose that $64 \nmid n$.

By Lemma 2.4, $3n-(a\delta)^2 \not\in E_0$ for some $\delta \in \{0,1,8\}$ satisfying (2.4). In view of (2.6), for some $x,y,z \in \mathbb{Z}$ we have $3n-(a\delta)^2 = 3x^2+6y^2+2(3z-a\delta)^2$ and hence

$$n = x^2 + (y+z)^2 + (z-y)^2 + (2z-a\delta)^2.$$

When $z \geqslant a\delta/2$, obviously $y+z \geqslant 0$, $2z-a\delta \geqslant 0$,

$$(y+z)+(z-y)-(2z-a\delta) = a\delta \in \{\pm a8^k:\ k \in \mathbb{N}\} \cup \{0\} \quad \text{if } y \leqslant z,$$

and

$$(y-z)+(2z-a\delta)-(y+z)=-a\delta\in\{\pm a8^k:\ k\in\mathbb{N}\}\cup\{0\}\quad\text{if } y>z.$$

If $z<a\delta/2$ and $y\geqslant |z|$, then $\{a\delta-2z, y+z, y-z\}\subseteq\mathbb{N}$ and

$$(a\delta-2z)+(y+z)-(y-z)=a\delta\in\{\pm a8^k:\ k\in\mathbb{N}\}\cup\{0\}.$$

If $z<a\delta/2$ and $z\leqslant -y$, then $\{a\delta-2z, y-z, -y-z\}\subseteq\mathbb{N}$ and

$$(y-z)+(-y-z)-(a\delta-2z)=-a\delta\in\{\pm a8^k:\ k\in\mathbb{N}\}\cup\{0\}.$$

Now we consider the remaining case $y<z<a\delta/2$. Since $a\delta>2$, we have $\delta=8$, and hence $a=1$ and $16\mid n$ by (2.4). In the case $y<z<8/2=4$, the ordered triple $(y+z, z-y, 8-2z)$ is among

$$(1,1,6),\ (2,2,4),\ (3,3,2),\ (3,1,4),\ (4,2,2),\ (5,1,2).$$

Note that $2+2-4=0=3+1-4$. If

$$(y+z, z-y, 8-2z)\in\{(1,1,6), (3,3,2), (5,1,2)\},$$

then

$$n=x^2+(y+z)^2+(z-y)^2+(8-2z)^2\equiv x^2+2\not\equiv 0\pmod 4$$

which contradicts $16\mid n$. This concludes the proof of Theorem 1.3(ii). $\square$

Lemma 2.6. *Suppose that $n\in\mathbb{Z}^+$ is the sum of three squares. Then $n=a^2+b^2+c^2$ for some $a,b,c\in\mathbb{Z}$ with $a+b\equiv 1\pmod 3$.*

Proof. Write $n=x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$. If $x\equiv y\equiv z\equiv 0\pmod 3$, then $9\mid n$ and hence by [S16, Lemma 2.2(ii)] we can write n as $\bar{x}^2+\bar{y}^2+\bar{z}^2$ with $\bar{x},\bar{y},\bar{z}\in\mathbb{Z}$ and $3\nmid\bar{x}\bar{y}\bar{z}$. So, without loss of generality we may assume that $3\nmid x$. If $x+y\equiv 0\pmod 3$, then $-x+y\not\equiv 0\pmod 3$. Thus, we may simply suppose that $x+y\not\equiv 0\pmod 3$ and hence $x+y$ is congruent to 1 or -1 modulo 3. If $x+y\equiv -1\pmod 3$, then $n=(-x)^2+(-y)^2+z^2$ with $(-x)+(-y)\equiv 1\pmod 3$. Therefore, there are $a,b,c\in\mathbb{Z}$ such that $n=a^2+b^2+c^2$ and $a+b\equiv 1\pmod 3$. $\square$

Proof of Theorem 1.4. (i) We prove the desired result by induction. For $n=1,2,\ldots,42$ we can verify the result directly via a computer.

Now let $n\geqslant 43$ and assume that any $m=1,\ldots,n-1$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{Z})$ with $x+y+2z=c4^k$ for some $k\in\mathbb{N}$.

If $16\mid n$, then by the induction hypothesis there are $x,y,z,w\in\mathbb{Z}$ and $k\in\mathbb{N}$ such that $n/16=x^2+y^2+z^2+w^2$ and $x+y+2z=c4^k$, hence $n=(4x)^2+(4y)^2+(4z)^2+(4w)^2$ with $4x+4y+2(4z)=c4^{k+1}$.

Now suppose that $16 \nmid n$.

Clearly, $3n-2 \not\equiv 14 \pmod{16}$. If $4 \nmid 3n-2$, then $3n-2 \notin E(1,3,6)$ by (2.4). If $4 \mid 3n-2$, then $4 \nmid 3n-32$. If $3n-32 \equiv 14 \pmod{16}$, then $3n-2 \equiv 12 \pmod{16}$ and hence $3n-2 \notin E(1,3,6)$. Thus, for some $\delta \in \{0,1\}$, we can write

$$3n - 2 \times 16^\delta = 3x^2 + 6y^2 + (3z - 4^\delta)^2 = 3x^2 + 6y^2 + 9z^2 - 6 \times 4^\delta z + 16^\delta$$

with $x, y, z \in \mathbb{Z}$. It follows that

$$n = x^2 + 2y^2 + 3z^2 - 2 \times 4^\delta z + 16^\delta = x^2 + (y+z)^2 + (z-y)^2 + (4^\delta - z)^2$$

with $(y+z) + (z-y) + 2(4^\delta - z) = 2 \times 4^\delta$. This proves the desired result with $c = 2$.

Below we show the desired result for $c = 1$.

Case 1. $4 \nmid n$.

In this case, $6n - 1 \not\equiv 7 \pmod 8$ and hence $6n - 1 \notin E(2,3,6)$ by (2.6). So, for some $x, y, z \in \mathbb{Z}$ we have

$$6n - 1 = 6x^2 + 3(2y+1)^2 + 2(3z+1)^2$$

and hence

$$n = x^2 + 2y^2 + 2y + 3z^2 + 1 = x^2 + (y+z+1)^2 + (z-y)^2 + (-z)^2$$

with $(y+z+1) + (z-y) + 2(-z) = 1 = 4^0$.

Case 2. $4 \| n$.

In this case, $3n - 8 \equiv 4 \pmod 8$ and hence $3n - 8 \notin E(1,3,6)$ by (2.5). Thus, for some $x, y, z \in \mathbb{Z}$ we have $3n - 8 = 3x^2 + 6y^2 + (3z-2)^2$ and hence

$$n = x^2 + 2y^2 + 3z^2 - 4z + 4 = x^2 + (y+z)^2 + (z-y)^2 + (2-z)^2$$

with $(y+z) + (z-y) + 2(2-z) = 4$.

Case 3. $8 \| n$.

