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SOME UNIVERSAL QUADRATIC SUMS OVER THE INTEGERS

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Abstract. Let $a,b,c,d,e,f\in\mathbb{N}$ with $a\ge c\ge e>0$, $b\le a$ and $b\equiv a\pmod 2$, $d\le c$ and $d\equiv c\pmod 2$, $f\le e$ and $f\equiv e\pmod 2$. If any nonnegative integer can be written as $x(ax+b)/2+y(cy+d)/2+z(ez+f)/2$ with $x,y,z\in\mathbb{Z}$, then the ordered tuple (a,b,c,d,e,f) is said to be universal over $\mathbb{Z}$. Recently, Z.-W. Sun found all candidates for such universal tuples over $\mathbb{Z}$. In this paper, we use the theory of ternary quadratic forms to show that 44 concrete tuples (a,b,c,d,e,f) in Sun's list of candidates are indeed universal over $\mathbb{Z}$. For example, we prove the universality of $(16,4,2,0,1,1)$ over $\mathbb{Z}$ which is related to the form $x^2+y^2+32z^2$.

1. Introduction

Those $p_3(x)=x(x+1)/2$ with $x\in\mathbb{Z}$ are called triangular numbers. In 1796 Gauss proved Fermat's assertion that each $n\in\mathbb{N}=\{0,1,2,\ldots\}$ can be expressed as the sum of three triangular numbers.

For polynomials $f_1(x),f_2(x),f_3(x)$ with $f_i(\mathbb{Z})=\{f_i(x):\ x\in\mathbb{Z}\}\subseteq\mathbb{N}$ for $i=1,2,3$, if any $n\in\mathbb{N}$ can be written as $f_1(x)+f_2(y)+f_3(z)$ with $x,y,z\in\mathbb{Z}$ then we call the sum $f_1(x)+f_2(y)+f_3(z)$ *universal over* $\mathbb{Z}$. For example, $p_3(x)+p_3(y)+p_3(z)$ is universal over $\mathbb{Z}$ by Gauss' result.

In 1862 Liouville (cf. [1, p. 82]) determined all universal sums $ap_3(x)+bp_3(y)+cp_3(z)$ over $\mathbb{Z}$ with $a,b,c\in\mathbb{Z}^+=\{1,2,3,\ldots\}$. Z.-W. Sun [19, 20] studied universal sums of the form $ap_i(x)+bp_j(y)+cp_k(z)$ with $a,b,c\in\mathbb{N}$ and $i,j,k\in\{3,4,\ldots\}$, where $p_m(x)$ denotes the generalized polygonal number

$$(m-2)\binom{x}{2}+x=\frac{x((m-2)x-(m-4))}{2};$$

see also [8, 16, 7, 15, 13] for subsequent work on some of Sun's conjectures posed in [19, 20]. In 2017 Sun [22] investigated universal sums $x(ax+1)+y(by+1)+z(cz+1)$ over $\mathbb{Z}$ with $a,b,c\in\mathbb{Z}^+$, and universal sums $x(ax+b)+y(ay+c)+z(az+d)$ over $\mathbb{Z}$ with $a,b,c,d\in\mathbb{N}$ and $a\ge b\ge c\ge d$. Quite recently, Sun [24] considered more general questions and investigated for what tuples (a,b,c,d,e,f) with $a\ge c\ge e\ge 1$, $b\equiv a\pmod 2$ and $0\le b\le a$, $d\equiv c\pmod 2$ and $0\le d\le c$, $f\equiv e\pmod 2$ and

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$0 \le f \le e$, the sum
$$\frac{x(ax+b)}{2}+\frac{y(cy+d)}{2}+\frac{z(ez+f)}{2}$$
is universal over $\mathbb{Z}$. Such (ordered) tuples (a,b,c,d,e,f) are said to be universal over $\mathbb{Z}$. He showed such tuples with $b<a$, $d<c$, $f<e$, and $b \ge d$ if $a=c$, and $d \ge f$ if $c=e$, must be in his list of 12082 candidates (cf. [23, A286944] and [25]), and conjectured that all such candidates are indeed universal over $\mathbb{Z}$. Note that
$$\{p_3(x):\ x\in\mathbb{Z}\}=\left\{\frac{x(4x+2)}{2}=x(2x+1):\ x\in\mathbb{Z}\right\}.$$
Sun [24] proved that some candidates (a,b,c,d,e,f) are universal over $\mathbb{Z}$, e.g., $(5,1,3,1,1,1)$ (equivalent to $(5,1,4,2,3,1)$) is universal over $\mathbb{Z}$. Sun even conjectured that any $n\in\mathbb{N}$ can be written as $x(x+1)/2+y(3y+1)/2+z(5z+1)/2$ with $x,y,z\in\mathbb{N}$.

In this paper, via the theory of ternary quadratic forms, we establish the universality (over $\mathbb{Z}$) of 44 concrete tuples (a,b,c,d,e,f) on Sun's list of candidates.

Theorem 1.1. *The tuples*
$$(5,1,2,2,1,1),\ (6,0,3,3,3,1),\ (6,2,5,5,1,1),\ (6,6,3,3,3,1),$$
$$(8,2,3,1,1,1),\ (8,6,3,1,1,1),\ (8,8,3,1,1,1)$$
are universal over $\mathbb{Z}$.

Remark 1. Sun [20] conjectured that any $n\in\mathbb{N}$ can be written as $p_3(x)+2p_3(y)+p_7(z)$ with $x,y,z\in\mathbb{N}$, and J. Ju, B.-K. Oh and B. Seo [13] proved that $p_3(x)+2p_3(y)+p_7(z)$ (or the tuple $(5,3,2,2,1,1)$) is universal over $\mathbb{Z}$.

Theorem 1.2. *The tuples*
$$(6,0,5,1,3,1),\ (6,0,5,3,3,1),\ (7,1,1,1,1,1),\ (7,1,2,0,1,1),$$
$$(7,1,2,2,1,1),\ (7,1,3,1,1,1),\ (7,3,1,1,1,1),\ (7,3,2,0,1,1),$$
$$(7,3,2,2,1,1),\ (7,3,3,1,1,1),\ (7,5,1,1,1,1),\ (7,5,3,1,1,1),$$
$$(15,3,3,1,1,1),\ (15,5,1,1,1,1),\ (15,5,3,1,2,0),\ (15,5,3,1,2,2),$$
$$(15,9,3,1,1,1),\ (21,7,3,1,2,2)$$
are universal over $\mathbb{Z}$.

Remark 2. Our proof of Theorem 1.2 involves the theory of genera of ternary quadratic forms. Sun [20] conjectured that any $n\in\mathbb{N}$ can be written as $p_3(x)+y^2+p_9(z)$ (or $p_3(x)+2p_3(y)+p_9(z)$) with $x,y,z\in\mathbb{N}$; along this line Ju, Oh and Seo [13] proved that $p_3(x)+y^2+p_9(z)$ and $p_3(x)+2p_3(y)+p_9(z)$ are universal over $\mathbb{Z}$, i.e., the tuples $(7,5,2,0,1,1)$ and $(7,5,2,2,1,1)$ are universal over $\mathbb{Z}$.

Theorem 1.3. (i) *The tuples* $(5,5,3,1,3,1)$, $(5,5,3,3,3,1)$, $(6,4,5,5,1,1)$ *and* $(7,7,3,1,1,1)$ *are universal over* $\mathbb{Z}$.

(ii) *All the five tuples*
$$(6,2,5,1,1,1),\ (6,2,5,5,1,1),\ (6,4,5,1,1,1),\ (15,5,6,2,1,1),\ (15,5,6,4,1,1)$$
are universal over $\mathbb{Z}$.

Remark 3. Our proof of Theorem 1.3(i) employs the Minkowski-Siegel formula (cf. [14, pp. 173-174]). Sun [20] conjectured that any $n\in\mathbb{N}$ can be written as $p_3(x)+p_7(y)+2p_5(z)$ (or $p_3(x)+p_7(y)+p_8(z)$) with $x,y,z\in\mathbb{N}$; along this line

Ju, Oh and Seo [13] proved that $p_3(x)+p_7(y)+2p_5(z)$ and $p_3(x)+p_7(y)+p_8(z)$ are universal over $\mathbb{Z}$, i.e., the tuples $(6,2,5,3,1,1)$ and $(6,4,5,3,1,1)$ are universal over $\mathbb{Z}$.

Similarly to [24, Theorem 1.4], we observe that

$$\{p_3(x)+p_5(y):\ x,y\in\mathbb{Z}\}=\{p_5(x)+3p_5(y):\ x,y\in\mathbb{Z}\}. \tag{1}$$

In fact,

$$\begin{aligned}&n\in\{p_3(x)+p_5(y):\ x,y\in\mathbb{Z}\}\\ \iff&24n+4\in\{3(2x+1)^2+(6y-1)^2:\ x,y\in\mathbb{Z}\}\\ \iff&24n+4\in\{3u^2+v^2:\ u,v\in\mathbb{Z}\ \&\ 2\nmid uv\}\end{aligned}$$

and

$$\begin{aligned}&n\in\{3p_5(x)+p_5(y):\ x,y\in\mathbb{Z}\}\\ \iff&24n+4\in\{3(6x-1)^2+(6y-1)^2:\ x,y\in\mathbb{Z}\}\\ \iff&24n+4\in\{3u^2+v^2:\ u,v\in\mathbb{Z},\ 2\nmid uv\ \&\ 3\nmid u\}.\end{aligned}$$

If u and v are odd integers with $3\mid u$ and $3\nmid v$, then

$$3u^2+v^2=3\left(\frac{u\pm v}{2}\right)^2+\left(\frac{3u\mp v}{2}\right)^2$$

with $(u\pm v)/2$ not divisible by 3. Therefore (1) holds. In view of (1.1) and Theorems 1.1-1.3, we have the following consequence.

Corollary 1. *The tuples*

$$\begin{gathered}(9,3,7,1,3,1),\ (9,3,7,3,3,1),\ (9,3,7,5,3,1),\\ (9,3,7,7,3,1),\ (9,3,8,2,3,1),\ (9,3,8,6,3,1),\\ (9,3,8,8,3,1),\ (15,3,9,3,3,1),\ (15,9,9,3,3,1)\end{gathered}$$

are universal over $\mathbb{Z}$.

