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A NEW EXTENSION OF THE SUN-ZAGIER RESULT INVOLVING BELL NUMBERS AND DERANGEMENT NUMBERS

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ABSTRACT. Let p be any prime and let a and n be positive integers with $p \nmid n$. We show that

$$\sum_{k=1}^{p^a-1} \frac{B_k}{(-n)^k} \equiv a(-1)^{n-1} D_{n-1} \pmod{p},$$

where $B_0, B_1, \ldots$ are the Bell numbers and $D_0, D_1, \ldots$ are the derangement numbers. This extends a result of Sun and Zagier published in 2011. Furthermore, we prove that

$$(-x)^n \sum_{k=1}^{p^a-1} \frac{B_k(x)}{(-n)^k} \equiv -\sum_{r=1}^{a} x^{p^r} \sum_{k=0}^{n-1} \frac{(n-1)!}{k!}(-x)^k \pmod{p\mathbb{Z}_p[x]},$$

where $B_k(x) = \sum_{l=0}^{k} S(k,l)x^l$ is the Bell polynomial of degree k with $S(k,l)$ $(0 \leqslant l \leqslant k)$ the Stirling numbers of the second kind, and $\mathbb{Z}_p$ is the ring of all p-adic integers.

1. INTRODUCTION

Let $B_0 = 1$. For each $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ let B_n denote the number of partitions of a set of cardinality n. For example, $B_3 = 5$ since there are totally 5 partitions of $\{1, 2, 3\}$:

$$\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\},\ \{\{1, 3\}, \{2\}\},\ \{\{2, 3\}, \{1\}\}, \{\{1, 2, 3\}\}.$$

The Bell numbers $B_0, B_1, \ldots$, named after Bell who studied them in the 1930s, play important roles in combinatorics. Here are the values of $B_1, \ldots, B_7$:

$$B_1 = 1,\ B_2 = 2,\ B_3 = 5,\ B_4 = 15,\ B_5 = 52,\ B_6 = 203,\ B_7 = 877.$$

It is known that

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1} \text{ and } B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \ (n = 0, 1, 2, \ldots).$$

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The author's conjecture (cf. [6, Conjecture 3.2]) that the sequence $(\sqrt[n+1]{B_{n+1}}/\sqrt[n]{B_n})_{n\geqslant 1}$ is strictly decreasing (with limit 1), is still open. For any prime p and $m,n\in\mathbb{N}=\{0,1,2,\ldots\}$, we have the classical Touchard congruence (cf. [8])

$$B_{p^m+n}\equiv mB_n+B_{n+1}\pmod p.$$

Let $D_0=1$, and define D_n $(n\in\mathbb{Z}^+)$ by

$$D_n=|\{\pi\in S_n:\ \pi(k)\neq k \text{ for all } k=1,\ldots,n\}|,$$

where S_n is the symmetric group of all permutations on $\{1,\ldots,n\}$. Those $D_0,D_1,D_2,\ldots$ are called the derangement numbers, and they were first introduced by Euler. It is well known that

$$D_n=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}\ \text{ for all } n\in\mathbb{N}.$$

In 2011, the author and Zagier [7] showed that for any prime p and $n\in\mathbb{Z}^+$ with $p\nmid n$ we have

$$\sum_{k=1}^{p-1}\frac{B_k}{(-n)^k}\equiv(-1)^{n-1}D_{n-1}\pmod p,\tag{1.1}$$

which relates the Bell numbers to the derangement numbers. The surprising congruence (1.1) was called the Sun-Zagier congruence by Sun, Wu and Zhuang [5] who used the umbral calculus to give a generalization, by Mező and Ramirez [3] in 2017 who extended it to the so-called r-Bell numbers, and by Mu [4] in 2018 who re-proved via an identity of Clarke and Sved [1] relating the Bell numbers to the derangement numbers.

In this paper we extend the fundamental Sun-Zagier result in a new way.

Theorem 1.1. *Let p be any prime and let a be a positive integer. For any $n\in\mathbb{Z}^+$ with $p\nmid n$, we have*

$$\sum_{k=1}^{p^a-1}\frac{B_k}{(-n)^k}\equiv a(-1)^{n-1}D_{n-1}\pmod p.\tag{1.2}$$

Remark 1.1. Note that (1.2) in the case $a=1$ gives (1.1).

