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NEW SERIES FOR POWERS OF π AND RELATED CONGRUENCES

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Abstract. Via symbolic computation we deduce 97 new type series for powers of π related to Ramanujan-type series. Here are three typical examples:

$$\sum_{k=0}^{\infty}\frac{P(k)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-640320)^{3k}}=\frac{18\times 557403^3\sqrt{10005}}{5\pi}$$

with

$$P(k)=637379600041024803108k^2+657229991696087780968k\\+19850391655004126179,$$

$$\sum_{k=1}^{\infty}\frac{(3k+1)16^k}{(2k+1)^2k^3\binom{2k}{k}^3}=\frac{\pi^2-8}{2},$$

and

$$\sum_{n=0}^{\infty}\frac{3n+1}{(-100)^n}\sum_{k=0}^{n}\binom{n}{k}^2T_k(1,25)T_{n-k}(1,25)=\frac{25}{8\pi},$$

where the generalized central trinomial coefficient $T_k(b,c)$ denotes the coefficient of x^k in the expansion of $(x^2+bx+c)^k$. We also formulate a general characterization of rational Ramanujan-type series for $1/\pi$ via congruences, and pose 117 new conjectural series for powers of π via looking for corresponding congruences. For example, we conjecture that

$$\sum_{k=0}^{\infty}\frac{39480k+7321}{(-29700)^k}T_k(14,1)T_k(11,-11)^2=\frac{6795\sqrt{5}}{\pi}.$$

Eighteen of the new series in this paper involve some imaginary quadratic fields with class number 8.

1. Introduction and our main results

The classical rational Ramanujan-type series for π^{-1} (cf. [1, 2, 8, 27] and a nice introduction by S. Cooper [10, Chapter 14]) have the form

$$\sum_{k=0}^{\infty}\frac{bk+c}{m^k}a(k)=\frac{\lambda\sqrt{d}}{\pi},\qquad (*)$$

where b,c,m are integers with $bm\neq 0$, d is a positive squarefree number, λ is a nonzero rational number, and $a(k)$ is one of the products

$$\binom{2k}{k}^3,\ \binom{2k}{k}^2\binom{3k}{k},\ \binom{2k}{k}^2\binom{4k}{2k},\ \binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}.$$

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In 1997 Van Hamme [47] conjectured that such a series $(*)$ has a p-adic analogue of the form
$$\sum_{k=0}^{p-1}\frac{bk+c}{m^k}a(k)\equiv cp\left(\frac{\varepsilon_d d}{p}\right)\pmod{p^3},$$
where p is any odd prime with $p\nmid dm$ and $\lambda\in\mathbb{Z}_p$, $\varepsilon_1\in\{\pm1\}$ and $\varepsilon_d=-1$ if $d>1$. (As usual, $\mathbb{Z}_p$ denotes the ring of all p-adic integers, and $(\frac{\cdot}{p})$ stands for the Legendre symbol.) W. Zudilin [53] followed Van Hamme's idea to provide more concrete examples. Sun [33] realized that many Ramanujan-type congruences are related to Bernoulli numbers or Euler numbers. In 2016 the author [44] thought that all classical Ramanujan-type congruences have their extensions like
$$\frac{\sum_{k=0}^{pn-1}(21k+8)\binom{2k}{k}^3-p\sum_{k=0}^{n-1}(21k+8)\binom{2k}{k}^3}{(pn)^3\binom{2n}{n}^3}\in\mathbb{Z}_p,$$
where p is an odd prime, and $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$. See Sun [45, Conjectures 21-24] for more such examples and further refinements involving Bernoulli or Euler numbers.

During the period 2002–2010, some new Ramanujan-type series of the form $(*)$ with $a(k)$ not a product of three nontrivial parts were found (cf. [3, 4, 9, 29]). For example, H. H. Chan, S. H. Chan and Z. Liu [3] proved that
$$\sum_{n=0}^{\infty}\frac{5n+1}{64^n}D_n=\frac{8}{\sqrt{3}\,\pi},$$
where D_n denotes the Domb number $\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$; Zudilin [53] conjectured its p-adic analogue:
$$\sum_{k=0}^{p-1}\frac{5k+1}{64^k}D_k\equiv p\left(\frac{p}{3}\right)\pmod{p^3}\quad\text{for any prime }p>3.$$
The author [45, Conjecture 77] conjectured further that
$$\frac{1}{(pn)^3}\left(\sum_{k=0}^{pn-1}\frac{5k+1}{64^k}D_k-\left(\frac{p}{3}\right)p\sum_{k=0}^{n-1}\frac{5k+1}{64^r}D_k\right)\in\mathbb{Z}_p$$
for each odd prime p and positive integer n.