Theorem 1.4. *The tuple* $(16,4,2,0,1,1)$ *is universal over* $\mathbb{Z}$. *In other words, any* $n\in\mathbb{N}$ *can be written as* $p_3(x)+y^2+2z(4z+1)$ *with* $x,y,z\in\mathbb{Z}$.

Remark 4. This result is closely related to the form $x^2+y^2+32z^2$. Sun [24] even conjectured that any $n\in\mathbb{N}$ can be written as $p_3(x)+y^2+2z(4z-1)$ with $x,y,z\in\mathbb{N}$.

We will show Theorems 1.1-1.4 in Sections 2-5 respectively.

2. Proof of Theorem 1.1

Lemma 2.1. (i) *For any* $n\in\mathbb{N}$, *we can write* $12n+5$ *as* $x^2+y^2+(6z)^2$ *with* $x,y,z\in\mathbb{Z}$.

(ii) *Let* $n\in\mathbb{Z}^+$ *and* $\delta\in\{0,1\}$. *Then we can write* $6n+1$ *as* $x^2+3y^2+6z^2$ *with* $x,y,z\in\mathbb{Z}$ *and* $x\equiv\delta\pmod 2$.

Remark 5. Lemma 2.1 is a known result due to the second author, see [20, Theorem 1.7(iii) and Lemma 3.3] and [22, Remark 3.1].

The following lemma (cf. [11, pp. 12-14]) occurred in a 1993 letter of J.S. Hsia to I. Kaplansky.

Lemma 2.2. *For each $n \in \mathbb{N}$, we can write $6n+5$ as $x^2+y^2+10z^2$ with $x,y,z \in \mathbb{Z}$.*

For $a,b,c \in \mathbb{Z}^+$, we define

$$E(a,b,c) = \{n \in \mathbb{N} :\ n \neq ax^2+by^2+cz^2 \text{ for any } x,y,z \in \mathbb{Z}\}.$$

L.E. Dickson [4, pp. 112-113] listed 102 diagonal quadratic forms $ax^2+by^2+cz^2$ for which the structure of $E(a,b,c)$ is known explicitly. For example, the Gauss-Legendre theorem asserts that $E(1,1,1) = \{4^k(8l+7) :\ k,l \in \mathbb{N}\}$.

In 1996 W. Jagy [9] showed the following result (cf. [11, pp. 25-26]).

Lemma 2.3. *We have*

$$E(1,4,9) = \{2\} \cup \bigcup_{k,l\in\mathbb{N}} \{4^k(8l+7),\ 8l+3,\ 9l+3\}.$$

Proof of Theorem 1.1. (i) We want to prove the universality of $(5,r,2,2,1,1)$ over $\mathbb{Z}$ for $r \in \{1,3\}$. Let $n \in \mathbb{N}$. Clearly,

$$\begin{aligned} &n = p_3(x) + y(y+1) + \frac{z(5z+r)}{2} \\ \iff &40n + r^2 + 15 = 5(2x+1)^2 + 10(2y+1)^2 + (10z+r)^2. \end{aligned}$$

Since

$$E(1,5,10) = \{25^l m :\ l,m \in \mathbb{N} \text{ and } m \equiv 2,3 \pmod 5\}$$

by Dickson [4, pp. 112-113], we have $40n+r^2+15 \in \{u^2+5v^2+10w^2 :\ u,v,w \in \mathbb{N}\}$. Thus we can write

$$40n + r^2 + 15 = (2^k x_0)^2 + 5(2^k y_0)^2 + 10(2^k z_0)^2 = 4^k(x_0^2 + 5y_0^2 + 10z_0^2)$$

with $k \in \mathbb{N}$, $x_0, y_0, z_0 \in \mathbb{Z}$, and x_0, y_0, z_0 not all even. In the case $k = 0$, if $2 \mid z_0$ then $x_0^2 + 5y_0^2 \equiv r^2 + 15 \equiv 0 \pmod 8$ and hence $x_0 \equiv y_0 \equiv 0 \pmod 2$ which contradicts that x_0, y_0, z_0 are not all even, thus $2 \nmid z_0$ and also $2 \nmid x_0 y_0$ since $x_0^2 + 5y_0^2 \equiv r^2 + 15 - 10z_0^2 \equiv 6 \pmod 8$.

It is easy to verify the following new identity:

$$4^2(x^2+5y^2+10z^2) = (x \pm 5y - 10z)^2 + 5(x \mp 3y - 2z)^2 + 10(x \pm y + 2z)^2. \tag{2}$$

If x,y,z are odd integers, then $(x+\varepsilon y)/2 + z$ is odd for some $\varepsilon \in \{\pm 1\}$, hence by (2) we have

$$4(x^2+5y^2+10z^2) = \tilde{x}^2 + 5\tilde{y}^2 + 10\tilde{z}^2$$

with

$$\tilde{x} = \frac{x+\varepsilon y}{2} + 2\varepsilon y - 5z,\ \tilde{y} = \frac{x+\varepsilon y}{2} - 2\varepsilon y - z,\ \tilde{z} = \frac{x+\varepsilon y}{2} + z$$

all odd. Thus, if $2 \nmid x_0 y_0 z_0$ then

$$40n + r^2 + 15 = 4^k(x_0^2 + 5y_0^2 + 10z_0^2) \in \{x^2+5y^2+10z^2 :\ x,y,z \text{ are odd}\}. \tag{3}$$

If $x_0 \not\equiv y_0 \pmod 2$, then $x_0^2 + 5y_0^2 + 10z_0^2 \equiv 1 \pmod 2$ and $k \geq 2$ since $40n + r^2 + 15 \equiv 0 \pmod 8$, hence by (2) we have

$$4^2(x_0^2 + 5y_0^2 + 10z_0^2) = \bar{x}_0^2 + 5\bar{y}_0^2 + 10\bar{z}_0^2$$

with $\bar{x}_0 = x_0 - 5y_0 - 10z_0$, $\bar{y}_0 = x_0 + 3y_0 - 2z_0$ and $\bar{z}_0 = x_0 - y_0 + 2z_0$ all odd, and therefore (3) holds.

Now we suppose that $k > 0$, $2 \mid x_0 y_0 z_0$ and $x_0 \equiv y_0 \pmod 2$. By (2),

$$4(x_0^2 + 5y_0^2 + 10z_0^2) = x_1^2 + 5y_1^2 + 10z_1^2$$

with

$$x_1 = \frac{x_0 - y_0}{2} - 2y_0 - 5z_0,\ y_1 = \frac{x_0 - y_0}{2} + 2y_0 - z_0,\ z_1 = \frac{x_0 - y_0}{2} + z_0$$

If x_0 and y_0 are odd, then we may assume $x_0 \not\equiv y_0 - 2z_0 \pmod 4$ without loss of generality (otherwise we replace x_0 by $-x_0$), and hence x_1, y_1, z_1 are all odd. If $x_0, y_0, (x_0 - y_0)/2$ are all even, then z_0 is odd and so are x_1, y_1, z_1. If x_0 and y_0 are even with $x_0 \not\equiv y_0 \pmod 4$, then z_0 is odd and we may assume $z_0 \equiv (y_0 - x_0)/2 \pmod 4$ without loss of generality (otherwise we replace z_0 by $-z_0$), hence $z_1 \equiv 0 \pmod 4$, $y_1 = z_1 + 2(y_0 - z_0) \equiv 0 \pmod 2$ and $(x_1 - y_1)/4 \equiv -y_0 - z_0 \equiv 1 \pmod 2$, therefore by (2) we have

$$x_1^2 + 5y_1^2 + 10z_1^2 = x_2^2 + 5y_2^2 + 10z_2^2$$

with

$$x_2 = \frac{x_1 - 5y_1 - 10z_1}{4},\ y_2 = \frac{x_1 + 3y_1 - 2z_1}{4},\ z_2 = \frac{x_1 - y_1 + 2z_1}{4}$$

all odd. So we still have (3).

By the above, there always exist odd integers x, y, z such that $40n + r^2 + 15 = x^2 + 5y^2 + 10z^2$. Write $y = 2u+1$ and $z = 2v+1$ with $u, v \in \mathbb{Z}$. As $x^2 \equiv r^2 \pmod 5$, either x or $-x$ has the form $10w + r$ with $w \in \mathbb{Z}$. Therefore

$$40n + r^2 + 15 = (10w + r)^2 + 5(2u + 1)^2 + 10(2v + 1)^2$$

and hence $n = p_3(u) + v(v + 1) + w(5w + r)/2$. This proves the universality of $(5, r, 2, 2, 1, 1)$ over $\mathbb{Z}$.

(ii) Let $n \in \mathbb{N}$ and $r \in \{1, 3\}$. It is easy to see that

$$\begin{aligned} & n = p_3(x) + \frac{y(3y + 1)}{2} + z(4z + r) \\ \iff & 48n + 3r^2 + 8 = 6(2x + 1)^2 + 2(6y + 1)^2 + 3(8z + r)^2. \end{aligned}$$

Since

$$E(2, 3, 6) = \{3q + 1 :\ q \in \mathbb{N}\} \cup \{4^k(8l + 7) :\ k, l \in \mathbb{N}\}$$

by Dickson [4, pp. 112-113], we see that $48n + 3r^2 + 8 = 2x^2 + 3y^2 + 6z^2$ for some $x, y, z \in \mathbb{Z}$. Clearly, $y^2 + 2z^2 \neq 0$, and hence by [20, Lemma 2.1] we have $y^2 + 2z^2 = y_0^2 + 2z_0^2$ for some $y_0, z_0 \in \mathbb{Z}$ not all divisible by 3. Thus, without any loss of generality, we simply assume that $3 \nmid y$ or $3 \nmid z$. Note that $3 \nmid x$, $2 \nmid y$, and $x \equiv z \pmod 2$ since $2(x^2 + z^2) \equiv 2x^2 + 6z^2 \equiv 3r^2 + 8 - 3y^2 \equiv 0 \pmod 4$. If $3 \mid y$ and $3 \nmid z$, then z or $-z$ is congruent to $x + y$ modulo 3. If $3 \nmid y$ and $3 \mid z$, then y or $-y$ is congruent to $x + z$ modulo 3. If $3 \nmid yz$, then $\varepsilon_1 y \equiv \varepsilon_2 z \equiv x \pmod 3$ for some $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. So, without loss of generality, we may assume that $x + y + z \equiv 0 \pmod 3$ (otherwise we may change signs of x, y, z suitably). Note that

$$48n + 3r^2 + 8 = 2x^2 + 3y^2 + 6z^2 = 2a^2 + 3b^2 + 6c^2,$$

where $a = y + z$, $b = (2x - y + 2z)/3$ and $c = (x + y - 2z)/3$ are integers. If $x \equiv z \equiv 1 \pmod 2$, then x, y, z are all odd. If $x \equiv z \equiv 0 \pmod 2$, then a, b, c are all odd.