For $n\in\mathbb{Z}^+$ and $k\in\{0,\ldots,n\}$, the Stirling number $S(n,k)$ of the second kind denotes the number of ways to partition the set $\{1,\ldots,n\}$

into k disjoint nonempty parts. In addition, we adopt the usual convention $S(0,0)=1$. For $n \geqslant k \geqslant 0$, it is well known that

$$k!S(n,k)=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}j^n. \tag{1.3}$$

For any $n \in \mathbb{N}$, the Bell polynomial (or the Touchard polynomial) of degree n is given by

$$B_n(x)=\sum_{k=0}^{n}S(n,k)x^k. \tag{1.4}$$

Clearly, $B_n(1)=B_n$ for all $n \in \mathbb{N}$, and $B_n(x)=x\sum_{k=1}^{n}S(n,k)x^{k-1}$ for all $n \in \mathbb{Z}^+$. Theorem 1.1 actually follows from our following theorem concerning the Bell polynomials.

Theorem 1.2. *Let a be any positive integer. For any $n \in \mathbb{Z}^+$ and prime $p \nmid n$, we have*

$$(-x)^n\sum_{k=1}^{p^a-1}\frac{B_k(x)}{(-n)^k} \equiv -\sum_{r=1}^{a}x^{p^r}\sum_{k=0}^{n-1}\frac{(n-1)!}{k!}(-x)^k \pmod{p\mathbb{Z}_p[x]}, \tag{1.5}$$

where $\mathbb{Z}_p$ denotes the ring of all p-adic integers.

Remark 1.2. The congruence (1.5) in the case $a=1$ was deduced by Sun and Zagier [7] via the usual explicit formula (1.3) for Stirling numbers of the second kind. Our Theorem 1.2 can be further extended in the spirit of [5, 3], we omit the details.

We will show Theorem 1.2 in the next section.

2. Proof of Theorem 1.2

Lemma 2.1. *Let p be a prime and let $a \in \mathbb{Z}^+$.*

(i) *For any $j,k \in \mathbb{N}$ with $j+k \leqslant p^a-1$, we have*

$$\binom{p^a-1-k}{j}\Big/\binom{-1-k}{j} \equiv 1 \pmod{p}.$$

In particular,

$$\binom{p^a-1}{j} \equiv (-1)^j \pmod{p} \quad \textit{for all } j=0,\dots,p^a-1.$$

(ii) *We have*

$$B_{p^a}(x) \equiv \sum_{r=0}^{a}x^{p^r} \pmod{p\mathbb{Z}_p[x]}.$$

Proof. (i) Since $j+k<p^a$ we have
$$\frac{\binom{p^a-1-k}{j}}{\binom{-1-k}{j}}=\frac{\prod_{0<i\leqslant j}\frac{p^a-i-k}{i}}{\prod_{0<i\leqslant j}\frac{-i-k}{i}}=\prod_{0<i\leqslant j}\left(1-\frac{p^a}{i+k}\right)\equiv 1 \pmod p.$$
When $k=0$, this yields
$$\binom{p^a-1}{j}\equiv\binom{-1}{j}=(-1)^j \pmod p.$$

(ii) By Gertsch and Robert's extension [2] of Touchard's congruence, for any $n\in\mathbb{N}$ we have
$$B_{p^a+n}(x)\equiv B_{n+1}(x)+B_n(x)\sum_{r=1}^{a}x^{p^r} \pmod{p\mathbb{Z}_p[x]}.$$
In particular,
$$B_{p^a}(x)\equiv B_1(x)+B_0(x)\sum_{r=1}^{a}x^{p^r}=x+\sum_{r=1}^{a}x^{p^r}=\sum_{r=0}^{a}x^{p^r} \pmod{p\mathbb{Z}_p[x]}.$$

In view of the above, we have completed the proof of Lemma 2.1. □

Proof of Theorem 1.2. It is known that
$$B_{m+1}(x)=x\sum_{k=0}^{m}\binom{m}{k}B_k(x)\quad\text{for all } m\in\mathbb{N}. \tag{2.1}$$
In light of this and Lemma 2.1, for any prime p we have
$$\begin{aligned}\sum_{k=1}^{p^a-1}(-1)^kB_k(x)\equiv\sum_{k=1}^{p^a-1}\binom{p^a-1}{k}B_k(x)&=\frac{B_{p^a}(x)}{x}-B_0(x)\\ &\equiv\sum_{r=1}^{a}x^{p^r-1} \pmod{p\mathbb{Z}_p[x]}.\end{aligned}$$
So the desired result holds when $n=1$.