Let $b,c\in\mathbb{Z}$. For each $n\in\mathbb{N}=\{0,1,2,\ldots\}$, we denote the coefficient of x^n in the expansion of $(x^2+bx+c)^n$ by $T_n(b,c)$, and call it a *generalized central trinomial coefficient*. In view of the multinomial theorm, we have
$$T_n(b,c)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}b^{n-2k}c^k.$$
Note also that
$$T_0(b,c)=1,\quad T_1(b,c)=b,$$
and
$$(n+1)T_{n+1}(b,c)=(2n+1)bT_n(b,c)-n(b^2-4c)T_{n-1}(b,c)$$
for all $n\in\mathbb{Z}^+$. Clearly, $T_n(2,1)=\binom{2n}{n}$ for all $n\in\mathbb{N}$. Those $T_n:=T_n(1,1)$ with $n\in\mathbb{N}$ are the usual central trinomial coefficients, and they play important roles in enumerative combinatorics. We view $T_n(b,c)$ as a natural generalization of central binomial and central trinomial coefficients.

For $n\in\mathbb{N}$ the Legendre polynomial of degree n is defined by

$$P_n(x):=\sum_{k=0}^n\binom nk\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k.$$

It is well-known that if $b,c\in\mathbb{Z}$ and $b^2-4c\neq 0$ then

$$T_n(b,c)=(\sqrt{b^2-4c})^nP_n\left(\frac{b}{\sqrt{b^2-4c}}\right)\quad\text{for all } n\in\mathbb{N}.$$

Via the Laplace-Heine asymptotic formula for Legendre polynomials, for any positive real numbers b and c we have

$$T_n(b,c)\sim\frac{(b+2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}\quad\text{as } n\to+\infty$$

(cf. [40]). For any real numbers b and $c<0$, S. Wagner [48] confirmed the author's conjecture that

$$\lim_{n\to\infty}\sqrt[n]{|T_n(b,c)|}=\sqrt{b^2-4c}.$$

In 2011, the author posed over 60 conjectural series for $1/\pi$ of the following new types with a,b,c,d,m integers and $mbcd(b^2-4c)$ nonzero (cf. Sun [34, 40]).

`Type I.` $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{2k}{k}^2T_k(b,c)$.
`Type II.` $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{2k}{k}\binom{3k}{k}T_k(b,c)$.
`Type III.` $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{4k}{2k}\binom{2k}{k}T_k(b,c)$.
`Type IV.` $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{2k}{k}^2T_{2k}(b,c)$.
`Type V.` $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{2k}{k}\binom{3k}{k}T_{3k}(b,c)$.
`Type VI.` $\sum_{k=0}^\infty\frac{a+dk}{m^k}T_k(b,c)^3$,
`Type VII.` $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{2k}{k}T_k(b,c)^2$,

In general, the corresponding p-adic congruences of these seven-type series involve linear combinations of two Legendre symbols. The author's conjectural series of types I-V and VII were studied in [6, 49, 54]. The author's three conjectural series of type VI and two series of type VII remain open. For example, the author conjectured that

$$\sum_{k=0}^\infty\frac{3990k+1147}{(-288)^{3k}}T_k(62,95^2)^3=\frac{432}{95\pi}(94\sqrt2+195\sqrt{14})$$

as well as its p-adic analogue

$$\sum_{k=0}^{p-1}\frac{3990k+1147}{(-288)^{3k}}T_k(62,95^2)^3\equiv\frac{p}{19}\left(4230\left(\frac{-2}{p}\right)+17563\left(\frac{-14}{p}\right)\right)\pmod{p^2},$$

where p is any prime greater than 3.

In 1905, J. W. L. Glaisher [15] proved that

$$\sum_{k=0}^\infty\frac{(4k-1)\binom{2k}{k}^4}{(2k-1)^4256^k}=-\frac{8}{\pi^2}.$$

This actually follows from the following finite identity observed by the author [38]:

$$\sum_{k=0}^n\frac{(4k-1)\binom{2k}{k}^4}{(2k-1)^4256^k}=-(8n^2+4n+1)\frac{\binom{2n}{n}^4}{256^n}\quad\text{for all } n\in\mathbb{N}.$$

Motivated by Glaisher's identity and Ramanujan-type series for $1/\pi$, we obtain the following theorem.