Write $n = 8q$ with q odd. Note that $q \geqslant 6$ and $3n > 128$ since $n \geqslant 43$. By Lemma 2.2, for some $\delta \in \{0,1\}$ we have $3q - 16^\delta \notin E_0$ and hence

$$3n - 8 \times 16^\delta = 8(3q - 16^\delta) \notin E(1,3,6)$$

by (2.5). Thus, for some $x, y, z \in \mathbb{Z}$ we have

$$3n - 8 \times 16^\delta = 3x^2 + 6y^2 + (3z - 2 \times 4^\delta)^2$$

and hence

$$n = x^2 + 2y^2 + 3z^2 - 4^{\delta+1} z + 4 \times 16^\delta = x^2 + (y+z)^2 + (z-y)^2 + (2 \times 4^\delta - z)^2$$

with $(y+z)+(z-y)+2(2\times 4^{\delta}-z)=4^{\delta+1}$.

The induction proof of Theorem 1.4(i) is now completed.

(ii) Let $m\in\{2,3\}$. For $n=1,2,\ldots,4^m-1$ we can easily verify that n^{m-1} can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{Z}$ and $x+2y+2z\in\{2^{km}:\ k\in\mathbb{N}\}$.

Now let $n\geqslant 4^m$ and assume that for each $n_0=1,2,\ldots,n-1$ we can write n_0^{m-1} as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{Z})$ with $x+2y+2z$ a power of 2^m.

If $4^m\mid n^{m-1}$, then by the induction hypothesis there are $x,y,z,w\in\mathbb{Z}$ and $k\in\mathbb{N}$ for which $n^{m-1}/4^m=x^2+y^2+z^2+w^2$ with $x+2y+2z=2^{km}$, and hence $n^{m-1}=(2^mx)^2+(2^my)^2+(2^mz)^2+(2^mw)^2$ with $2^mx+2(2^my)+2(2^mz)=2^{(k+1)m}$.

Now we suppose that $4^m\nmid n^{m-1}$. By Lemmas 2.1-2.2, if $16\nmid n^{m-1}$, then for some $\delta\in\{0,1\}$ we can write $9n^{m-1}-4^{\delta m}$ as $a^2+b^2+c^2$ with $a,b,c\in\mathbb{Z}$. When $16\mid n^{m-1}$, we must have $m=3$ and $4\| n$, thus by taking $\delta=1$ we find that $9n^{m-1}-4^{\delta m}=4^2(9(n/4)^2-4)$ can be written as $a^2+b^2+c^2$ with $a,b,c\in\mathbb{Z}$. Clearly, we cannot have $3\nmid abc$. Without loss of generality, we assume that $c=3w$ with $w\in\mathbb{Z}$. As $a^2+b^2\equiv-16^{\delta}\equiv 2\pmod 3$, we must have $3\nmid ab$. We may simply suppose that $a=3u+2^{\delta m+1}$ and $b=3v-2^{\delta m+1}$ with $u,v\in\mathbb{Z}$. (Note that if $x\equiv 1\equiv -2\pmod 3$ then $-x\equiv 2\pmod 3$.) Since

$$\begin{aligned}&12\times 2^{\delta m}u-12\times 2^{\delta m}v+8(2^{\delta m})^2\\ \equiv&(3u+2\times 2^{\delta m})^2+(3v-2\times 2^{\delta m})^2=a^2+b^2\equiv-(2^{\delta m})^2\pmod 9,\end{aligned}$$

we must have $u\equiv v\pmod 3$. Set

$$y=\frac{2u+v}{3}\quad\text{and}\quad z=\frac{u+2v}{3}.$$

Then

$$\begin{aligned}9n^{m-1}-4^{\delta m}=&a^2+b^2+c^2=(3u+2^{\delta m+1})^2+(3v-2^{\delta m+1})^2+9w^2\\ =&(3(2y-z)+2^{\delta m+1})^2+(3(2z-y)-2^{\delta m+1})^2+9w^2\\ =&9(2y-z)^2+9(2z-y)^2+3\times 2^{\delta m+2}((2y-z)-(2z-y))\\ &+8\times 4^{\delta m}+9w^2\end{aligned}$$

and hence

$$\begin{aligned}n^{m-1}=&(2y-z)^2+(2z-y)^2+2^{\delta m+2}(y-z)+w^2+4^{\delta m}\\ =&(2y-2z+2^{\delta m})^2+(-y)^2+z^2+w^2\end{aligned}$$

with $(2y-2z+2^{\delta m})+2(-y)+2z=2^{\delta m}$. This proves Theorem 1.4(ii).

(iii) For any $k\in\mathbb{N}$, we obviously have

$$2^{2k+1}=(2^k)^2+(2^k)^2+0^2+0^2\ \text{with}\ 2^k+2\times 2^k+2\times 0=3\times 2^k,$$

and

$$2^{2k+2} = (-2^k)^2 + (2^k)^2 + (2^k)^2 + (2^k)^2 \text{ with } (-2^k) + 2 \times 2^k + 2 \times 2^k = 3 \times 2^k.$$

So, it suffices to prove the first assertion in Theorem 1.4(iii).

For $n \in \{1, \ldots, 15\}$ with $n \neq 1, 8$, we can easily verify that n can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + 2y + 2z \in \{3 \times 4^k : k \in \mathbb{N}\}$. For example, $4 = (-1)^2 + 1^2 + 1^2 + 1^2$ with $(-1) + 2 \times 1 + 2 \times 1 = 3$.

Now let $n \geqslant 16$ with $n \notin S = \bigcup_{k \in \mathbb{N}} \{2^{4k}, 2^{4k+3}\}$, and assume that each $m = 1, 2, \ldots, n-1$ with $m \notin S$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + 2y + 2z \in \{3 \times 4^k : k \in \mathbb{N}\}$.

If $16 \mid n$, then by the induction hypothesis there are $x, y, z, w \in \mathbb{Z}$ and $k \in \mathbb{N}$ for which $n/16 = x^2 + y^2 + z^2 + w^2$ with $x + 2y + 2z = 3 \times 4^k$, and hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $4x + 2(4y) + 2(4z) = 3 \times 4^{k+1}$.

Now we suppose that $16 \nmid n$. By Lemmas 2.1-2.2, for some $\delta \in \{0, 1\}$ we can write $n - 16^\delta$ as the sum of three squares. Combining this with Lemma 2.6, we see that $n - 16^\delta = a^2 + b^2 + c^2$ for some $a, b, c \in \mathbb{Z}$ with $a + b \equiv 1 \pmod 3$. Let $u = a - 2^{2\delta+1}$ and $v = b - 2^{2\delta+1}$. Then $u + v \equiv a - 2 + (b - 2) \equiv 0 \pmod 3$, and

$$y = \frac{2u - v}{3} \quad \text{and} \quad z = \frac{2v - u}{3}$$

are both integers. Observe that

$$n - 16^\delta = (u + 2 \times 4^\delta)^2 + (v + 2 \times 4^\delta)^2 + c^2 = (2y + z + 2 \times 4^\delta)^2 + (y + 2z + 2 \times 4^\delta)^2 + c^2$$

and hence

$$n = (2y + 2z + 3 \times 4^\delta)^2 + (-y)^2 + (-z)^2 + c^2$$

with $(2y + 2z + 3 \times 4^\delta) + 2(-y) + 2(-z) = 3 \times 4^\delta$.

So far we have completed the proof of Theorem 1.4. □

3. Proofs of Theorems 1.5 and 1.6

Let us first recall some known results on ternary quadratic forms.