By the above, $48n + 3r^2 + 8 = 2a^2 + 3b^2 + 6c^2$ for some odd integers a, b, c. Since $3b^2 \equiv 3r^2 + 8 - 2a^2 - 6c^2 \equiv 3r^2 \pmod{16}$, we can write b or $-b$ as $8w + r$ with $w \in \mathbb{Z}$. Clearly, a or $-a$ has the form $6u + 1$ with $u \in \mathbb{Z}$, and $c = 2v + 1$ for some $v \in \mathbb{Z}$. Therefore

$$48n + 3r^2 + 8 = 2(6u + 1)^2 + 3(8w + r)^2 + 6(2v + 1)^2$$

and hence $n = u(3u+1)/2 + p_3(v) + w(4w+r)$. This proves the universality of $(8, 2r, 3, 1, 1, 1)$ over $\mathbb{Z}$.

(iii) Let $n \in \mathbb{N}$. By Lemma 2.1(ii), we can write $6n+7$ in the form $x^2+3y^2+6z^2$ with $x, y, z \in \mathbb{Z}$ and $x \equiv n+1 \pmod 2$. Clearly, $y \equiv n \pmod 2$. Since $6z^2 \equiv 6n+7-(n+1)^2-3n^2 \equiv 6 \pmod 4$, we have $2 \nmid z$. Hence

$$24n + 28 = 4(6n+7) = 4(x^2+3y^2+6z^2) = (x-3y)^2 + 3(x+y)^2 + 24z^2$$

with $x - 2y$, $x + 2y$ and z all odd. Note that $x - 3y$ or $3y - x$ has the form $6w + 1$ with $w \in \mathbb{Z}$. Write $x + y = 2u + 1$ and $z = 2v + 1$ with $u, v \in \mathbb{Z}$. Then

$$24n + 28 = (6w+1)^2 + 3(2u+1)^2 + 24(2v+1)^2$$

and hence $n = w(3w+1)/2 + p_3(u) + 8p_3(v)$. This proves the universality of $(8, 8, 3, 1, 1, 1)$.

(iv) Let $n \in \mathbb{N}$. By Lemma 2.2, we can write $6n + 5$ as $x^2 + y^2 + 10z^2$ with $x, y, z \in \mathbb{Z}$. Clearly, $x \not\equiv y \pmod 2$. Since $x^2 + y^2 + z^2 \equiv 2 \pmod 3$, exactly one of x, y, z is divisible by 3. Without loss of generality, we may assume that $x + y + z \equiv 0 \pmod 3$ (otherwise we adjust signs of x, y, z suitably to meet our purpose). Observe that

$$4(x^2+y^2+10z^2) = 2(x-y)^2 + 3\left(\frac{x+y+10z}{3}\right)^2 + 15\left(\frac{x+y-2z}{3}\right)^2.$$

So, $4(6n+5) = 2a^2+3b^2+15c^2$ for some odd integers a, b, c. As $3 \nmid a$, we may write a or $-a$ as $6w+1$ with $w \in \mathbb{Z}$. Write $b = 2u+1$ and $c = 2v+1$ with $u, v \in \mathbb{Z}$. Then

$$24n + 20 = 2(6w+1)^2 + 3(2u+1)^2 + 15(2v+1)^2$$

and so $n = p_3(u)+5p_3(v)+w(3w+1)$. This proves the universality of $(6, 2, 5, 5, 1, 1)$ over $\mathbb{Z}$.

(v) Let $n \in \mathbb{N}$. By Lemma 2.1(i), we can write $12n+5$ in the form $x^2+y^2+(6z)^2$ with $x, y, z \in \mathbb{Z}$. It follows that $24n + 10 = (x+y)^2 + (x-y)^2 + 72z^2$. As $(x+y)^2 + (x-y)^2 \equiv 10 \equiv 2 \pmod 4$, both $x + y$ and $x - y$ are odd. Since $(x+y)^2 + (x-y)^2 \equiv 10 \equiv 1 \pmod 3$, exactly one of $x+y$ and $x-y$ is divisible by 3. So $(x+y)^2 + (x-y)^2 = (6u+1)^2 + (6v+3)^2$ for some $u, v \in \mathbb{Z}$. Therefore

$$24n + 10 = (6u+1)^2 + (6v+3)^2 + 72z^2,$$

i.e., $n = u(3u+1)/2 + 3p_3(v) + 3z^2$. This proves the universality of $(6, 0, 3, 3, 3, 1)$ over $\mathbb{Z}$.

By Lemma 2.3, we can write $12n+14$ in the form $x^2+4y^2+9z^2$ with $x, y, z \in \mathbb{Z}$. Since $x^2 + z^2 \equiv 14 \pmod 4$, we have $2 \nmid xz$. Observe that

$$24n + 28 = 2(x^2+4y^2+9z^2) = (x-2y)^2 + (x+2y)^2 + 18z^2$$

with $x \pm 2y$ and z all odd. Clearly, exactly one of $x - 2y$ and $x + 2y$ is divisible by 3. So, for some $u, v, w \in \mathbb{Z}$ we have

$$24n + 28 = (6u+1)^2 + 9(2v+1)^2 + 18(2w+1)^2$$

and hence $n = u(3u+1)/2 + 3p_3(v) + 6p_3(w)$. This proves the universality of $(6, 6, 3, 3, 3, 1)$ over $\mathbb{Z}$.

The proof of Theorem 1.1 is now complete. □

3. Proof of Theorem 1.2

The following lemma is one of the most important theorems about integral representations of quadratic forms (cf. [2, p. 129]).

Lemma 3.1. *Let f be a nonsingular integral quadratic form and let m be a nonzero integer which is represented by f over the real field $\mathbb{R}$ and the ring $\mathbb{Z}_p$ of p-adic integers for each prime p. Then m is represented by some form f^* over $\mathbb{Z}$ where f^* is in the genus of f.*

Lemma 3.2. (i) [20, Lemma 3.2] *If $x^2+3y^2 \equiv 4 \pmod 8$ with $x, y \in \mathbb{Z}$, then $x^2+3y^2 = u^2+3v^2$ for some odd integers u and v.*

(ii) [20, Lemma 3.6] *If $w = x^2+7y^2 > 0$ with $x, y \in \mathbb{Z}$ and $8 \mid w$, then $w = u^2+7v^2$ for some odd integers u and v.*

(iii) [24, Lemma 5.1] *If $w = 3x^2+5y^2 > 0$ with $x, y \in \mathbb{Z}$ and $8 \mid w$, then $w = 3u^2+5v^2$ for some odd integers u and v.*

Proof of Theorem 1.2. (i) Let $n \in \mathbb{N}$. Clearly,

$$n = p_3(x)+p_3(y)+5z(3z+1)/2 \iff 24n+11 = 3(2x+1)^2+3(2y+1)^2+5(6z+1)^2.$$

There are two classes in the genus of $3x^2+3y^2+5z^2$, and the one not containing $3x^2+3y^2+5z^2$ has the representative

$$\begin{aligned}3x^2+2y^2+8z^2-2yz =& 3x^2+3\left(\frac{y}{2}+z\right)^2+5\left(\frac{y}{2}-z\right)^2\\ =& 3x^2+3\left(\frac{y-3z}{2}\right)^2+5\left(\frac{y+z}{2}\right)^2\end{aligned}$$

If $24n+11 = 3x^2+2y^2+8z^2-2yz$ with y odd and z even, then $3x^2 \equiv 11-2y^2 \equiv 9 \pmod 4$ which is impossible. Thus, if $24n+11 \in \{3x^2+2y^2+8z^2-2yz:\ x,y,z \in \mathbb{Z}\}$ then $24n+11 \in \{3x^2+3y^2+5z^2:\ x,y,z \in \mathbb{Z}\}$. With the help of Lemma 3.1, there are $x,y,z \in \mathbb{Z}$ such that $24n+11 = 3x^2+3y^2+5z^2$. As $5z^2 \not\equiv 11 \pmod 4$, x and y cannot be both even. Without loss of generality, we assume that $2 \nmid x$. Then $3y^2+5z^2 \equiv 11-3x^2 \equiv 0 \pmod 8$ and $3y^2+5z^2 \neq 0$. By Lemma 3.2(iii), $3y^2+5z^2 = 3y_0^2+5z_0^2$ for some odd integers y_0 and z_0. Write $x = 2u+1$ and $y_0 = 2v+1$ with $u, v \in \mathbb{Z}$. As $2 \nmid z_0$ and $3 \nmid z_0$, z_0 or $-z_0$ has the form $6w+1$ with $w \in \mathbb{Z}$. Thus $24n+11 = 3(2u+1)^2+3(2v+1)^2+5(6w+1)^2$ and hence $n = p_3(u)+p_3(v)+5w(3w+1)/2$. This proves the universality of $(15,5,1,1,1,1)$ over $\mathbb{Z}$.

