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EACH POSITIVE RATIONAL NUMBER HAS THE FORM $\varphi(m^2)/\varphi(n^2)$

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Abstract. In this note we show that each positive rational number can be written as $\varphi(m^2)/\varphi(n^2)$, where φ is Euler's totient function and m and n are positive integers.

Let φ be Euler's totient function. For distinct primes $p_1,\ldots,p_k$ and positive integers $a_1,\ldots,a_k$, it is well known that

$$\varphi(p_1^{a_1}\cdots p_k^{a_k})=\prod_{i=1}^k p_i^{a_i-1}(p_i-1).$$

(See, e.g., [1, p. 20].) Thus, if n has the prime factorization $\prod_{i=1}^k p_i^{a_i}$ (where $p_1,\ldots,p_k$ are distinct primes and $a_1,\ldots,a_k$ are positive integers), then

$$\varphi(n^2)=\prod_{i=1}^k p_i^{2a_i-1}(p_i-1)=n\varphi(n).$$

For positive integers m and n with $\varphi(m^2)=\varphi(n^2)$, we have $m=n$ by comparing the prime factorizations of $\varphi(m^2)$ and $\varphi(n^2)$. The sequence $\varphi(n^2)$ $(n=1,2,3,\ldots)$ is available from [2].

Section 4 of [3] contains many challenging conjectures on representations of positive rational numbers. For example, Sun [3, Conjecture 4.4] conjectured that any positive rational number can be written as m/n, where m and n are positive integers such that the sum of the mth prime and the nth prime is a square. Motivated by this, in this note we establish the following new result.

Theorem 1. *Any positive rational number can be written as $\varphi(m^2)/\varphi(n^2)$, where m and n are positive integers.*

Proof. We claim a stronger result: If $p_1<p_2<\cdots<p_k$ are distinct primes and $a_1,\ldots,a_k$ are integers, then there are positive integers m and n with mn not divisible by any prime greater than p_k such that $p_1^{a_1}\cdots p_k^{a_k}=\varphi(m^2)/\varphi(n^2)$.

We prove the claim by induction on p_k.

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The base of the induction is $p_k = 2$. For any $a \in \mathbb{Z}$, clearly

$$2^{2a} = \frac{2^{2(a+b)-1}}{2^{2b-1}} = \frac{\varphi(2^{2(a+b)})}{\varphi(2^{2b})}$$

for each integer $b > |a|$, and

$$2^{2a+1} = \begin{cases} \varphi(2^{2(a+1)})/\varphi(1^2) & \text{if } a \ge 0, \\ \varphi(1^2)/\varphi(2^{-2a}) & \text{if } a < 0. \end{cases}$$

Now let q be an odd prime and assume that the claim holds whenever $p_k < q$. Let $q_1 < \cdots < q_k = q$ be distinct primes and let $r = \prod_{i=1}^k q_i^{a_i}$ with $a_1, \ldots, a_k \in \mathbb{Z}$. Set $r_0 = r/q_k^{a_k}$ if $2 \mid a_k$, and $r_0 = r/((q_k - 1)q_k^{a_k})$ if $2 \nmid a_k$. Clearly, all the primes in the factorization of r_0 are smaller than $q_k = q$. By the induction hypothesis, there are positive integers m_0 and n_0 with $m_0 n_0$ not divisible by any prime $p \ge q_k$ such that

$$\frac{\varphi(m_0^2)}{\varphi(n_0^2)} = r_0.$$

Obviously, we may take $m_0 = n_0 = 1$ if $r_0 = 1$.

Case 1. $2 \mid a_k$.

In this case, we take positive integers b and c with $b - c = a_k/2$, and set $m = m_0 q^b$ and $n = n_0 q^c$. Then

$$\frac{\varphi(m^2)}{\varphi(n^2)} = \frac{\varphi(m_0^2)}{\varphi(n_0^2)} \times \frac{q^{2b-1}(q-1)}{q^{2c-1}(q-1)} = r_0 q^{2(b-c)} = \prod_{i=1}^k q_i^{a_i} = r.$$

Case 2. $2 \nmid a_k$.

When $a_k > 0$, for $m = m_0 q^{(a_k+1)/2}$ and $n = n_0$, we have

$$\frac{\varphi(m^2)}{\varphi(n^2)} = \frac{\varphi(m_0^2)}{\varphi(n_0^2)} \times q^{a_k}(q-1) = r_0 q^{a_k}(q-1) = \prod_{i=1}^k q_i^{a_i} = r.$$

If $a_k < 0$, then there are positive integers m and n with mn not divisible by any prime greater than q_k such that $\prod_{i=1}^k q_i^{-a_i} = \varphi(n^2)/\varphi(m^2)$ and hence $\prod_{i=1}^k q_i^{a_i} = \varphi(m^2)/\varphi(n^2)$.

In view of the above, the claim holds and hence so does the theorem. $\square$

Examples. We have

$$\frac{19}{47} = \frac{19 \times 19673280}{47 \times 19673280} = \frac{\varphi(39330^2)}{\varphi(55836^2)}$$

with

$$39330 = 2 \times 3^2 \times 5 \times 19 \times 23 \quad \text{and} \quad 55836 = 2^2 \times 3^3 \times 11 \times 47.$$

Also,

$$\frac{47}{58} = \frac{47 \times 1700160}{58 \times 1700160} = \frac{\varphi(14476^2)}{\varphi(20010^2)}$$

with

$$14476 = 2^2 \times 7 \times 11 \times 47 \text{ and } 20010 = 2 \times 3 \times 5 \times 23 \times 29.$$

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