Lemma 3.1. *We have*

$$E(1,2,4) = \{4^k(16l + 14) : k, l \in \mathbb{N}\}, \tag{3.1}$$

$$E(1,6,9) = \{3q + 2 : q \in \mathbb{N}\} \cup \{9^k(9l + 3) : k, l \in \mathbb{N}\}. \tag{3.2}$$

$$E(2,3,12) = \{16q + 6 : q \in \mathbb{N}\} \cup \{9^k(3l + 1) : k, l \in \mathbb{N}\}, \tag{3.3}$$

$$E(1,5,10) = \bigcup_{k,l \in \mathbb{N}} \{25^k(5l + 2), 25^k(5l + 3)\}, \tag{3.4}$$

$$E(2,5,10) = \{8q + 3 : q \in \mathbb{N}\} \cup \bigcup_{k,l \in \mathbb{N}} \{25^k(5l + 1), 25^k(5l + 4)\}. \tag{3.5}$$

Remark 3.1. (3.1)-(3.5) can be found in L. E. Dickson [D39, pp. 112-113].

Lemma 3.2. *Let $n \in \mathbb{Z}^+$ and $\delta \in \{0,1\}$. Then $6n+1 = x^2+3y^2+6z^2$ for some $x,y,z \in \mathbb{Z}$ with $x \equiv \delta \pmod 2$.*

Remark 3.2. This appeared in Sun [S17a, Remark 3.1].

We also need the following result in [S15].

Lemma 3.3. (i) *Any $n \in \mathbb{N}$ with $n \equiv 4 \pmod{12}$ can be written as $x^2+3y^2+3z^2$ with $x,y,z \in \mathbb{Z}$ and $2 \nmid x$.*

(ii) *For $n \in \mathbb{N}$ with $n \equiv 4 \pmod 8$, we have*

$$|\{(x,y) \in \mathbb{Z}^2 :\ x^2+3y^2 = n \text{ and } 2 \nmid xy\}| = \frac{2}{3}|\{(x,y) \in \mathbb{Z}^2 :\ x^2+3y^2 = n\}|. \tag{3.6}$$

Remark 3.3. For parts (i) and (ii) one may consult Theorem 1.7(iii) and Lemma 3.2 of Sun [S15].

Lemma 3.4. *Let $n \in \mathbb{Z}^+$ with $n \equiv 1,2,4 \pmod 7$ and $n \not\equiv 2 \pmod 3$. Then $n = x^2+7y^2+14z^2$ for some $x,y,z \in \mathbb{Z}$.*

Proof. In light of [BIS], the genus G (of discriminant -392) containing the class of the form $x^2+7y^2+14z^2$ contains only one other class, namely the class containing the form $2x^2+7y^2+7z^2$. By Jones [J31a, p. 99] and [J31b, p. 123], a positive integer is represented by a form in G if and only if it does not belong to the set

$$\left\{7^{2k}(7m+r) :\ k,m \in \mathbb{N} \text{ and } r \in \{3,5,6\}\right\}.$$

As $n \not\equiv 3,5,6 \pmod 7$, it is represented by a form in G, i.e., n is represented by $x^2+7y^2+14z^2$ or $2x^2+7y^2+7z^2$ or both.

Suppose that $n = 2x^2+7y^2+7z^2$ for some $x,y,z \in \mathbb{Z}$. If $y^2, z^2 \not\equiv x^2 \pmod 3$, then $y^2 \equiv z^2 \pmod 3$ and hence $2x^2+7y^2+7z^2 \equiv 2x^2+2y^2 \equiv 2 \pmod 3$. As $n \not\equiv 2 \pmod 3$, x^2 is congruent to y^2 or z^2 modulo 3. Without loss of generality we may simply assume that $x \equiv z \pmod 3$. (If $x \equiv -z \pmod 3$ then we may replace z by $-z$.) Thus both $u = (2x+7z)/3$ and $v = (x-z)/3$ are integers. Note that $u^2+7y^2+14v^2 = 2x^2+7y^2+7z^2 = n$.

In view of the above, we immediately obtain the desired result. $\square$

Remark 3.4. I. Kaplansky [K] reported that 2, 74 and 506 are the only positive integers $n \leqslant 10^5$ with $n \equiv 1,2,4 \pmod 7$ which cannot be represented by $x^2+7y^2+14z^2$ with $x,y,z \in \mathbb{Z}$. We guess that any positive integer $n \equiv 1 \pmod 7$ can be written as $x^2+7y^2+14z^2$ with $x,y,z \in \mathbb{Z}$.

Proof of Theorem 1.5. (i) By (3.1), for each $n \in \mathbb{Z}^+$ we can write $2n-1 = x^2+2z^2+4w^2$ with $x,z,w \in \mathbb{N}$. As x is odd, we may write $x = 2y+1$ with $y \in \mathbb{N}$. Thus $2n-1 = (2y+1)^2+2z^2+4w^2$ and hence $n = x^2+y^2+z^2+2w^2$ with $x = y+1$.

By the above, we can write any $n\in\mathbb{Z}^+$ as $x^2+y^2+z^2+2w^2$ $(x,y,z,w\in\mathbb{Z})$ with $x+y=1$. Clearly, z is congruent to x or y modulo 2. Without loss of generality, we may assume that $y\equiv z\pmod 2$. Then

$$n=x^2+2\left(\frac{y+z}{2}\right)^2+2\left(\frac{y-z}{2}\right)^2+2w^2\ \text{ with } x+\frac{y+z}{2}+\frac{y-z}{2}=1.$$

(ii) Let $n\in\mathbb{Z}^+$. By (3.3), $6n-1\not\in E(2,3,12)$. So $6n-1=2u^2+3v^2+12w^2$ for some $u,v,w\in\mathbb{Z}$. As $2\nmid v$, we may write $v=2y+1$ with $y\in\mathbb{Z}$. Since $3\nmid u$, we may write u or $-u$ as $3z+1$ with $z\in\mathbb{Z}$. Thus

$$6n-1=2(3z+1)^2+3(2y+1)^2+12w^2$$

and hence

$$n=(y+z+1)^2+(z-y)^2+(-z)^2+2w^2$$

with $(y+z+1)+(z-y)+2(-z)=1$.

By the above, $n=x^2+y^2+z^2+2w^2$ for some $x,y,z,w\in\mathbb{Z}$ with $x+y+2z=1$. As $x+y$ is odd, x or y is even. Thus $n=u^2+(2v)^2+z^2+2w^2$ for some $u,v\in\mathbb{Z}$ with $u+2v+2z=1$. Clearly, x or y is congruent to z modulo 2. Without any loss of generality, we may assume that $y\equiv z\pmod 2$. Observe that

$$n=x^2+2\left(\frac{z-y}{2}\right)^2+2\left(\frac{y+z}{2}\right)^2+2w^2$$

with

$$x+\frac{z-y}{2}+3\frac{y+z}{2}=x+y+2z=1.$$

(iii) As $4n-1\not\in E(1,2,4)$ by (3.1), we have $4n-1=u^2+2v^2+4x^2$ for some $u,v,x\in\mathbb{Z}$. Since u or $-u$ is congruent to 1 modulo 4, without loss of generality we may assume that $u=4y+1$ with $y\in\mathbb{Z}$. As $u^2\not\equiv -1\pmod 4$, we have $v=2z+1$ for some $z\in\mathbb{Z}$. Thus $4n-1=4x^2+(4y+1)^2+2(2z+1)^2$ and hence

$$n=x^2+4y^2+2y+2z^2+2z+1=x^2+(y+z+1)^2+(y-z)^2+2(-y)^2$$

with $(y+z+1)+(y-z)+2(-y)=1$.

