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ON THE SET $\{\pi(kn):\ k=1,2,3,\ldots\}$

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Abstract. An open conjecture of Z.-W. Sun states that for any integer $n>1$ there is a positive integer $k\le n$ such that $\pi(kn)$ is prime, where $\pi(x)$ denotes the number of primes not exceeding x. In this paper, we show that for any positive integer n the set $\{\pi(kn):\ k=1,2,3,\ldots\}$ contains infinitely many P_2-numbers which are products of at most two primes. We also prove that under the Bateman–Horn conjecture the set $\{\pi(4k):\ k=1,2,3,\ldots\}$ contains infinitely many primes.

1. Introduction

For $x\ge 0$, let $\pi(x)$ denote the number of primes not exceeding x. For the asymptotic behavior of the prime-counting function $\pi(x)$, by the Prime Number Theorem we have

$$\pi(x)\sim\frac{x}{\log x}\quad\text{as } x\to+\infty.$$

Since there are no simple closed formula for the exact values of $\pi(x)$ with $x>0$, it is difficult to obtain combinatorial properties of the prime-counting function $\pi(x)$.

In 1962, S. Golomb [2]] proved that for any integer $m>1$ there is an integer $n>1$ with $n/\pi(n)=m$ (i.e., $\pi(n)=n/m$). In 2017 Z.-W. Sun [6] obtained the following general result: For any $a\in\mathbb{Z}$ and $m\in\mathbb{Z}^+=\{1,2,3,\ldots\}$, we have

$$\pi(n)=\frac{n+a}{m}\quad\text{for some integer } n>1$$

if and only if $a\le s_m$, where

$$s_m:=\max\{km-p_k:\ k\in\mathbb{Z}^+\}=\max\{km-p_k:\ k=1,2,\ldots,\lfloor e^{m+1}\rfloor\}$$

with p_k the k-th prime. This implies that for any integer $m>4$ we have $\pi(mn)=m+n$ for some $n\in\mathbb{Z}^+$ (cf. [6, Corollary 1.2]).

On Feb. 9, 2014, Z.-W. Sun [4] made the following conjecture.

Conjecture 1.1. (Sun [5, Conjecture 2.1(i)]) *For any integer $n>1$, there is a positive integer $k\le n$ such that $\pi(kn)$ is prime.*

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This conjecture was verified by Sun for all $n = 2, 3, \ldots, 2 \times 10^7$ (cf. [4]). For $n = 10$, among the ten numbers

$$\pi(10) = 4,\ \pi(20) = 8,\ \pi(30) = 10,\ \pi(40) = 12,\ \pi(50) = 15,$$
$$\pi(60) = 17,\ \pi(70) = 19,\ \pi(80) = 22,\ \pi(90) = 24,\ \pi(100) = 25,$$

only $\pi(60) = 17$ and $\pi(70) = 19$ are prime. Note also that among the 13 numbers

$$\pi(13) = 6,\ \pi(2 \times 13) = 9,\ \pi(3 \times 13) = 12,\ \pi(4 \times 13) = 15,\ \pi(5 \times 13) = 18,$$
$$\pi(6 \times 13) = 21,\ \pi(7 \times 13) = 24,\ \pi(8 \times 13) = 27,\ \pi(9 \times 13) = 30,$$
$$\pi(10 \times 13) = 31,\ \pi(11 \times 13) = 34,\ \pi(12 \times 13) = 36,\ \pi(13 \times 13) = 39$$

only $\pi(10 \times 13) = 31$ is prime.

Motivated by Conjecture 1.1, for any $n \in \mathbb{Z}^+$ we introduce the set

$$\mathcal{A}_n = \{\pi(kn) :\ k \in \mathbb{Z}^+\}. \tag{1.1}$$

Clearly, $\mathcal{A}_1$ coincides with $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\mathcal{A}_2 = \mathbb{Z}^+$ since $\pi(p_j + 1) = j$ for all $j = 2, 3, \ldots$. As $\lim_{k\to+\infty} \pi(3k) = +\infty$, and $\pi(3(k + 1)) - \pi(3k) \in \{0, 1\}$ for all $k \in \mathbb{Z}^+$, we see that

$$\mathcal{A}_3 = \{m \in \mathbb{Z}^+ :\ m \geq \pi(3)\} = \{2, 3, \ldots\}.$$

It is not known whether $\mathcal{A}_4$ contains infinitely many primes.

Throughout this paper, for any $A \subseteq \mathbb{Z}^+$ and $x \geq 0$, we define

$$A(x) := \{a \leq x :\ a \in A\}. \tag{1.2}$$

Now we present our first theorem.

Theorem 1.1. *Let $S \subseteq \mathbb{Z}^+$ with*

$$\lim_{x\to+\infty} \frac{|S(x)|}{x/\log x} = +\infty.$$

Then, for any $n \in \mathbb{Z}^+$ the set $\mathcal{A}_n$ contains infinitely many elements of S.