Now we fix $n\in\mathbb{Z}^+$ and assume that (1.5) holds for every prime $p\nmid n$.

Let p be any prime not dividing $n+1$. If $p\mid n$, then $n!/k!\equiv 0 \pmod p$ for all $k=0,\ldots,n-1$, and hence
$$\begin{aligned}(-x)^{n+1}\sum_{k=1}^{p^a-1}\frac{B_k(x)}{(-n-1)^k}\equiv&(-x)^{n+1}\sum_{k=1}^{p^a-1}\frac{B_k(x)}{(-1)^k}\equiv(-x)^{n+1}\sum_{r=1}^{a}x^{p^r-1}\\ &\equiv-\sum_{r=1}^{a}x^{p^r}\sum_{k=0}^{n}\frac{n!}{k!}(-x)^k \pmod{p\mathbb{Z}_p[x]}.\end{aligned}$$

Now we suppose that $p \nmid n$. In view of (2.1) and Lemma 2.1, we have

$$\begin{aligned}\sum_{k=1}^{p^a-1}\frac{B_k(x)/x}{(-n)^k} &= \sum_{k=1}^{p^a-1}\frac{\sum_{l=0}^{k-1}\binom{k-1}{l}B_l(x)}{(-n)^k} = \sum_{l=0}^{p^a-2}\frac{B_l(x)}{(-n)^l}\sum_{k=l+1}^{p^a-1}\frac{\binom{k-1}{l}}{(-n)^{k-l}}\\ &= \sum_{l=0}^{p^a-2}\frac{B_l(x)}{(-n)^{l+1}}\sum_{r=1}^{p^a-1-l}\frac{\binom{l+r-1}{r-1}}{(-n)^{r-1}}\\ &= \sum_{l=0}^{p^a-2}\frac{B_l(x)}{(-n)^{l+1}}\sum_{r=1}^{p^a-1-l}\frac{\binom{-l-1}{r-1}}{n^{r-1}}\\ &\equiv \sum_{l=0}^{p^a-2}\frac{B_l(x)}{(-n)^{l+1}}\sum_{r=1}^{p^a-1-l}\binom{p^a-1-l}{r-1}n^{-(r-1)}\\ &\equiv \sum_{l=0}^{p^a-1}\frac{B_l(x)}{(-n)^{l+1}}\left(\left(1+\frac{1}{n}\right)^{p^a-1-l}-\frac{1}{n^{p^a-1-l}}\right)\\ &\equiv \sum_{l=0}^{p^a-1}\frac{n^lB_l(x)}{(-n)^{l+1}}\left(\frac{1}{(n+1)^l}-1\right) \pmod{p}\end{aligned}$$

with the aid of Fermat's little theorem. Therefore

$$-n\sum_{k=1}^{p^a-1}\frac{B_k(x)/x}{(-n)^k} \equiv \sum_{k=1}^{p^a-1}\frac{B_k(x)}{(-n-1)^k} - \sum_{l=1}^{p^a-1}\frac{B_l(x)}{(-1)^l} \pmod{p\mathbb{Z}_p[x]}$$

and hence

$$\sum_{k=1}^{p^a-1}\frac{B_k(x)}{(-n-1)^k} \equiv -n\sum_{k=1}^{p^a-1}\frac{B_k(x)/x}{(-n)^k} + \sum_{r=1}^{a}x^{p^r-1} \pmod{p\mathbb{Z}_p[x]}.$$

Combining this with (1.5), we obtain

$$\begin{aligned}(-x)^{n+1}\sum_{k=1}^{p^a-1}\frac{B_k(x)}{(-n-1)^k} &\equiv -\,n\sum_{r=1}^{a}x^{p^r}\sum_{k=0}^{n-1}\frac{(n-1)!}{k!}(-x)^k + (-x)^{n+1}\sum_{r=1}^{a}x^{p^r-1}\\ &= -\sum_{r=1}^{a}x^{p^r}\sum_{k=0}^{n}\frac{n!}{k!}(-x)^k \pmod{p\mathbb{Z}_p[x]}.\end{aligned}$$

This concludes the induction step.

By the above, the proof of Theorem 1.1 is now complete. □

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