By the above, there are $x,y,z,w\in\mathbb{Z}$ with $n=x^2+y^2+z^2+2w^2$ and $y+z+2w=1$. As $y+z$ is odd, one of y and z is odd and another is even. If $z=2t$ with $t\in\mathbb{Z}$, then $n=x^2+y^2+4t^2+2w^2$ with $y+2t+2w=1$. Clearly x is congruent to y or z modulo 2; if $x\equiv y\pmod 2$ then

$$n=2\left(\frac{x+y}{2}\right)^2+2\left(\frac{y-x}{2}\right)^2+z^2+2w^2$$

with
$$\frac{x+y}{2} + \frac{y-x}{2} + z + 2w = y + z + 2w = 1.$$
This proves Theorem 1.5(iii).

(iv) We first suppose that $n \not\equiv 1 \pmod 8$. Then $5n-2 \not\equiv 3 \pmod 8$ and hence $5n-2 \notin E(2,5,10)$ by (3.5). So $5n-2 = 2v^2 + 5x^2 + 10z^2$ for some $v, x, z \in \mathbb{Z}$. As $v^2 \equiv -1 \equiv 2^2 \pmod 5$, v or $-v$ is congruent to -2 modulo 5. Without any loss of generality, we may assume that $v = 5y - 2$ with $y \in \mathbb{Z}$. Thus
$$5n = 2 + 2(5y-2)^2 + 5x^2 + 10z^2$$
and hence
$$n = x^2 + 10y^2 - 8y + 2 + 2z^2 = x^2 + (y+z)^2 + (y-z)^2 + 2(1-2y)^2$$
with $(y+z) + (y-z) + (1-2y) = 1$.

Now we handle the case $n \equiv 1 \pmod 8$. As $10n - 4 \notin E(1,5,10)$ by (3.4), there are $u, v, x \in \mathbb{Z}$ with $10n - 4 = u^2 + 5v^2 + 10x^2$. If u and v are both even, then $10x^2 \equiv 10n - 4 \equiv 2 \pmod 4$ and hence $2 \nmid x$, thus
$$\left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 \equiv \left(\frac{u}{2}\right)^2 + 5\left(\frac{v}{2}\right)^2 = \frac{10(n-x^2)-4}{4} \equiv -1 \pmod 4$$
which is impossible. Thus $2 \nmid uv$ and we can write $v = 2z + 1$ with $z \in \mathbb{Z}$. Since $u^2 \equiv 1 \pmod 5$, without loss of generality we may assume that $u = 10y + 1$ for some $y \in \mathbb{Z}$. Thus $10n = 4 + (10y+1)^2 + 5(2z+1)^2 + 10x^2$ and hence
$$n = x^2 + 10y^2 + 2y + 2z^2 + 2z + 1 = x^2 + (y+z+1)^2 + (y-z)^2 + 2(-2y)^2$$
with $(y+z+1) + (y-z) + (-2y) = 1$.

(v) Let $n \in \mathbb{Z}^+$. By Lemma 3.2, we can write $6n - 5 = u^2 + 3v^2 + 6x^2$ with $u, v, x \in \mathbb{Z}$ and $2 \nmid u$. As u or $-u$ is congruent to -1 modulo 6, we may simply assume that $u = 6w - 1$ for some $w \in \mathbb{Z}$. Since v is even, $v = 2z$ for some $z \in \mathbb{Z}$. Thus
$$6n - 5 = (6w-1)^2 + 3(2z)^2 + 6x^2 = 36w^2 - 12w + 1 + 12z^2 + 6x^2$$
and hence
$$n = 6w^2 - 2w + 1 + 2z^2 + x^2 = x^2 + (1-w)^2 + 2z^2 + 5w^2$$
with $(1-w) + w = 1$.

Suppose that $n \in \mathbb{Z}^+$ and $n \not\equiv 2 \pmod 3$. Let δ be 1 or 2. By Lemma 3.4, we have $7n - 6/\delta = v^2 + 7x^2 + 14z^2$ for some $v, x, z \in \mathbb{Z}$. Since $v^2 \equiv \delta^2 \pmod 7$, without loss of generality we may assume that $v = 7w - \delta$ for some $w \in \mathbb{Z}$. Thus
$$\begin{aligned} n =& \frac{7x^2 + 14z^2 + (7w-\delta)^2 + 6/\delta}{7} = x^2 + 2z^2 + 7w^2 - 2\delta w + \frac{1}{7}\left(\delta^2 + \frac{6}{\delta}\right) \\ =& x^2 + (1-\delta w)^2 + 2z^2 + \frac{6}{\delta} w^2 \end{aligned}$$

with $(1-\delta w)+\delta w=1$.

(vi) As $3n-2\not\in E(1,6,9)$ by (3.2), there are $u,v,w\in\mathbb{Z}$ such that
$$3n-2=(3u+2)^2+6(v+1)^2+9w^2$$
and hence
$$n=3u^2+4u+4+2v^2+4v+3w^2=(u+v+2)^2+(u-v)^2+(-u)^2+3w^2$$
with $(u+v+2)+(u-v)+2(-u)=2$.

(vii) Note that $5=1^2+(-1)^2+1^2+2\times 1^2$ with $1+(-1)+1=1^2$.

Below we let $n\in\mathbb{N}$ with $n\geqslant 6$. Choose $c\in\{1,4\}$ such that $6n-2c^2\equiv 0\pmod 4$. Note that $6n-2c^2\equiv c^2\equiv 1\pmod 3$ and hence $6n-2c^2\equiv 4\pmod{12}$. By Lemma 3.3(i), there are $r,s,t\in\mathbb{Z}$ with r odd such that $6n-2c^2=r^2+3s^2+3t^2$. As $s\not\equiv t\pmod 2$, without loss of generality we simply assume that $2\nmid s$ and $t=2w$ with $w\in\mathbb{Z}$. Since $r^2+3s^2\equiv 1+3=4\pmod 8$, by Lemma 3.3(ii) we can write $r^2+3s^2=u^2+3v^2$ with $u,v\in\mathbb{Z}$ and $u\equiv v\equiv c\pmod 2$. Clearly $v=2y+c$ for some $y\in\mathbb{Z}$. As u or $-u$ is congruent to c modulo 3, we may write u or $-u$ as $6z+c$ with $z\in\mathbb{Z}$. Thus
$$6n-2c^2=(6z+c)^2+3(2y+c)^2+3(2w)^2=12y^2+12c(y+z)+36z^2+12w^2+4c^2$$
and hence
$$n=2y^2+2z^2+2c(y+z)+c^2+4z^2+2w^2=(y+z+c)^2+(z-y)^2+(-2z)^2+2w^2$$
with $(y+z+c)+(z-y)+(-2z)=c\in\{t^2:\ t=1,2\}$.