(ii) Let $n \in \mathbb{N}$ and $r \in \{1,3\}$. Obviously,

$$\begin{aligned}& n = p_3(x)+\frac{y(3y+1)}{2}+3\frac{z(5z+r)}{2}\\ \iff & 120n+9r^2+20 = 15(2x+1)^2+5(6y+1)^2+9(10z+r)^2.\end{aligned}$$

There are two classes in the genus of $x^2+15y^2+5z^2$, and the one not containing $x^2+15y^2+5z^2$ has the representative

$$\begin{aligned}4x^2+4y^2+5z^2+2xy &= \left(\frac{x}{2}+2y\right)^2+15\left(\frac{x}{2}\right)^2+5z^2\\ &= \left(2x+\frac{y}{2}\right)^2+15\left(\frac{y}{2}\right)^2+5z^2.\end{aligned}$$

If $120n+9r^2+20 = 4x^2+4y^2+5z^2+2xy$ with $x,y,z \in \mathbb{Z}$, then $2xy \equiv 9r^2-5z^2 \equiv 0 \pmod 4$ and hence x or y is even. Thus, with the help of Lemma 3.1, we can always write $120n+9r^2+20=x^2+15y^2+5z^2$ with $x,y,z\in\mathbb{Z}$. Since $x^2+5z^2\equiv 20\equiv 2\pmod 3$, $x=3x_0$ for some $x_0\in\mathbb{Z}$. As $15y^2\not\equiv 9r^2\pmod 4$, x and z cannot be both even. If $2\nmid x$, then $5(3y^2+z^2)\equiv 9r^2+20-x^2\equiv 4\pmod 8$ and hence by Lemma 3.2(i) we can write $3y^2+z^2$ as $3y_0^2+z_0^2$ with y_0 and z_0 both odd. If $2\nmid z$, then $x^2+15y^2\neq 0$ and $x^2+15y^2=3(3x_0^2+5y^2)\equiv 9r^2+20-5z^2\equiv 0\pmod 8$, hence by Lemma 3.2(iii) we can write $3x_0^2+5y^2$ as $3x_1^2+5y_1^2$ with x_1 and y_1 both odd.

By the above, there are odd integers x,y,z such that $120n+9r^2+20=9x^2+15y^2+5z^2$. Write $y=2u+1$ with $u\in\mathbb{Z}$. As $3\nmid z$, we can write z or $-z$ as $6v+1$ with $v\in\mathbb{Z}$. Since $x^2\equiv r^2\pmod 5$, we can write x or $-x$ as $10w+r$ with $w\in\mathbb{Z}$. Thus

$$120n+9r^2+20=15(2u+1)^2+5(6v+1)^2+9(10z+r)^2$$

and hence $n=p_3(x)+y(3y+1)/2+3z(5z+r)/2$ with $x,y,z\in\mathbb{Z}$. This proves the universality of $(15,3r,3,1,1,1)$ over $\mathbb{Z}$.

(iii) Let $n\in\mathbb{N}$ and $r\in\{1,3\}$. Obviously,

$$\begin{aligned}&n=3x^2+\frac{y(3y+1)}{2}+\frac{z(5z+r)}{2}\\ \iff\ &120n+3r^2+5=360x^2+5(6y+s)^2+3(10z+r)^2.\end{aligned}$$

If $60n+(3r^2+5)/2=4x^2+4y^2+5z^2+2xy$ with $x,y,z\in\mathbb{Z}$, then x or y must be even. Thus, as in part (ii), $60n+(3r^2+5)/2=x^2+5y^2+15z^2$ for some $x,y,z\in\mathbb{Z}$. Note that $x^2+y^2\equiv z^2\pmod 4$. If y is odd, then $2\mid x$, $2\nmid z$ and we may assume $y\not\equiv z\pmod 4$ (otherwise it suffices to change the sign of z), hence

$$y^2+3z^2=\left(\frac{y-3z}{2}\right)^2+3\left(\frac{y+z}{2}\right)^2$$

with $y_1=(y-3z)/2$ and $z_1=(y+z)/2$ both even. So, without loss of generality, we may simply assume that $2\mid y$ and $x\equiv z\pmod 2$. Observe that

$$120n+3r^2+5=2(x^2+5y^2+15z^2)=3a^2+5b^2+10y^2.$$

with $a=(x+5z)/2$ and $b=(x-3z)/2$ both integral. Since $3a^2+5b^2\equiv 5s^2+3t^2-10y^2\equiv 0\pmod 8$ and $3a^2+5b^2>0$, by Lemma 3.2(iii) we can write $3a^2+5b^2=3c^2+5d^2$ with c and d both odd. Thus

$$120n+3r^2+5=3c^2+5d^2+40\left(\frac{y}{2}\right)^2.$$

As $(y/2)^2\equiv 5(1-d^2)\equiv d^2-1\pmod 3$, we must have $3\nmid d$ and $3\mid y$. Write $y=6u$ with $u\in\mathbb{Z}$. Clearly, d or $-d$ has the form $6v+1$ with $v\in\mathbb{Z}$. Since $c^2\equiv r^2\pmod 5$, we may write c or $-c$ as $10w+r$ with $w\in\mathbb{Z}$. Therefore

$$120n+3r^2+5=3(10w+r)^2+5(6v+1)^2+40(3u)^2$$

and hence $n=3u^2+v(3v+1)/2+w(5w+r)/2$. This proves the universality of $(6,0,5,r,3,1)$ over $\mathbb{Z}$.

(iv) Let $n\in\mathbb{N}$ and $\delta\in\{0,1\}$. Clearly,

$$\begin{aligned}&n=x(x+\delta)+\frac{y(3y+1)}{2}+5\frac{z(3z+1)}{2}\\ \iff\ &24n+6(\delta+1)=6(2x+\delta)^2+(6y+1)^2+5(6z+1)^2.\end{aligned}$$

There are two classes in the genus of $x^2+5y^2+6z^2$, and the one not containing $x^2+5y^2+6z^2$ has the representative $3x^2+3y^2+4z^2-2yz+2zx$. If $24n+6(\delta+1)=3x^2+3y^2+4z^2-2yz+2zx$, then $u=(x+y)/2$ and $v=(x-y)/2$ are integers, and

$$24n+6(\delta+1)=6u^2+6v^2+4z^2+4vz=6u^2+5v^2+(v+2z)^2.$$

Thus, by Lemma 3.1, $24n+6(\delta+1)=x^2+5y^2+6z^2$ for some $x,y,z\in\mathbb{Z}$. Since $x^2\equiv-5y^2\equiv y^2 \pmod 3$, we may assume that $x\equiv y\pmod 3$ without loss of generality. If $z\not\equiv\delta\pmod 2$, then $x^2+5y^2\equiv 6(\delta+1)-6z^2\equiv 6(\delta+1)-6(1-\delta)\equiv 4\delta\pmod 8$, hence both x and y are even and $(x-y)/2\equiv\delta\pmod 2$, and thus

$$x^2+5y^2+6z^2=\left(z-\frac{5(x-y)}{6}\right)^2+5\left(\frac{x-y}{6}+z\right)^2+6\left(\frac{x-y}{6}+y\right)^2$$

with $(x-y)/6+y\equiv(x-y)/2\equiv\delta\pmod 2$.

By the above, $24n+6(\delta+1)=x^2+5y^2+6z^2$ for some $x,y,z\in\mathbb{Z}$ with $x,y,z\in\mathbb{Z}$ with $z\equiv\delta\pmod 2$. Since x^2+5y^2 is a positive multiple of 3, by [20, Lemma 2.1] we can write $x^2+5y^2=x_0^2+5y_0^2$ with $x_0y_0\in\mathbb{Z}$ and $3\nmid x_0y_0$. So, there are $x,y,z\in\mathbb{Z}$ with $x\equiv y\not\equiv 0\pmod 3$ and $z\equiv\delta\pmod 2$ such that $24n+6(\delta+1)=x^2+5y^2+6z^2$. Write $z=2w+\delta$ with $w\in\mathbb{Z}$. Since $x^2+5y^2\equiv 6\pmod 8$, both x and y are odd. Thus x or $-x$ has the form $6u+1$ with $u\in\mathbb{Z}$, and y or $-y$ has the form $6v+1$ with $v\in\mathbb{Z}$. Therefore

$$24n+6(\delta+1)=(6u+1)^2+5(6v+1)^2+6(2w+\delta)^2$$

and hence $n=w(w+\delta)+u(3u+1)/2+5v(3v+1)/2$. This proves the universality of $(15,5,3,1,2,2\delta)$ over $\mathbb{Z}$.

(v) Let $n\in\mathbb{N}$. Apparently,

$$\begin{aligned}&n=x(x+1)+\frac{y(3y+1)}{2}+7\frac{z(3z+1)}{2}\\ \iff &24n+14=6(2x+1)^2+(6y+1)^2+7(6z+1)^2.\end{aligned}$$

There are two classes in the genus of $x^2+6y^2+7z^2$, and the one not containing $x^2+6y^2+7z^2$ has the representative

$$2x^2+5y^2+5z^2-4yz=2x^2+10u^2+10v^2-4(u+v)(u-v)=2x^2+6u^2+14v^2$$

with $u=(y+z)/2$ and $v=(y-z)/2$. If $24n+14=2x^2+6u^2+14v^2$ for some $x,u,v\in\mathbb{Z}$ with $x\not\equiv v\pmod 2$, then $14\equiv 2+6u^2\pmod 8$ which is impossible. If $24n+14=2x^2+6u^2+14v^2$ with $x,u,v\in\mathbb{Z}$ and $x\equiv v\pmod 2$, then

$$24n+14=6u^2+\left(\frac{x-7v}{2}\right)^2+7\left(\frac{x+v}{2}\right)^2.$$

By the above and Lemma 3.1, there are $x,y,z\in\mathbb{Z}$ such that $24n+14=6x^2+y^2+7z^2$. If $2\mid x$, then $y^2+7z^2\equiv 6-6x^2\equiv 6\pmod 8$ which is impossible. So $x=2u+1$ for some $u\in\mathbb{Z}$. Note that $y^2+7z^2\equiv 6-6x^2\equiv 0\pmod 8$ and $y^2+7z^2\neq 0$. Applying Lemma 3.2(ii) we can write y^2+7z^2 as $y_0^2+7z_0^2$ with y_0 and z_0 both odd. Note that $y_0^2+z_0^2\equiv y_0^2+7z_0^2\equiv 14\equiv 2\pmod 3$. So y_0 or $-y_0$ can be written as $6v+1$ with $v\in\mathbb{Z}$, and z_0 or $-z_0$ has the form $6w+1$ with $w\in\mathbb{Z}$. Thus

$$24n+14=6x^2+y_0^2+7z_0^2=6(2u+1)^2+(6v+1)^2+7(6w+1)^2$$

and hence $n=u(u+1)+v(3v+1)/2+7z(3z+1)/2$. This proves the universality of $(21,7,3,1,2,2)$.