If $S = \{a \in \mathbb{Z}^+ :\ a \equiv r \pmod m\}$ with $m, r \in \mathbb{Z}^+$, then

$$\lim_{x\to+\infty} \frac{|S(x)|}{x} = \frac{1}{m} \quad \text{and hence} \quad \lim_{x\to+\infty} \frac{|S(x)|}{x/\log x} = +\infty.$$

Thus Theorem 1.1 yields the following corollary.

Corollary 1.1. *For any $n, m, r \in \mathbb{Z}^+$, there are infinitely many $a \in \mathcal{A}_n$ with $a \equiv r \pmod m$.*

In contrast, Sun [5, Conjecture 2.2] conjectured that for each $n \in \mathbb{Z}^+$ we have $n \mid \pi(kn)$ for some $k = 1, \ldots, p_n - 1$, and also $\{\pi(kn) :\ k = 1, \ldots, 2p_n\}$ contains a complete system of residues modulo n.

As usual, for any $r \in \mathbb{Z}^+$, we call $n \in \mathbb{Z}^+$ a P_r-number if it is a product of at most r primes. It is known (cf. [7, Theorem 6.4]) that the number of P_2-numbers up to X is $\gg \frac{X}{\log X} \log\log X$ for $X > 1$. So Theorem 1.1 has the following consequence.

Corollary 1.2. *For any $n \in \mathbb{Z}^+$, the set $\mathcal{A}_n$ contains infinitely many P_2-numbers.*

The following conjecture extends a conjecture of Hardy and Littlewood concerning twin primes.

Conjecture 1.2 (P. T. Bateman and R. A. Horn [1])**.** *For $N \in \mathbb{Z}^+$ let $V(N)$ denote the number of positive integers $n \le N$ with $4n+1$ and $4n+3$ twin prime. Then*

$$V(N) = 4\mathfrak{S}\frac{N}{\log^2 N}\Big(1+o(1)\Big) \quad \text{as } N \to +\infty,$$

where the twin prime constant $\mathfrak{S}$ is given by

$$\mathfrak{S} = \prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big) \approx 0.6601618$$

with p in the product runs over all odd primes.

Now we state our second theorem.

Theorem 1.2. *Assuming the truth of Conjecture 1.2, there are infinitely many primes in $\mathcal{A}_4$.*

We are going to prove Theorems 1.1 and 1.2 in Sections 2 and 3 respectively.

2. Proof of Theorem 1.1

For $A \subseteq \mathbb{Z}^+$, we write A^c for $\mathbb{Z}^+ \setminus A$, the complement of A.

Lemma 2.1. *For integers $K \ge 3$, we have*

$$|\mathcal{A}_n^c(X)| \ll_n \frac{X}{\log X}, \tag{2.1}$$

where $X = \pi(Kn)$ with $n \in \mathbb{Z}^+$.

Proof. Note that

$$\begin{aligned} |\mathcal{A}_n^c(X)| &= \left|\left\{a \in \mathbb{Z}^+ :\ a \in \bigcup_{k=1}^{K}(\pi((k-1)n), \pi(kn))\right\}\right| \\ &= \sum_{\substack{1\le k\le K \\ \pi(kn)-\pi(kn-n)\ge 2}} \Big(\pi(kn)-\pi(kn-n)-1\Big). \end{aligned} \tag{2.2}$$

Since $\pi(kn)-\pi(kn-n)-1 \le n$, we have

$$|\mathcal{A}_n^c(X)| \le n|\mathcal{K}|,$$

where

$$\mathcal{K} = \{1 \le k \le K :\ \pi(kn)-\pi(kn-n) \ge 2\}.$$

For each $k \in \mathcal{K}$, there exist two primes p and q such that $kn - n < p < q \leq kn$ and hence $2 \leq q - p < n$. Thus

$$|\mathcal{K}| \leq \sum_{2 \leq h < n} |\{p \leq Kn :\ p \text{ and } p+h \text{ are both prime}\}|.$$

It is well known (cf. [3, Theorem 6.7]) that

$$\pi_h(Kn) := |\{p \leq Kn :\ p \text{ and } p+h \text{ are both prime}\}| \ll_h \frac{Kn}{\log^2(Kn)},$$

where the implied constant may depend on $h \geq 2$. Combining the above, we obtain

$$|\mathcal{A}_n^c(X)| \leq n|\mathcal{K}| \ll_n \frac{K}{\log^2 K}. \tag{2.3}$$

In view of the Prime Number Theorem,

$$X = \frac{Kn}{\log(Kn)}\Big(1 + o(1)\Big). \tag{2.4}$$

Now (2.1) follows from (2.3) and (2.4). This concludes the proof. $\square$

Proof of Theorem 1.1. By Lemma 2.1, there is a constant $C_n > 0$ such that for any $K \in \{3, 4, \ldots\}$ we have

$$|\mathcal{A}_n^c(X)| \leq C_n \frac{X}{\log X}, \tag{2.5}$$

where $X = Kn$. As $\lim_{x \to +\infty} |S(x)|/(x/\log x) = +\infty$, if $K \in \mathbb{Z}^+$ is large enough then

$$\frac{|S(X)|}{X/\log X} \geq 2C_n. \tag{2.6}$$

.