(viii) For $n=8,9,10$ we can easily verify that n can be written as $x^2+y^2+z^2+2w^2$ $(x,y,z,w\in\mathbb{Z})$ with $x+y+2z\in\{2t^2:\ t=1,2\}$.

Now assume that $n\geqslant 11$. As $6n-2^2\not\equiv 6n-8^2\pmod{16}$, by (3.3) we have $6n-c^2\not\in E(2,3,12)$ for some $c\in\{2t^2:\ t=1,2\}$. Then we can write $6n-c^2=2(3z+c)^2+3(2y+c)^2+12w^2$ with $y,z,w\in\mathbb{Z}$. It follows that
$$n=(y+z+c)^2+(z-y)^2+(-z)^2+2w^2$$
with $(y+z+c)+(z-y)+2(-z)=c\in\{2t^2:\ t=1,2\}$.

Combining the above, we have completed the proof of Theorem 1.5. $\square$

Lemma 3.5. *We have the new identity*
$$\begin{aligned}&(a^2+ab+b^2)(av^2+b(s^2+t^2+u^2))\\ =&a(b(s+t-u)+(a-b)v)^2+b(bs+au+av)^2\\ &+b(at+bu-av)^2+b(as-bt-av)^2.\end{aligned}\tag{3.7}$$
In particular,
$$7(s^2+t^2+u^2+2v^2)=x^2+y^2+z^2+2w^2,\tag{3.8}$$
where
$$x=s+2u+2v,\ y=-2t-u+2v,\ z=2s-t-2v,\ w=s+t-u+v.\tag{3.9}$$

Proof. By expanding and simplifying the right-hand side of (3.7), we see that (3.7) does hold. Putting $a=2$ and $b=1$ in (3.7), we immediately obtain (3.8). $\square$

Lemma 3.6. *Let $m \in \{2,3\}$. For any integer $n \geqslant 2 \times 7^{2m-1}$, there are integers x, y, z and $w \in \{0, 7^m\}$ such that*

$$7n = x^2 + y^2 + z^2 + 2w^2 \quad \text{with } x + 3y \equiv 0 \pmod 7. \tag{3.10}$$

Proof. Note that $7n \geqslant 2 \times 7^{2m} = 2 \times 49^m$. If $7n \in E(1,1,1)$, then $7n = 4^k(8l+7)$ for some $k, l \in \mathbb{N}$, and hence $7n - 2 \times 49^m \notin E(1,1,1)$. So, for some $w \in \{0, 7^m\}$ we can write $7n - 2w^2$ as $x^2 + y^2 + z^2$ with $x, y, z \in \mathbb{Z}$. Note that $x^2 + y^2 + z^2 \equiv 0 \pmod 7$.

Since $x^2 + y^2 \equiv 6z^2 \pmod 7$, we have $x^2 \equiv (2z)^2$ and $y^2 \equiv (3z)^2$, or $x^2 \equiv (3z)^2$ and $y^2 \equiv (2z)^2$. Without any loss of generality, we may assume that $x \equiv 2z \pmod 7$ and $y \equiv -3z \pmod 7$, hence $x + 3y \equiv 2z - 9z \equiv 0 \pmod 7$. This concludes our proof. □

Proof of Theorem 1.6. For $n = 0, 1, \ldots, 2 \times 7^{2m-1} - 1$ we can verify the desired results via a computer.

Below we fix an integer $n \geqslant 2 \times 7^{2m-1}$.

(i) By Lemma 3.6, there are $x, y, z \in \mathbb{Z}$ and $w \in \{0, 7^m\}$ satisfying (3.10). Note that $x + 3y \equiv w \equiv 0 \pmod 7$ and

$$\begin{aligned} z^2 = 7n - x^2 - y^2 - 2w^2 \equiv & - x^2 - y^2 \\ \equiv & 6x^2 + 27y^2 = (2x - 3y)^2 + 2(x + 3y)^2 \\ \equiv & (2x - 3y)^2 \pmod 7. \end{aligned}$$

Without loss of generality, we may assume that $z \equiv 2x - 3y \pmod 7$. Define

$$\begin{cases} s = \frac{x+2z+2w}{7}, \\ t = \frac{2w-2y-z}{7}, \\ u = \frac{2x-y-2w}{7}, \\ v = \frac{x+y-z+w}{7}. \end{cases} \tag{3.11}$$

It is easy to see that (3.9) holds. Thus, by Lemma 3.5 we have (3.8) and hence

$$n = \frac{7n}{7} = \frac{x^2 + y^2 + z^2 + 2w^2}{7} = s^2 + t^2 + u^2 + 2v^2.$$

As $x \equiv -3y \pmod 7$ and $z \equiv 2x - 3y \pmod 7$, we see that $s, t, u, v \in \mathbb{Z}$ and that $s + t - u + v = w$ is an m-th power.

(ii) Choose $x \in \{0, 7^m\}$ such that $7n - x^2$ is odd. By Dickson [D39, pp. 112-113],

$$E(1,1,2) = \{4^k(16l + 14) : \ k, l \in \mathbb{N}\}.$$

So $7n - x^2 = y^2 + z^2 + 2w^2$ for some $y, z, w \in \mathbb{Z}$. As $y^2 + z^2 \equiv 5w^2 \pmod 7$, we have $y^2 \equiv (2w)^2 \pmod 7$ and $z^2 \equiv w^2 \pmod 7$, or $y^2 \equiv w^2 \pmod 7$ and $z^2 \equiv (2w)^2 \pmod 7$. Without any loss of generality, we may assume that $y \equiv -2w \pmod 7$ and $z \equiv -w \pmod 7$. Now it is easy to see that the numbers s, t, u, v given by (3.11) are all integral. Note that $n = s^2 + t^2 + u^2 + 2v^2$ by Lemma 3.5. Clearly, $s + 2u + 2v = x$ is an m-th power.

The proof of Theorem 1.6 is now complete. □

4. Some open conjectures

In this section, we pose 16 open conjectures on sums of squares for further research.

Conjecture 4.1. (i) *Any integer $n > 1$ can be written as the sum of two squares, a power of three and a power of five; in other words, we have*

$$\{a^2 + b^2 + 3^c + 5^d : \ a, b, c, d \in \mathbb{N}\} = \{2, 3, 4, \dots\}.$$

(ii) *Each integer $n > 1$ can be written as the sum of two squares and two central binomial coefficients; in other words, we have*

$$\left\{a^2 + b^2 + \binom{2c}{c} + \binom{2d}{d} : \ a, b, c, d \in \mathbb{N}\right\} = \{2, 3, 4, \dots\}.$$

(iii) *Any integer $n > 5$ can be written as $a^2 + b^2 + 2^c + 5 \times 2^d$ with $a, b, c, d \in \mathbb{N}$.*

Remark 4.1. See [S, A303656, A303540 and A303637] for related data, and note that $\binom{2k}{k} \sim 4^k/\sqrt{k\pi}$ as $k \to +\infty$. We have verified parts (i)-(iii) for n up to 2×10^{10}, 10^{10} and 5×10^9 respectively. The author would like to offer 3500 US dollars as the prize for the first proof of part (i) of Conjecture 4.1. In contrast with Conjecture 4.1(iii), R. Crocker [C] showed in 2008 that there are infinitely many positive integers not representable as the sum of two squares and at most two powers of 2 (see also [PT] for a simple proof). We also conjecture that any integer $n > 1$ can be written as the sum of two triangular numbers and two powers of 5 (cf. [S, A303389]).