(vi) Let $r\in\{1,3,5\}$ and $n\in\mathbb{N}$. Clearly,

$$n=p_3(x)+p_3(y)+\frac{z(7z+r)}{2}\iff 56n+14+r^2=7(2x+1)^2+7(2y+1)^2+(14z+r)^2.$$

There are two classes in the genus of $x^2+7y^2+7z^2$, and the one not containing $x^2+7y^2+7z^2$ has the representative

$$\begin{aligned}2x^2+4y^2+7z^2+2xy&=\left(\frac{x}{2}+2y\right)^2+7\left(\frac{x}{2}\right)^2+7z^2.\\&=\left(\frac{x-3y}{2}\right)^2+7\left(\frac{x+y}{2}\right)^2+7z^2\end{aligned}$$

If $56n+14+r^2=2x^2+4y^2+7z^2+2xy$ with x odd and y even, then $15\equiv 14+r^2\equiv 2x^2+7z^2\equiv 9\pmod 4$ which is impossible. Thus, if $56n+14+r^2\in\{2x^2+4y^2+7z^2+2xy:\ x,y,z\in\mathbb{Z}\}$ then $56n+14+r^2\in\{x^2+7y^2+7z^2:\ x,y,z\in\mathbb{Z}\}$. With the help of Lemma 3.1, there are $x,y,z\in\mathbb{Z}$ such that $56n+14+r^2=x^2+7y^2+7z^2$. As $x^2\not\equiv 14+r^2\equiv 15\pmod 4$, y and z cannot be both even. Without loss of generality, we assume that $2\nmid z$. Then $x^2+7y^2\equiv 14+r^2-7z^2\equiv 0\pmod 8$ and $x^2+7y^2\neq 0$. By Lemma 3.2(ii), $x^2+7y^2=x_0^2+7y_0^2$ for some odd integers x_0 and y_0. Now $56n+14+r^2=x_0^2+7y_0^2+7z^2$. Clearly, x_0 or $-x_0$ has the form $14w+r$ with $w\in\mathbb{Z}$. Write $y_0=2u+1$ and $z=2v+1$ with $u,v\in\mathbb{Z}$. Then

$$56n+14+r^2=(14w+r)^2+7(2u+1)^2+7(2v+1)^2$$

and hence $n=p_3(u)+p_3(v)+w(7w+r)/2$. This proves the universality of $(7,r,1,1,1,1)$ over $\mathbb{Z}$.

(vii) Let $n\in\mathbb{N}$ and $t\in\{1,3,5\}$. Clearly,

$$\begin{aligned}&n=p_3(x)+\frac{y(3y+1)}{2}+\frac{z(7z+t)}{2}\\\iff&168n+28+3t^2=21(2x+1)^2+7(6y+1)^2+3(14z+t)^2.\end{aligned}$$

There are two classes in the genus of $3x^2+21y^2+7z^2$, and the one not containing $3x^2+21y^2+7z^2$ has the representative

$$\begin{aligned}6x^2+12y^2+7z^2+6xy=&3\left(\frac{x}{2}+2y\right)^2+21\left(\frac{x}{2}\right)^2+7z^2.\\=&3\left(\frac{x-3y}{2}\right)^2+21\left(\frac{x+y}{2}\right)^2+7z^2\end{aligned}$$

If $168n+28+3t^2=6x^2+12y^2+7z^2+6xy$ with x odd and y even, then $31\equiv 28+3t^2\equiv 6x^2+7z^2\equiv 13\pmod 4$ which is impossible. Thus, if $168n+28+3t^2\in\{6x^2+12y^2+7z^2+6xy:\ x,y,z\in\mathbb{Z}\}$ then $168n+28+3t^2\in\{3x^2+21y^2+7z^2:\ x,y,z\in\mathbb{Z}\}$. With the help of Lemma 3.1, there are $x,y,z\in\mathbb{Z}$ such that $168n+28+3t^2=3x^2+21y^2+7z^2$. As $21y^2\not\equiv 28+3t^2\equiv 31\pmod 4$, x and z cannot be both even. If $2\nmid x$, then $21y^2+7z^2\equiv 28+3t^2-3x^2\equiv 4\pmod 8$ and hence by Lemma 3.2(i) we can write $3y^2+z^2$ as $3y_0^2+z_0^2$ with y_0,z_0 odd integers. Note that $x^2+7y^2\neq 0$ since $7\nmid t$. If $2\nmid z$, then $3(x^2+7y^2)\equiv 28+3t^2-7z^2\equiv 0\pmod 8$ and hence by Lemma 3.2(ii) $x^2+7y^2=x_0^2+7y_0^2$ for some odd integers x_0 and y_0.

By the above, there are odd integers x,y,z such that $168n+28+3t^2=3x^2+7y^2+21z^2$. Write $z=2u+1$ with $u\in\mathbb{Z}$. As $y^2\equiv 1\pmod 3$, y or $-y$ has the form $6v+1$ with $v\in\mathbb{Z}$. Since $x^2\equiv t^2\pmod 7$, x or $-x$ has the form $14w+t$ with $w\in\mathbb{Z}$. Thus

$$168n+28+3t^2=3(14w+t)^2+7(6v+1)^2+21(2u+1)^2$$

and hence $n = p_3(u) + v(3v+1)/2 + w(7w+t)/2$. This proves the universality of $(7, t, 3, 1, 1, 1)$ over $\mathbb{Z}$.

(viii) Let $\delta \in \{0, 1\}$ and $r \in \{1, 3, 5\}$. Clearly,

$$\begin{aligned} & n = p_3(x) + y(y+\delta) + \frac{z(7z+r)}{2} \\ \iff & 56n + 14\delta + r^2 + 7 = 7(2x+1)^2 + 14(2y+\delta)^2 + (14z+r)^2. \end{aligned}$$

There are two classes in the genus of $x^2 + 7y^2 + 14z^2$, the one not containing $x^2 + 7y^2 + 14z^2$ has the representative

$$2x^2 + 7y^2 + 7z^2 = 2x^2 + 14\left(\frac{y+z}{2}\right)^2 + 14\left(\frac{y-z}{2}\right)^2.$$

If $56n + 14\delta + r^2 + 7 = 2x^2 + 14y^2 + 14z^2$ with $x, y, z \in \mathbb{Z}$ and $y, z \not\equiv x \pmod 2$, then $2x^2 \equiv 14\delta + r^2 + 7 \equiv 2\delta \pmod 4$, hence $x^2 \equiv \delta \pmod 4$ and also $y \equiv z \equiv \delta \pmod 2$ since

$$-2(y^2 + z^2) \equiv 14(y^2 + z^2) \equiv 14\delta + r^2 + 7 - 2\delta \equiv -4\delta \pmod 8,$$

this contradicts with $y, z \not\equiv x \pmod 2$. If $56n + 14\delta + r^2 + 7 = 2x^2 + 14y^2 + 14z^2$ with $x, y, z \in \mathbb{Z}$ and $x \equiv y \pmod 2$, then

$$56n + 14\delta + r^2 + 7 = \left(\frac{x-7y}{2}\right)^2 + 7\left(\frac{x+y}{2}\right)^2 + 14z^2.$$

In view of Lemma 3.1 and the above, there are $x, y, z \in \mathbb{Z}$ such that $56n + 14\delta + r^2 + 7 = x^2 + 7y^2 + 14z^2$. If $z \not\equiv \delta \pmod 2$, then

$$x^2 + 7y^2 \equiv 14\delta + r^2 + 7 - 14z^2 \equiv 14\delta - 14(1-\delta) \equiv 2 \pmod 4$$

which is impossible. Thus $z \equiv \delta \pmod 2$ and $x^2 + 7y^2 \equiv r^2 + 7 \equiv 0 \pmod 8$. Note that $x^2 + 7y^2 \neq 0$ since $7 \nmid r$. Applying Lemma 3.2(ii) we can write $x^2 + 7y^2$ as $x_0^2 + 7y_0^2$ with x_0 and y_0 both odd. Since $x_0^2 \equiv r^2 \pmod 7$, either x_0 or $-x_0$ has the form $14w + r$ with $w \in \mathbb{Z}$. Write $y_0 = 2u + 1$ and $z = 2v + \delta$ with $u, v \in \mathbb{Z}$. Then

$$56n + 14\delta + r^2 + 7 \equiv (14w + r)^2 + 7(2u+1)^2 + 14(2v+\delta)^2$$

and hence $n = p_3(u) + v(v+\delta) + w(7w+r)/2$. This proves the universality of $(7, r, 2, 2\delta, 1, 1)$ over $\mathbb{Z}$.

The proof of Theorem 1.2 is now complete. $\square$

4. Proof of Theorem 1.3

For a positive definite integral ternary quadratic form $f(x, y, z)$ and an integer n, as usual we define

$$r(n, f) := |\{(x, y, z) \in \mathbb{Z}^3 : \ f(x, y, z) = n\}|$$

and

$$r(n, \mathrm{gen}(f)) := \left(\sum_{f^* \in \mathrm{gen}(f)} \frac{1}{|\mathrm{Aut}(f^*)|}\right)^{-1} \sum_{f^* \in \mathrm{gen}(f)} \frac{r(n, f^*)}{|\mathrm{Aut}(f^*)|},$$

where the summation is over a set of representatives of the classes in $\mathrm{gen}(f)$, and $\mathrm{Aut}(f^*)$ is the group of integral isometries of f^*.

Lemma 4.1. *Let f be a positive definite ternary quadratic form with determinant $d(f)$. Let $m \in \{1,2\}$ and suppose that m is represented by the genus of f. Then, for each prime $p \nmid 2md(f)$, we have*

$$\frac{r(mp^2, \mathrm{gen}(f))}{r(m, \mathrm{gen}(f))} = p + 1 - \left(\frac{-md(f)}{p}\right), \tag{4}$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol.