Let $K \in \{3, 4, \ldots\}$ be large enough so that (2.6) holds. Then, for $X = \pi(Kn)$ we have

$$\frac{|S_1(X)|}{X/\log X} + \frac{|S_2(X)|}{X/\log X} = \frac{|S(X)|}{X/\log X} \geq 2C_n,$$

where $S_1 = S \cap \mathcal{A}_n$ and $S_2 = S \cap \mathcal{A}_n^c$. As

$$\frac{|S_2(X)|}{X/\log X} \leq \frac{|\mathcal{A}_n^c(X)|}{X/\log X} \leq C_n$$

by (2.5), we obtain

$$|S_1(X)| \geq C_n \frac{X}{\log X}.$$

In view of the above, $\lim_{x \to +\infty} |S_1(x)| = +\infty$. So $\mathcal{A}_n$ contains infinitely many elements of S. This completes the proof of Theorem 1.1. $\square$

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $X = \pi(4K)$ with $K \in \{3, 4, \ldots\}$. Applying (2.2) with $n = 4$, we get

$$|\mathcal{A}_4^c(X)| = \sum_{\substack{1 \le k \le K \\ \pi(4k) - \pi(4k-4) \ge 2}} \Big(\pi(4k) - \pi(4k-4) - 1\Big).$$

For each integer $k > 1$, the interval $(4k-4, 4k]$ contains at most two primes. Note also that $\pi(4) - \pi(0) = 2$. So we have

$$|\mathcal{A}_4^c(X)| = 1 + |\mathcal{V}|,$$

where

$$\mathcal{V} = \{1 \le k < K : \ \pi(4k+4) - \pi(4k) = 2\}.$$

For any $k = 1, \ldots, K-1$, clearly $\pi(4k+4) - \pi(4k) = 2$ if and only if both $4k+1$ and $4k+3$ are twin prime. Under Conjecture 1.2, we have

$$|\mathcal{V}| = V(K-1) = 4\mathfrak{S}\frac{K}{\log^2 K}\Big(1 + o(1)\Big)$$

and hence

$$|\mathcal{A}_4^c(X)| = 4\mathfrak{S}\frac{K}{\log^2 K}\Big(1 + o(1)\Big).$$

By the Prime Number Theorem,

$$X = \frac{4K}{\log K}\Big(1 + o(1)\Big) \quad \text{and} \quad \pi(X) = \frac{X}{\log X}(1 + o(1)).$$

Thus

$$\pi(X) - |\mathcal{A}_4^c(X)| = (1 - \mathfrak{S})\frac{X}{\log X}\Big(1 + o(1)\Big).$$

Note that $\mathfrak{S} < 1$. By the above,

$$|\{p \le X : \ p \text{ is a prime in } \mathcal{A}_4\}| \ge \pi(X) - |\mathcal{A}_4^c(X)| \to +\infty$$

as $X = 4K \to +\infty$. So $\mathcal{A}_4$ contains infinitely many primes. This concludes the proof. $\square$

References

[1] P. T. Bateman and R. A. Horn, *A heuristic asymptotic formula concerning the distribution of prime numbers*, Math. Comp. **16** (1962), 363–367.

[2] S. W. Golomb, *On the ratio of N to $\pi(N)$*, Amer. Math. Monthly **69** (1962), 36–37.

[3] H. Iwaniec and E. Kowalski, Analytic number theory, Amer. Math. Soc. Colloq. Publ. **53**, Amer. Math. Soc., Providence, RI, 2004.

[4] Z.-W. Sun, Sequence A237578 on OEIS (On-Line Encyclopedia of Integer Sequences), `http://oeis.org/A237578`, Feb. 9, 2014.

[5] Z.-W. Sun, *Problems on combinatorial properties of primes*, in: Number Theory: Plowing and Starring through High Wave Forms, 169–187, Ser. Number Theory Appl., 11, World Sci. Publ., Singapore, 2015.
[6] Z.-W. Sun, *A new theorem on the prime-counting function*, Ramanujan J. **42** (2017), 59–67.
[7] G. Tenenbaum, Introduction to Analytic and Probabilistic Number theory, Graduate Studies in Mathematics. **163**, Amer. Math. Soc., Providence, RI, 2014.

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