We also have the following conjecture on restricted sums of three squares.

Conjecture 4.2. (i) *Any $n \in \mathbb{Z}^+$ with $\mathrm{ord}_2(n)$ odd can be written as $x^2 + y^2 + z^2$ $(x, y, z \in \mathbb{Z})$ with $x + 3y + 5z$ a square (or twice a square).*

(ii) *Any $n \in \mathbb{N}$ not of the form $4^k(8l+7)$ $(k, l \in \mathbb{N})$ can be written as $x^2+y^2+z^2$ with $x, y, z \in \mathbb{Z}$ such that $x + 2y + 3z$ is a square or twice a square.*

(iii) *Let $n \in \mathbb{N}$. Then $8n + 1$ can be written as $x^2 + y^2 + z^2$ with $x, y \in \mathbb{Z}$ and $z \in \mathbb{Z}^+$ such that $x + 3y$ is a square. Also, we can write $8n + 6$ as $x^2 + y^2 + z^2$ with $x \in \mathbb{Z}$, $y, z \in \mathbb{N}$ and $2 \nmid z$ such that $x + 2y$ is a square.*

(iv) *We can write any positive odd integer as $x^2 + 2y^2 + 3z^2$ with $x, y, z \in \mathbb{Z}$ such that $x + y + z$ is a square or twice a square.*

Remark 4.2. See [S, A283269, A283273 and A283299] for related data. It is known that any positive odd integer can be written as $x^2 + 2y^2 + 3z^2$ with $x, y, z \in \mathbb{Z}$ (cf. [D39, pp. 112-113]).

As $2(x^2 + y^2 + z^2 + w^2) = (x + y)^2 + (x - y)^2 + (z + w)^2 + (z - w)^2$, Lagrange's four-square theorem is equivalent to the fact that each positive odd integer can be written as the sum of four squares. Our following conjecture provides some refinements of Lagrange's four-square theorem involving primes.

Conjecture 4.3. (i) *Any positive odd integer can be written as* $x^2+y^2+z^2+w^2$ *with* $x,y,z,w\in\mathbb{Z}$ *such that* $p=x^2+3y^2+5z^2+7w^2$ *and* $p-2$ *are twin prime.*

(ii) *Any integer* $n>1$ *not divisible by* 4 *can be written as* $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ *such that* $p=x+2y+5z$, $p-2$, $p+4$ *and* $p+10$ *are all prime.*

(iii) *Any positive odd integer can be written as* $x^2+y^2+z^2+4w^2$ *with* $x,y,z,w\in\mathbb{N}$ *such that* $2^x+2^y+2^z+1$ *is prime.*

(iv) *Any odd integer* $n>1$ *can be written as* $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ *such that* $2^{x+y}+2^{z+w}+1$ *is prime.*

Remark 4.3. See [S, A290935, A291635, A291150 and A291191] for related data. For example, $39=1^2+3^2+5^2+2^2$ with $1^2+3\cdot3^2+5\cdot5^2+7\cdot2^2=181$ and $181-2=179$ twin prime, $143=1^2+5^2+9^2+4\cdot3^2$ with $2^1+2^5+2^9+1=547$ prime, and

$$2\times 6998538+1=122^2+220^2+208^2+3727^2$$

with $2^{122+220}+2^{208+3727}+1=2^{342}+2^{3935}+1$ a prime of 1185 decimal digits. Clearly, part (i) of Conjecture 4.3 unifies Lagrange's four-square theorem and the twin prime conjecture.

The following Conjectures 4.4-4.6 are mainly motivated by Theorems 1.2-1.4.

Conjecture 4.4. (i) *Any* $n\in\mathbb{Z}^+$ *can be written as* $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ *with* $P(x,y,z)\in\{2^k:\ k\in\mathbb{N}\}$, *whenever* $P(x,y,z)$ *is among the polynomials*

$$\begin{gathered}2x-y,\ 2x-3y,\ x+(y-z)/3,\ 2x+(y-z)/3,\ 2x-2y-z,\\ 4x-2y-z,\ 4x-3y-z,\ 4x-4y-3z,\ x+y-z,\ x+y-2z,\ x+2y-z,\\ x+3y-z,\ x+3y-2z,\ x+3y-3z,\ x+3y-4z,\ x+3y-5z,\\ x+4y-z,\ x+4y-2z,\ x+4y-3z,\ x+4y-4z,\ x+5y-z,\ x+5y-2z,\\ x+5y-4z,\ x+5y-5z,\ x+6y-3z,\ x+7y-4z,\ x+7y-7z,\ x+8y-z,\\ x+9y-2z,\ 2x+3y-z,\ 2x+3y-3z,\ 2x+3y-4z,\ 2x+5y-z,\\ 2x+5y-3z,\ 2x+5y-4z,\ 2x+5y-5z,\ 2x+7y-z,\ 2x+7y-3z,\\ 2x+7y-7z,\ 2x+9y-3z,\ 2x+11y-5z,\ 3x+4y-3z,\ 7x+8y-7z.\end{gathered}$$

(ii) *Any* $n\in\mathbb{Z}^+$ *can be written as* $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ *with*

$Q(x,y,z,w) \in \{2^k:\ k \in \mathbb{N}\}$, *whenever* $Q(x,y,z,w)$ *is among the polynomials*

$$x+y+2z-2w,\ x+y+2z-3w,\ x+y+2z-4w,\ x+2y+2z-3w,$$
$$x+2y+3z-4w,\ x+2y+4z-3w,\ x+2y+6z-7w,\ x+3y+4z-4w,$$
$$x+4y+6z-5w,\ 2x+3y+5z-4w,\ 2x+y-z-w,\ 2x+y-2z-w,$$
$$2x+y-3z-w,\ 2x+2y-3z-2w,\ 3x+y-3z-2w,\ 3x+y-4z-2w,$$
$$3x+2y-2z-w,\ 3x+2y-3z-w,\ 3x+2y-3z-2w,\ 3x+y-2z-w,$$
$$3x+2y-6z-w,\ 4x+y-2z-w,\ 4x+2y-2z-w,\ 4x+3y-2z-w,$$
$$4x+3y-3z-w,\ 4x+3y-4z-3w,\ 4x+3y-5z-w,\ 4x+3y-5z-2w,$$
$$5x+2y-2z-w,\ 5x+2y-3z-2w,\ 5x+2y-4z-3w,\ 5x+4y-5z-w,$$
$$7x+y-6z-2w,\ 8x+3y-3z-2w,\ 8x+3y-10z-w,\ 9x+y-4z-w.$$