Proof. By the Minkowski-Siegel formula [14, pp. 173-174], for any $n \in \mathbb{Z}^+$ we have

$$r(n, \mathrm{gen}(f)) = 2\pi\sqrt{\frac{n}{d(f)}} \prod_q \alpha_q(n, f),$$

where q runs over all primes and α_q is the local density. As $p \nmid 2md(f)$, by [26] we have

$$\alpha_p(mp^2, f) = 1 + \frac{1}{p} - \frac{1}{p^2} + \left(\frac{-md(f)}{p}\right)\frac{1}{p^2},$$
$$\alpha_p(m, f) = 1 + \left(\frac{-md(f)}{p}\right)\frac{1}{p}.$$

Thus

$$\frac{r(mp^2, \mathrm{gen}(f))}{r(m, \mathrm{gen}(f))} = p\frac{\alpha_p(mp^2, f)}{\alpha_p(m, f)} = p + 1 - \left(\frac{-md(f)}{p}\right).$$

This concludes the proof. □

Lemma 4.2. *Let $w = u^2 + 15v^2 > 0$ with $u, v \in \mathbb{Z}$ and $8 \mid w$. Then $w = x^2 + 15y^2$ for some odd integers x and y.*

Proof. Let k be the 2-adic order of $\gcd(u, v)$, and write $u = 2^k u_0$ and $v = 2^k v_0$ with $u_0, v_0 \in \mathbb{Z}$ not all even. If $k = 0$, then both u_0 and v_0 are odd since w is even. Below we assume $k > 0$.

We observe the identity

$$4^2(x^2 + 15y^2) = (x - 15y)^2 + 15(x + y)^2.$$

If $u_0 \not\equiv v_0 \pmod 2$, then $k \geq 2$ (since $8 \mid w$) and $4^2(u_0^2 + 15v_0^2) = s^2 + 15t^2$ with $s = u_0 - 15v_0$ and $t = u_0 + v_0$ both odd. For $j \in \mathbb{N}$, if $4^j(u_0^2 + 15v_0^2) = u_j^2 + 15v_j^2$ for some odd integers u_j and v_j, then we may assume $u_j \equiv v_j \pmod 4$ without loss of generality (otherwise we may replace v_j by $-v_j$), and hence

$$4^{j+1}(u_0^2 + 15v_0^2) = 4(u_j^2 + 15v_j^2) = u_{j+1}^2 + 15v_{j+1}^2$$

with $u_{j+1} = (u_j - 15v_j)/2$ and $v_{j+1} = (u_j + v_j)/2$ both odd. Thus, for some odd integers u_k and v_k, we have

$$w = 4^k(u_0^2 + 15v_0^2) = u_k^2 + 15v_k^2.$$

This concludes the proof. □

Proof of Theorem 1.3(i). (a) We first prove that $(7, 7, 3, 1, 1, 1)$ is universal over $\mathbb{Z}$. Let $n \in \mathbb{N}$. Clearly,

$$n = p_3(x) + 7p_3(y) + \frac{z(3z+1)}{2}$$
$$\iff 24n + 25 = 3(2x+1)^2 + 21(2y+1)^2 + (6z+1)^2.$$

There are two classes in the genus of $x^2+3y^2+21z^2$ and the one not containing $x^2+3y^2+21z^2$ has the representative

$$x^2+6y^2+12z^2-6yz = x^2+3\left(\frac{y}{2}-2z\right)^2+21\left(\frac{y}{2}\right)^2 = x^2+3\left(\frac{y+3z}{2}\right)^2+21\left(\frac{y-z}{2}\right)^2. \tag{5}$$

If $24n+25=x^2+6y^2+12z^2-6yz$ with $x,y,z\in\mathbb{Z}$, then the equality modulo 4 yields $y(y-z)\equiv 0 \pmod 2$. Thus, by (5) and Lemma 3.1, we have

$$24n+25\in\{x^2+3y^2+21z^2:\ x,y,z\in\mathbb{Z}\}. \tag{6}$$

Now we claim that $24n+25=x^2+3y^2+21z^2$ for some $x,y,z\in\mathbb{Z}$ with $y^2+7z^2>0$. This holds by (6) if $24n+25$ is not a square. Suppose that $24n+25=m^2$ with $m\in\mathbb{Z}^+$. Let p be any prime divisor of m. Clearly, $p\geq 5$. Note that $r(7^2,x^2+3y^2+21z^2)>2$ since $7^2=(\pm5)^2+3\times(\pm1)^2+21\times(\pm1)^2$. If $p\neq 7$ and $r(p^2,x^2+6y^2+12z^2-6yz)>2$, then $p^2=x^2+6y^2+12z^2-6yz$ for some $x,y,z\in\mathbb{Z}$ with $2\mid y(y-z)$ and $y^2+z^2>0$, hence by (5) we have $p^2=x^2+3u^2+21v^2$ for some $x,u,v\in\mathbb{Z}$ with $u^2+7v^2>0$, and thus $r(p^2,x^2+3y^2+21z^2)>2$. By Lemma 4.1, if $p\neq 7$ then

$$\frac{r(p^2,\mathrm{gen}(x^2+3y^2+21z^2))}{r(1,\mathrm{gen}(x^2+3y^2+21z^2))}=p+1-\left(\frac{-7}{p}\right)$$

and hence

$$r(p^2,x^2+3y^2+21z^2)+r(p^2,x^2+6y^2+12z^2-6yz)=4\left(p+1-\left(\frac{-7}{p}\right)\right)>4.$$

So we still have $r(p^2,x^2+3y^2+21z^2)>2$ if $r(p^2,x^2+6y^2+12z^2-6yz)\leq 2$. As $r(m^2,x^2+3y^2+21z^2)\geq r(p^2,x^2+3y^2+21z^2)>2$, we can write $24n+25=m^2$ as $x^2+3y^2+21z^2$ with $x,y,z\in\mathbb{Z}$ and $y^2+7z^2>0$. This proves the claim.

By the claim, there are $x,y,z\in\mathbb{Z}$ such that $24n+25=x^2+3y^2+21z^2$ and $y^2+7z^2>0$. As $3y^2\not\equiv 25\equiv 1 \pmod 4$, either x or z is odd. If $2\nmid x$, then $3(y^2+7z^2)\equiv 25-x^2\equiv 0 \pmod 8$ and hence by Lemma 3.2(ii) we can write y^2+7z^2 as $y_0^2+7z_0^2$ with y_0 and z_0 both odd. If $2\nmid z$, then $x^2+3y^2\equiv 25-21z^2\equiv 4 \pmod 8$ and hence by Lemma 3.2(i) we can write x^2+3y^2 as $x_1^2+3y_1^2$ with x_1 and y_1 both odd. Thus $24n+25=a^2+3b^2+21c^2$ for some odd integers a,b,c. As $3\nmid a$, either a or $-a$ has the form $6w+1$ with $w\in\mathbb{Z}$. Write $b=2u+1$ and $c=2v+1$ with $u,v\in\mathbb{Z}$. Then

$$24n+25=(6w+1)^2+3(2u+1)^2+21(2v+1)^2$$

and hence $n=p_3(u)+7p_3(v)+w(3w+1)/2$. This proves the universality of $(7,7,3,1,1,1)$ over $\mathbb{Z}$.

(b) Let $n\in\mathbb{N}$ and $r\in\{1,3\}$. Clearly,

$$n=5p_3(x)+\frac{y(3y+1)}{2}+\frac{z(3z+r)}{2}$$
$$\iff 24n+r^2+16=15(2x+1)^2+(6y+1)^2+(6z+r)^2.$$

There are two classes in the genus of $x^2+y^2+15z^2$, and the one not containing $x^2+y^2+15z^2$ has the representative

$$x^2+4y^2+4z^2-2yz = x^2+\left(\frac{y}{2}-2z\right)^2+15\left(\frac{y}{2}\right)^2 = x^2+\left(2y-\frac{z}{2}\right)^2+15\left(\frac{z}{2}\right)^2. \tag{7}$$

If $24n+r^2+16=x^2+4y^2+4z^2-2yz$ with $x,y,z\in\mathbb{Z}$, then $2\nmid x$ and $2\mid yz$. Thus, in view of (7) and Lemma 3.1, we have

$$24n+r^2+16\in\{x^2+y^2+15z^2:\ x,y,z\in\mathbb{Z}\}. \tag{8}$$

We claim that $24n+r^2+16=x^2+y^2+15z^2$ for some $x,y,z\in\mathbb{Z}$ with $(x^2+15z^2)(y^2+15z^2)>0$. This holds by (8) if $24n+r^2+16$ is not a square. Now suppose that $24n+r^2+16=m^2$ with $m\in\mathbb{Z}^+$. Let p be any prime divisor of m. Clearly, $p\ge 5$. Note that $r(5^2,x^2+y^2+15z^2)>4$ since

$$5^2=(\pm5)^2+0^2+15\times0^2=0^2+(\pm5)^2+15\times0^2=(\pm3)^2+(\pm4)^2+15\times0^2.$$

If $r(p^2,x^2+4y^2+4z^2-2yz)>2$, then $p^2=x^2+4y^2+4z^2-2yz$ for some $x,y,z\in\mathbb{Z}$ with $2\mid yz$ and $y^2+z^2>0$, hence by (7) $p^2=x^2+u^2+15v^2$ for some $x,u,v\in\mathbb{Z}$ with $(x^2+15v^2)(u^2+15v^2)>0$, and thus $r(p^2,x^2+y^2+15z^2)>4$. When $p>5$, by Lemma 4.1 we have

$$\frac{r(p^2,\mathrm{gen}(x^2+y^2+15z^2))}{r(1,\mathrm{gen}(x^2+y^2+15z^2))}=p+1-\left(\frac{-15}{p}\right)$$

and hence

$$r(p^2,x^2+y^2+15z^2)+2r(p^2,x^2+4y^2+4z^2-2yz)=8\left(p+1-\left(\frac{-15}{p}\right)\right)>50.$$

Thus we still have $r(p^2,x^2+y^2+15z^2)>4$ if $r(p^2,x^2+4y^2+4z^2-2yz)\le 2$. As $r(m^2,x^2+y^2+15z^2)\ge r(p^2,x^2+y^2+15z^2)>4$, we can write $24n+r^2+16$ as $x^2+y^2+15z^2$ with $(x^2+15z^2)(y^2+15z^2)>0$. This proves the claim.