Conjecture 4.5. (i) *Each* $n \in \mathbb{N}$ *can be written as* $x^2+y^2+z^2+w^2$ *with* $x,y,z,w \in \mathbb{N}$ *and* $P(x,y,z,w) \in \{4^k:\ k \in \mathbb{N}\} \cup \{0\}$, *whenever* $P(x,y,z,w)$ *is among the linear polynomials*

$$2x-y,\ x+y-z,\ x-y-z,\ x+y-2z,\ \ 2x+y-z,\ 2x-y-z,$$
$$2x-2y-z,\ 2x+y-3z,\ 2x+2y-2z,\ 2x+2y-4z,\ 3x-2y-z,$$
$$x+3y-3z,\ 2x+3y-3z,\ 4x+2y-2z,\ 8x+2y-2z,$$
$$2(x-y)+z-w,\ 4(x-y)+2(z-w).$$

(ii) *Any* $n \in \mathbb{Z}^+$ *can be written as* $x^2+y^2+z^2+w^2$ *with* $x,y,z,w \in \mathbb{N}$ *and* $|Q(x,y,z,w)| \in \{4^k:\ k \in \mathbb{N}\}$, *whenever* $Q(x,y,z,w)$ *is among the polynomials*

$$x+y-3z,\ x+2y-3z,\ x+2y-4z,\ x+2y-5z,\ x+3y-3z,$$
$$x+3y-5z,\ x+4y-2z,\ x+5y-2z,\ x+5y-7z,\ 2x+3y-4z,$$
$$2x+4y-6z,\ 2x+4y-10z,\ 2x+5y-4z,\ 4x+6y-6z,$$
$$4x+6y-14z,\ x+4y-2z-w,\ x+8y-3z-w,$$
$$2x+3y-3z-w,\ x+y+2z-2w,\ x+2y+3z-4w.$$

(iii) *Each* $n \in \mathbb{N}$ *can be written as* $x^2+y^2+z^2+w^2$ *with* $x,y,z,w \in \mathbb{N}$ *and* $R(x,y,z,w) \in \{\pm 8^k:\ k \in \mathbb{N}\} \cup \{0\}$, *whenever* $R(x,y,z,w)$ *is among the linear polynomials*

$$x+2y-2z,\ x+3y-3z,\ 2x+3y-3z,\ 4x+6y-6z,\ 4x+8y-8z,$$
$$4x+2y-10z,\ 4x+12y-12z,\ 8x+12y-12z,\ 8x+4y-20z,\ 2x+y-z-w,$$
$$8x+4y-4z-4w,\ 2x+8y-4z-2w,\ 4x+y-2z-w,\ 4x+16y-8z-4w.$$

Conjecture 4.6. (i) *Let $a \in \mathbb{Z}^+$ and let $b,c,d \in \mathbb{N}$ with $a \geqslant b \geqslant c \geqslant d$ and $4 \nmid \gcd(a,b,c,d)$. Then any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{Z})$ with $ax+by+cz+dw \in \{8^k : k \in \mathbb{N}\}$, if and only if (a,b,c,d) is among the quadruples*

$$(7,3,2,1),\ (7,5,2,1),\ (8,4,2,1),\ (8,4,3,2),\ (8,5,4,2),\ (8,6,2,1),$$
$$(8,6,3,2),\ (8,6,5,1),\ (8,6,5,2),\ (9,8,7,4),\ (10,4,3,1),\ (12,4,3,1).$$

(ii) *Let $a \in \mathbb{Z}^+$ and let $b,c,d \in \mathbb{N}$ with $a \geqslant b \geqslant c \geqslant d$ and $2 \nmid \gcd(a,b,c,d)$. Then any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{Z})$ with $ax+by+cz+dw \in \{2 \times 8^k : k \in \mathbb{N}\}$, if and only if (a,b,c,d) is among the quadruples*

$$(7,3,2,1),\ (7,5,2,1),\ (8,4,2,1),\ (8,4,3,2),\ (8,5,4,2),\ (8,6,2,1),$$
$$(8,6,3,2),\ (8,6,5,2),\ (9,5,4,2),\ (14,3,2,1),\ (14,5,2,1),\ (14,7,3,2).$$

Conjecture 4.7. (i) (24-Conjecture) *Any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ such that both x and $x+24y$ are squares.*

(ii) *Each $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ such that both x and $49x+48(y-z)$ are squares. Also, any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ such that both x and $121x+48(y-z)$ are squares.*

(iii) *Every $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ such that both x and $-7x-8y+8z+16w$ are squares.*

(iv) *Each $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ and $x \equiv y \pmod 2$ such that both x and $x^2+62xy+y^2$ are squares.*

Remark 4.4. See [S, A281976, A281977, A281980, A282013, A282014, A282226 and A282463] for related data. We verified the 24-Conjecture for all $n = 1, \ldots, 10^7$, and Qing-Hu Hou extended the verification for n up to 10^{10}. The author would like to offer 2400 US dollars as the prize for the first solution of the 24-Conjecture. We verified parts (ii) and (iii) of Conjecture 4.7 for n up to 10^7 and 10^6 respectively, and later Qing-Hu Hou extended the verification of parts (ii) and (iii) of Conjecture 4.7 for n up to 10^9 and 10^8 respectively. Note that any $n \in \mathbb{N}$ is the sum of a fourth power and three squares as proved by the author [S17b].

Conjecture 4.8. (i) *Any $n \in \mathbb{Z}^+$ can be written as $x^4+y^2+z^2+w^2$ with $x,y,z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $9y^2-8yz+8z^2$ is a square. Also, any $n \in \mathbb{N}$ can be written as $4x^4+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ such that $79y^2-220yz+205z^2$ is a square.*

(ii) *Each $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z \in \mathbb{Z}$ and $w \in \mathbb{Z}^+$ such that both $2x+y$ and $2x+z$ are squares.*

(iii) *Any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,w \in \mathbb{N}$ and $y,z \in \mathbb{Z}$ such that both $x+2y$ and $z+2w$ are squares. Also, any $n \in \mathbb{N}$ can be written as*

$x^2+y^2+z^2+w^2$ with $x,y,z\in\mathbb{Z}$ and $w\in\mathbb{N}$ such that both $x+3y$ and $z+3w$ are squares.

Remark 4.5. See [S, A282933, A282972, A283170, A283196, A283204 and A283205] for related data. The author [S17b] proved that any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{Z}$ such that $x+2y$ is a square, and that any $n\in\mathbb{N}$ can be written as $4x^4+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{Z}$.

Conjecture 4.9. (i) *Any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ such that x or y is a square, and $x-y$ is also a square.*

(ii) *Any positive integer can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ such that $x+3y+5z$ is a positive square, and one of $2x,y,z$ (or $3x,y,z$) is a square.*

(iii) *Any $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z\in\mathbb{N}$ and $w\in\mathbb{Z}^+$ such that $(3x)^2+(4y)^2+(12z)^2$ is a square, and also one of $z,2z,3z$ is a square. Also, each $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z\in\mathbb{N}$ and $w\in\mathbb{Z}^+$ such that $(12x)^2+(15y)^2+(20z)^2$ is a square, and also one of x,y,z is a square, and any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ such that $(12x)^2+(21y)^2+(28z)^2$ is a square, and also one of $x,2y,z$ is a square,*

Remark 4.6. See [S, A281975, A300708, A300139, A300666, A300667, A300712, A300751, A300752, A300791, A300844, A300908] for related data or similar conjectures.

Conjecture 4.10. *Any $n\in\mathbb{N}\setminus\{71,85\}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{Z}$ such that $9x^2+16y^2+24z^2+48w^2$ is a square.*

Remark 4.7. We have verified this for n up to 2×10^5. See [S, A281659] for related data and other similar conjectures.

Conjecture 4.11. (i) *Let $a,b\in\mathbb{Z}^+$ with $\gcd(a,b)$ squarefree. Then, each $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ such that $ax^3+b(y-z)^3$ is a square, if and only if (a,b) is among the ordered pairs*

$$(1,1),\ (1,9),\ (2,18),\ (8,1),\ (9,5),\ (9,8),$$
$$(9,40),\ (16,2),\ (18,16),\ (25,16),\ (72,1).$$

(ii) *Let $a,b\in\mathbb{Z}^+$ with $a\leqslant b$ and $\gcd(a,b)$ squarefree. Then, each $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{Z})$ with ax^3+by^3 a square, if and only if (a,b) is among the ordered pairs*

$$(1,2),\ (1,8),\ (2,16),\ (4,23),\ (4,31),$$
$$(5,9),\ (8,9),\ (8,225),\ (9,47),\ (25,88),\ (50,54).$$

Remark 4.8. See [S, A282863 and A283617] for related data or similar conjectures.

Conjecture 4.12. (i) *Any $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+z^2+3w^2$ $(x,y,z,w\in\mathbb{Z})$ with $P(x,y,z,w)=1$, whenever $P(x,y,z,w)$ is among the polynomials*

$$y+z+w,\ 2y+z+w,\ 2y+z+3w,\ 2x+2y+z+w,$$
$$2x+2y+z+3w,\ 4x+2y+z+dw\ (d=1,3,5).$$

(ii) *Any $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+2z^2+3w^2$ $(x,y,z,w\in\mathbb{Z})$ with $P(x,y,z,w)=1$, whenever $P(x,y,z,w)$ is among the polynomials*

$$x+2y+w,\ y+z+w,\ y+2z+w,\ y+2z+3w,\ x+2y+2z+w,\ x+2y+2z+3w.$$

(iii) *Any $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+3z^2+4w^2$ $(x,y,z,w\in\mathbb{Z})$ with $y+z+2w=1$. Also, any $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+2z^2+5w^2$ $(x,y,z,w\in\mathbb{Z})$ with $y+2z+w=1$.*

Conjecture 4.13. (i) *Any $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+z^2+2w^2$ $(x,y,z,w\in\mathbb{Z})$ with $P_1(x,y,z,w)=1$, whenever $P_1(x,y,z,w)$ is among the polynomials*

$$x+2y+3z,\ x+2y+5z,\ x+3y+4z,\ y+3z+2w,\ y+3z+4w,\ 2y+z+w,$$
$$x+y+2z+2w,\ x+2y+2z+2w,\ x+2y+3z+dw\ (d=1,2,4),$$
$$x+2y+5z+2w,\ x+2y+5z+6w,\ x+3y+4z+2w,\ x+3y+4z+4w.$$

(ii) *Any $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+z^2+2w^2$ $(x,y,z,w\in\mathbb{Z})$ with $P_2(x,y,z,w)=2$, whenever $P_2(x,y,z,w)$ is among the polynomials*

$$x+y+2z+2w,\ x+y+2z+6w,\ x+2y+3z+2w,$$
$$x+2y+3z+6w,\ x+2y+4z+4w,\ x+2y+5z+2w,\ 3x+3y+2z+2w.$$

(iii) *Any $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+z^2+2w^2$ $(x,y,z,w\in\mathbb{Z})$ with $P_3(x,y,z,w)=3$, whenever $P_3(x,y,z,w)$ is among the polynomials*

$$x+2y+3z+2w,\ x+2y+3z+4w,\ x+2y+3z+6w.$$

Conjecture 4.14. *Any $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+z^2+2w^2$ with $x,y,z,w\in\mathbb{N}$ and $x+2y\in\{4^k:\ k\in\mathbb{N}\}$. Also, we may replace $x+2y$ by $y-z+3w$ (or $y+2z-w$).*

Remark 4.9. It is easy to show that any $n\in\mathbb{Z}^+$ can be written as $x^2+y^2+z^2+2w^2$ with $x,y,z,w\in\mathbb{Z}$ and $x+2y=1$.

Now we pose a conjecture similar to the 1-3-5 conjecture of Sun [S17b, Conjecture 4.3(i)].

Conjecture 4.15 (1-2-3 Conjecture). (i) *Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y + 3z$ is a square.*

(ii) *For each $n \in \mathbb{Z}^+$ we can write n^2 as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y + 3z \in \{4^k : k \in \mathbb{N}\}$.*

Remark 4.10. See [S, A275344 and A299924] for related data. Each of the numbers

$$0,\ 1,\ 3,\ 5,\ 7,\ 14,\ 15,\ 16,\ 25,\ 30,\ 84,\ 169,\ 225$$

has a unique representation $x^2 + y^2 + z^2 + 2w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 2y + 3z$ a square. For example,

$$\begin{aligned}
33 =& 1^2 + 0^2 + 0^2 + 2 \times 4^2 \quad \text{with } 1 + 2 \times 0 + 3 \times 0 = 1^2,\\
84 =& 4^2 + 6^2 + 0^2 + 2 \times 4^2 \quad \text{with } 4 + 2 \times 6 + 3 \times 0 = 4^2,\\
169 =& 10^2 + 6^2 + 1^2 + 2 \times 4^2 \quad \text{with } 10 + 2 \times 6 + 3 \times 1 = 5^2,\\
225 =& 10^2 + 6^2 + 9^2 + 2 \times 2^2 \quad \text{with } 10 + 2 \times 6 + 3 \times 9 = 7^2.
\end{aligned}$$

Also, for each $n \in \{1, 2, 3, 7, 11, 13, 14, 17, 49, 61\}$ there is a unique way to write n^2 as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x + 2y + 3z \in \{4^k : k \in \mathbb{N}\}$. For example,

$$\begin{aligned}
11^2 =& 2^2 + 1^2 + 4^2 + 10^2 \quad \text{with } 2 + 2 \times 1 + 4 \times 3 = 4^2,\\
49^2 =& 22^2 + 3^2 + 12^2 + 42^2 \quad \text{with } 22 + 2 \times 3 + 3 \times 12 = 4^3.
\end{aligned}$$

We conjecture that if a, b, c are positive integers with $\gcd(a, b, c)$ squarefree and any positive square can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $ax + by + cz \in \{4^k : k \in \mathbb{N}\}$ then we must have $\{a, b, c\} = \{1, 2, 3\}$.

Conjecture 4.16. *Let $\delta \in \{0, 1\}$.*

(i) *For any integer $n > \delta$, we can write n^2 as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{N}$ and $\{x, 4x - 3y\} \subseteq \{2^{2k+\delta} : k \in \mathbb{N}\}$.*

(ii) *For any $n \in \mathbb{Z}^+$ we can write $2n^2 = x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 3y + 5z + 15w \in \{2^{2k+\delta} : k \in \mathbb{N}\}$.*

Remark 4.11. We have verified part (i) for n up to 10^7. See [S, A300219, A299537, A299794, A300360, A300396, A301891] for related data or similar conjectures.

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