By the claim, there are $x,y,z\in\mathbb{Z}$ such that $24n+r^2+16=x^2+y^2+15z^2$ and $(x^2+15z^2)(y^2+15z^2)>0$. Since $15z^2\not\equiv r^2\equiv 1 \pmod 4$, either x or y is odd. Without any loss of generality, we assume that $2\nmid x$. Since $y^2+15z^2>0$ and $y^2+15z^2\equiv r^2-x^2\equiv 0 \pmod 8$, by Lemma 4.2 we can write $y^2+15z^2=y_0^2+15z_0^2$ with y_0 and z_0 both odd. Now, $24n+r^2+16=x^2+y_0^2+15z_0^2$. Since $x^2+y_0^2\equiv r^2+1 \pmod 3$, one of x^2 and y_0^2 is congruent to r^2 modulo 3 and the other one is congruent to 1 modulo 3. Thus $x^2+y_0^2=(6u+r)^2+(6v+1)^2$ for some $u,v\in\mathbb{Z}$. Write $z_0=2w+1$ with $v\in\mathbb{Z}$. Then

$$24n+r^2+16=(6u+r)^2+(6v+1)^2+15(2w+1)^2$$

and hence $n=u(3u+r)/2+v(3v+1)/2+5p_3(w)$. This proves the universality of $(5,5,3,r,3,1)$ over $\mathbb{Z}$.

(c) Let $n\in\mathbb{N}$. Apparently,

$$n=p_3(x)+5p_3(y)+z(3z+2)$$
$$\iff 24n+26=3(2x+1)^2+15(2y+1)^2+2(6z+2)^2.$$

There are two classes in the genus of $2x^2+3y^2+15z^2$, and the one not containing $2x^2+3y^2+15z^2$ has the representative

$$g(x,y,z)=2x^2+5y^2+11z^2+2yz+2x(y-z)=2\left(x+v\right)^2+3(u-2v)^2+15u^2 \tag{9}$$

with $u=(y+z)/2$ and $v=(y-z)/2$. If $24n+26=g(x,y,z)$ with $x,y,z\in\mathbb{Z}$, then $y\equiv z \pmod 2$, and hence by (9) we have $24n+26=2a^2+3b^2+15c^2$ for some $a,b,c\in\mathbb{Z}$. So, in view of Lemma 3.1, we always have

$$24n+26\in\{2x^2+3y^2+15z^2:\ x,y,z\in\mathbb{Z}\}. \tag{10}$$

We claim that $24n+26=2x^2+3y^2+15z^2$ for some $x,y,z\in\mathbb{Z}$ with $y^2+5z^2>0$. This holds by (10) if $12n+13$ is not a square. Now suppose that $12n+13=m^2$ with $m\in\mathbb{Z}^+$. Let p be any prime divisor of m. Clearly, $p\geq 5$. Note that $r(2\times 5^2, 2x^2+3y^2+15z^2)>2$ since

$$2\times 5^2=2\times(\pm 5)^2+3\times 0^2+15\times 0^2=2(\pm 1)^2+3(\pm 4)^2+30\times 0^2.$$

If $r(2p^2,g(x,y,z))>2$, then $2p^2=g(x,y,z)$ for some $x,y,z\in\mathbb{Z}$ with $y^2+z^2>0$, hence by (9) $2p^2=2x^2+3b^2+15c^2$ for some $x,b,c\in\mathbb{Z}$ with $b^2+c^2>0$, and thus $r(2p^2,2x^2+3y^2+15z^2)>2$. When $p>5$, by Lemma 4.1 we have

$$\frac{r(2p^2,\mathrm{gen}(2x^2+3y^2+15z^2))}{r(2,\mathrm{gen}(2x^2+3y^2+15z^2))}=p+1-\left(\frac{-5}{p}\right)$$

and hence

$$r(2p^2,2x^2+3y^2+15z^2)+2r(2p^2,g(x,y,z))=6\left(p+1-\left(\frac{-5}{p}\right)\right)>40.$$

Thus we still have $r(2p^2,2x^2+3y^2+15z^2)>2$ if $r(2p^2,g(x,y,z))\leq 2$. As $r(2m^2,2x^2+3y^2+15z^2)\geq r(2p^2,2x^2+3y^2+15z^2)>2$, we can write $24n+26$ as $2x^2+3y^2+15z^2$ with $y^2+5z^2>0$. This proves the claim.

By the claim, there are $x,y,z\in\mathbb{Z}$ such that $24n+26=2x^2+3(y^2+5z^2)$ and $y^2+5z^2>0$. By [20, Lemma 2.1], $y^2+5z^2=y_0^2+5z_0^2$ for some integers y_0 and z_0 not all divisible by 3. Without any loss of generality, we simply assume that $3\nmid y$ or $3\nmid z$. Note that $3\nmid x$ and $y\equiv z \pmod 2$. If $3\nmid yz$, then $\varepsilon_1 y\equiv\varepsilon_2 z\equiv x \pmod 3$ for some $\varepsilon_1,\varepsilon_2\in\{\pm 1\}$. If $3\mid y$ and $3\nmid z$ then $x+y+\varepsilon z\equiv 0 \pmod 3$ for some $\varepsilon\in\{\pm 1\}$; similarly, if $3\nmid y$ and $3\mid z$ then $x+\varepsilon y+z\equiv 0 \pmod 3$. So, without loss of generality we may suppose that $x+y+z\equiv 0 \pmod 3$ (otherwise we adjust signs of x,y,z suitably to meet our purpose). If $y\equiv z\equiv 0 \pmod 2$, then $2x^2\equiv 26 \pmod 4$, hence $2\nmid x$ and $y\equiv z \pmod 4$ since $y^2+5z^2\equiv 0 \pmod 8$, therefore

$$2x^2+3y^2+15z^2=2\left(\frac{y-5z}{2}\right)^2+3\left(\frac{2x+5y+5z}{6}\right)^2+15\left(\frac{2x-y-z}{6}\right)^2 \tag{11}$$

with $(2x+5y+5z)/6$ and $(2x-y-z)/6$ both odd.

By the above, $24n+26=2a^2+3b^2+15c^2$ for some $a,b,c\in\mathbb{Z}$ with $2\nmid bc$. As $3\nmid a$ and $2a^2\equiv 26-3-15\equiv 0 \pmod 8$, a or $-a$ has the form $2(3w+1)$ with $w\in\mathbb{Z}$. Write $b=2u+1$ and $c=2v+1$ with $u,v\in\mathbb{Z}$. Then

$$24n+26=2(2(3w+1))^2+3(2u+1)^2+15(2v+1)^2$$

and so $n=p_3(u)+5p_3(v)+w(3w+2)$. This proves the universality of $(6,4,5,5,1,1)$ over $\mathbb{Z}$. □

Proof of Theorem 1.3(ii). (a) Let $n\in\mathbb{N}$ and $r\in\{1,2\}$. It is easy to see that

$$n=p_3(x)+5\frac{y(3y+1)}{2}+z(3z+r)$$
$$\iff 24n+2r^2+8=3(2x+1)^2+5(6y+1)^2+2(6z+r)^2.$$

As mentioned in part (b) of the proof of Theorem 1.3(i), there are two classes in the genus of $x^2+y^2+15z^2$, and the one not containing $x^2+y^2+15z^2$ has the representative $x^2+4y^2+4z^2-2yz$. If $12n+r^2+4=x^2+4y^2+4z^2-2yz$ with $x,y,z\in\mathbb{Z}$, then $2\mid yz$ since $r^2\not\equiv x^2-2\pmod 4$. Thus, in view of (7) and Lemma 3.1, $12n+r^2+4=x^2+y^2+15z^2$ for some $x,y,z\in\mathbb{Z}$. If $x\equiv y\pmod 2$, then $z\equiv r\pmod 2$, $x^2+y^2\equiv r^2-15z^2\equiv 2r^2\pmod 4$ and hence $x\equiv y\equiv r\equiv z\pmod 2$. So, x or y has the same parity with z. Without loss of generality we may assume that $y\equiv z\pmod 2$. Since $y^2+15z^2\equiv 0\pmod 4$, we have $x\equiv r\pmod 2$. If $r=2$ and $y^2+15z^2=0$, then $12n+r^2+4=0^2+x^2+15\times 0^2$ with $x\equiv 0\equiv r\pmod 2$ and $x^2+15\times 0^2>0$. If $r=1$, then $12n^2+r^2+4=12n+5$ is congruent to 2 modulo 3 and hence not a square. Thus, without loss of generality we may assume that $y^2+15z^2>0$.

Observe that

$$24n+2r^2+8=2(x^2+y^2+15z^2)=2x^2+3u^2+5v^2$$

with $u=(y+5z)/2$ and $v=(y-3z)/2$ both odd. Since $3u^2+5v^2\equiv 2r^2-2x^2\equiv 0\pmod 8$ and $2(3u^2+5v^2)=y^2+15z^2>0$, by Lemma 3.2(iii) we can write $3u^2+5v^2$ as $3y_0^2+5z_0^2$ with y_0 and z_0 both odd. As $2(x^2+z_0^2)\equiv 2x^2+5z_0^2\equiv 2r^2+8\pmod 3$, we have $x^2+z_0^2\equiv r^2+1\equiv 2\pmod 3$ and hence we may write x or $-x$ as $6u+r$, z_0 or $-z_0$ as $6v+1$, and $y_0=2w+1$, where u,v,w are integers. Therefore

$$24n+2r^2+8=2x^2+3y_0^2+5z_0^2=2(6u+r)^2+3(2w+1)^2+5(6v+1)^2$$

and hence $n=u(3u+r)/2+5v(3v+1)/2+p_3(w)$. This proves the universality of $(15,5,6,2r,1,1)$ over $\mathbb{Z}$.

(b) Let $n\in\mathbb{N}$, $s\in\{1,3,5\}$ and $t\in\{1,2\}$ with $(s,t)\neq(5,2)$. Obviously,

$$\begin{aligned} &n=p_3(x)+\frac{y(5y+s)}{2}+z(3z+t)\\ \iff &120n+3s^2+10t^2+15=15(2x+1)^2+3(10y+s)^2+10(6z+t)^2.\end{aligned}$$

There are two classes in the genus of $3x^2+10y^2+15z^2$, and the one not containing $3x^2+10y^2+15z^2$ has the representative

$$\begin{aligned} g(x,y,z)=&7x^2+7y^2+12z^2+6(x+y)z+4xy\\ =&3\left(\frac{x+y}{2}+2z\right)^2+10\left(\frac{x-y}{2}\right)^2+15\left(\frac{x+y}{2}\right)^2. \end{aligned}\tag{12}$$

If $120n+3s^2+10t^2+15=g(x,y,z)$ with $x,y,z\in\mathbb{Z}$, then we obviously have $x\equiv y\pmod 2$. Thus, in view of (12) and Lemma 3.1, $120n+3s^2+10t^2+15=3x^2+10y^2+15z^2$ for some $x,y,z\in\mathbb{Z}$. If $x=z=0$, then $120n+3s^2+10t^2+15=10y^2$, hence $(s,t)=(5,1)$ and $y^2=12n+10\equiv 2\pmod 4$ which is impossible. So $x^2+5z^2>0$, and hence by [20, Lemma 2.1] we can rewrite x^2+5z^2 as $x_0^2+5z_0^2$ with $x_0,z_0\in\mathbb{Z}$ not all divisible by 3. Without loss of generality, we simply assume that $3\nmid x$ or $3\nmid z$. Note that $3\nmid y$ since $3\nmid t$. If $3\nmid xz$, then $\varepsilon_1x\equiv y\equiv\varepsilon_2z$ for some $\varepsilon_1,\varepsilon_2\in\{\pm1\}$. If $3\mid x$ and $3\nmid z$, then $x+y+\varepsilon z\equiv 0\pmod 3$ for some $\varepsilon\in\{\pm1\}$. If $3\nmid x$ and $3\mid z$, then $\varepsilon x+y+z\equiv 0\pmod 3$ for some $\varepsilon\in\{\pm1\}$. Without loss of generality, we just assume that $x+y+z\equiv 0\pmod 3$ (otherwise we may adjust signs of x,y,z suitably). Note that $x\equiv z\pmod 2$ and we have the identity

$$3x_1^2+10y_1^2+15z_1^2=3x^2+10y^2+15z^2,\tag{13}$$

where

$$x_1=\frac{x+10y-5z}{6},\quad y_1=\frac{x+z}{2}\quad\text{and}\quad z_1=\frac{x-2y-5z}{6}$$

are all integral.

If $x \equiv z \equiv 1 \pmod 2$, then $10y^2 = 120n + 3s^2 + 10t^2 + 15 - 3x^2 - 15z^2 \equiv 10t^2 \pmod 4$ and hence $y \equiv t \pmod 2$.

Now suppose that $x \equiv z \equiv 0 \pmod 2$. Then $2y^2 \equiv 10y^2 \equiv 3s^2 + 10t^2 + 15 \equiv 2(t^2+1) \pmod 4$ and hence $y \not\equiv t \pmod 2$. Observe that

$$2t^2 + 2 \equiv 120n + 3s^2 + 10t^2 + 15 = 3x^2 + 10y^2 + 15z^2 \equiv x^2 + z^2 + 2(t+1)^2 \pmod 8$$

and hence

$$y_1 = \frac{x+z}{2} \equiv \left(\frac{x}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = \frac{x^2+z^2}{4} \equiv t \pmod 2.$$

Thus

$$z_1 = x_1 - 2y \equiv x_1 \equiv \frac{x+z}{2} - 3z + 5y \equiv t + y \equiv 1 \pmod 2.$$

In view of the above, there are integers $x, y, z \in \mathbb{Z}$ with $x \equiv z \equiv 1 \pmod 2$ and $y \equiv t \pmod 2$ such that $120n + 3s^2 + 10t^2 + 15 = 3x^2 + 10y^2 + 15z^2$. Clearly, y or $-y$ has the form $6v + t$ with $v \in \mathbb{Z}$. Write $z = 2w + 1$ with $w \in \mathbb{Z}$. Since $x^2 \equiv s^2 \pmod 5$, we can write x or $-x$ as $10u + s$ with $w \in \mathbb{Z}$. Therefore

$$120n + 3s^2 + 10t^2 + 15 = 3(10u+s)^2 + 10(6v+t)^2 + 15(2w+1)^2$$

and hence $n = p_3(w) + u(5u+s)/2 + v(3v+t)$. This proves the universality of $(6, 2t, 5, s, 1, 1)$ over $\mathbb{Z}$. □

5. Proof of Theorem 1.4

B. W. Jones and G. Pall [12] proved the following celebrated result.

Lemma 5.1. *Let $n \in \mathbb{N}$ with $8n+1$ not a square. Then*

$$\begin{aligned}&|\{(x,y,z) \in \mathbb{Z}^3 : \ x^2 + y^2 + z^2 = 8n+1 \ \&\ 4 \mid x\}| \\ =&|\{(x,y,z) \in \mathbb{Z}^3 : \ x^2 + y^2 + z^2 = 8n+1 \ \&\ x \equiv 2 \pmod 4\}| > 0.\end{aligned}$$

A. G. Earnest [5, 6] showed the following useful result.

Lemma 5.2. *Let c be a primitive spinor exceptional integer for the genus of a positive ternary quadratic form $f(x,y,z)$, and let S be a spinor genus containing f. Let s be a fixed positive integer relatively prime to $2d(f)$ for which cs^2 can be primitively represented by S. If $t \in \mathbb{Z}^+$ is relatively prime to $2d(f)$, then ct^2 can be primitively represented by S if and only if*

$$\left(\frac{-cd(f)}{s}\right) = \left(\frac{-cd(f)}{t}\right).$$

Proof of Theorem 1.4. Fix $n \in \mathbb{N}$. Clearly,

$$n = p_3(x) + y^2 + 2z(4z+1) \iff 8n + 2 = (2x+1)^2 + 8y^2 + (8z+1)^2.$$

So, it suffices to show that $8n + 2 = x^2 + y^2 + 8z^2$ for some $x, y, z \in \mathbb{Z}$ with $x \equiv \pm 1 \pmod 8$.

Case 1. n is not twice a triangular number.

In this case, $4n+1$ is not a square. If $2 \mid n$, then by Lemma 5.1 we can write $4n+1$ as $x^2+y^2+z^2$ with $2 \nmid x$, $2 \mid y$ and $z \equiv 2 \pmod 4$. If $2 \nmid n$, then there are $x, y, z \in \mathbb{Z}$ with $2 \nmid x$ and $y \equiv z \equiv 0 \pmod 2$ such that $4n+1 = x^2+y^2+z^2$ and hence $y \not\equiv z \pmod 4$ since $y^2 + z^2 \equiv 5 - x^2 \equiv 4 \pmod 8$. So we can always write $4n+1 = x^2+y^2+z^2$ with $2 \nmid x$, $2 \mid y$ and $z \equiv 2n-2 \pmod 4$, hence

$$8n + 2 = 2(x^2+y^2+z^2) = (x+y)^2 + (x-y)^2 + 8\left(\frac{z}{2}\right)^2$$

with $z/2 \equiv n-1 \pmod 2$. Thus

$$(x+y)^2+(x-y)^2 \equiv 8n+2-8(n-1)=10 \not\equiv 3^2+3^2 \pmod{16}$$

and hence $x+\varepsilon y \equiv \pm 1 \pmod 8$ for some $\varepsilon \in \{\pm 1\}$.

Case 2. $n=2p_3(m)$ with $m \in \mathbb{N}$, and $2m+1$ has no prime factor of the form $4k+3$.

In this case, $2m+1$ can be expressed as the sum of two squares. If $4 \mid m$, then

$$8n+2=2(2m+1)^2=(2m+1)^2+(2m+1)^2+8\times 0^2$$

with $2m+1 \equiv 1 \pmod 8$. If $4 \nmid m$, then $2m+1=u^2+(2v)^2$ for some odd integers u and v, and hence

$$\begin{aligned} 8n+2 =&2(u^2+4v^2)^2=2((u^2-4v^2)^2+(4uv)^2) \\ =&(u^2-4v^2+4uv)^2+(u^2-4v^2-4uv)^2+8\times 0^2 \end{aligned}$$

with $u^2-4v^2 \pm 4uv \equiv 1 \pmod 8$.

Case 3. $n=2p_3(m)$ with $m \in \mathbb{N}$, and $2m+1$ has a prime factor $p \equiv 3 \pmod 4$.

By Lagrange's four-square theorem, we can write $p=a^2+b^2+c^2+d^2$, where a is an even number and b,c,d are odd numbers. Thus

$$\begin{aligned} p^2 =&(a^2+b^2-c^2-d^2)^2+4(a^2+b^2)(c^2+d^2) \\ =&(a^2+b^2-c^2-d^2)^2+(2ac+2bd)^2+(2ad-2bc)^2 \end{aligned}$$

and hence $(2m+1)^2=x^2+(2y)^2+(2z)^2$ for some odd integers x,y,z. Observe that

$$8n+2=2(2m+1)^2=(x+2y)^2+(x-2y)^2+8z^2$$

and $(x+2y)^2+(x-2y)^2 \equiv 2-8z^2 \equiv 10 \not\equiv 3^2+3^2 \pmod{16}$. So one of $x+2y$ and $x-2y$ is congruent to 1 or -1 modulo 8.

Now we give an alternative approach to Case 3. There are three classes in the genus of $x^2+y^2+32z^2$ with the three representatives

$$\begin{aligned} f_1(x,y,z) =&x^2+y^2+32z^2, \\ f_2(x,y,z) =&2x^2+2y^2+9z^2+2yz-2zx, \\ f_3(x,y,z) =&x^2+4y^2+9z^2-4yz. \end{aligned}$$

The class of f_1 and the class of f_2 constitute a spinor genus while another spinor genus in the genus only contains the class of f_3. Since 2 is a primitive spinor exceptional integer for this genus, by Lemma 5.2 we can write $2p^2$ as

$$f_3(u,v,w)=u^2+4v^2+9w^2-4vw=u^2+(2v-w)^2+8w^2$$

with $u,v,w \in \mathbb{Z}$. Since $2 \nmid uw$, we see that $8n+2=2(2m+1)^2=a^2+b^2+8c^2$ for some odd integers a,b,c. As $a^2+b^2 \equiv 2-8c^2 \equiv 10 \not\equiv 3^2+3^2 \pmod{16}$, a or b is congruent to 1 or -1 modulo 8. This concludes our discussion of Case 3.

In view of the above, we have completed the proof of Theorem 1.4. □

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