Theorem 1.1. *We have the following identities:*

$$\sum_{k=0}^{\infty}\frac{k(4k-1)\binom{2k}{k}^3}{(2k-1)^2(-64)^k}=-\frac{1}{\pi},\tag{1.1}$$

$$\sum_{k=0}^{\infty}\frac{(4k-1)\binom{2k}{k}^3}{(2k-1)^3(-64)^k}=\frac{2}{\pi},\tag{1.2}$$

$$\sum_{k=0}^{\infty}\frac{(12k^2-1)\binom{2k}{k}^3}{(2k-1)^2256^k}=-\frac{2}{\pi},\tag{1.3}$$

$$\sum_{k=0}^{\infty}\frac{k(6k-1)\binom{2k}{k}^3}{(2k-1)^3256^k}=\frac{1}{2\pi},\tag{1.4}$$

$$\sum_{k=0}^{\infty}\frac{(28k^2-4k-1)\binom{2k}{k}^3}{(2k-1)^2(-512)^k}=-\frac{3\sqrt{2}}{\pi},\tag{1.5}$$

$$\sum_{k=0}^{\infty}\frac{(30k^2+3k-2)\binom{2k}{k}^3}{(2k-1)^3(-512)^k}=\frac{27\sqrt{2}}{8\pi},\tag{1.6}$$

$$\sum_{k=0}^{\infty}\frac{(28k^2-4k-1)\binom{2k}{k}^3}{(2k-1)^24096^k}=-\frac{3}{\pi},\tag{1.7}$$

$$\sum_{k=0}^{\infty}\frac{(42k^2-3k-1)\binom{2k}{k}^3}{(2k-1)^34096^k}=\frac{27}{8\pi},\tag{1.8}$$

$$\sum_{k=0}^{\infty}\frac{(34k^2-3k-1)\binom{2k}{k}^2\binom{3k}{k}}{(2k-1)(3k-1)(-192)^k}=-\frac{10}{\sqrt{3}\,\pi},\tag{1.9}$$

$$\sum_{k=0}^{\infty}\frac{(64k^2-11k-7)\binom{2k}{k}^2\binom{3k}{k}}{(k+1)(2k-1)(3k-1)(-192)^k}=-\frac{125\sqrt{3}}{9\pi},\tag{1.10}$$

$$\sum_{k=0}^{\infty}\frac{(14k^2+k-1)\binom{2k}{k}^2\binom{3k}{k}}{(2k-1)(3k-1)216^k}=-\frac{\sqrt{3}}{\pi},\tag{1.11}$$

$$\sum_{k=0}^{\infty}\frac{(90k^2+7k+1)\binom{2k}{k}^2\binom{3k}{k}}{(k+1)(2k-1)(3k-1)216^k}=\frac{9\sqrt{3}}{2\pi},\tag{1.12}$$

$$\sum_{k=0}^{\infty}\frac{(34k^2-3k-1)\binom{2k}{k}^2\binom{3k}{k}}{(2k-1)(3k-1)(-12)^{3k}}=-\frac{2\sqrt{3}}{\pi},\tag{1.13}$$

$$\sum_{k=0}^{\infty}\frac{(17k+5)\binom{2k}{k}^2\binom{3k}{k}}{(k+1)(2k-1)(3k-1)(-12)^{3k}}=\frac{9\sqrt{3}}{\pi},\tag{1.14}$$

$$\sum_{k=0}^{\infty}\frac{(111k^2-7k-4)\binom{2k}{k}^2\binom{3k}{k}}{(2k-1)(3k-1)1458^k}=-\frac{45}{4\pi},\tag{1.15}$$

$$\sum_{k=0}^{\infty}\frac{(1524k^2+899k+263)\binom{2k}{k}^2\binom{3k}{k}}{(k+1)(2k-1)(3k-1)1458^k}=\frac{3375}{4\pi}, \tag{1.16}$$

$$\sum_{k=0}^{\infty}\frac{(522k^2-55k-13)\binom{2k}{k}^2\binom{3k}{k}}{(2k-1)(3k-1)(-8640)^k}=-\frac{54\sqrt{15}}{5\pi}, \tag{1.17}$$

$$\sum_{k=0}^{\infty}\frac{(1836k^2+2725k+541)\binom{2k}{k}^2\binom{3k}{k}}{(k+1)(2k-1)(3k-1)(-8640)^k}=\frac{2187\sqrt{15}}{5\pi}, \tag{1.18}$$

$$\sum_{k=0}^{\infty}\frac{(529k^2-45k-16)\binom{2k}{k}^2\binom{3k}{k}}{(2k-1)(3k-1)15^{3k}}=-\frac{55\sqrt{3}}{2\pi}, \tag{1.19}$$

$$\sum_{k=0}^{\infty}\frac{(77571k^2+68545k+16366)\binom{2k}{k}^2\binom{3k}{k}}{(k+1)(2k-1)(3k-1)15^{3k}}=\frac{59895\sqrt{3}}{2\pi}, \tag{1.20}$$

$$\sum_{k=0}^{\infty}\frac{(574k^2-73k-11)\binom{2k}{k}^2\binom{3k}{k}}{(2k-1)(3k-1)(-48)^{3k}}=-20\frac{\sqrt{3}}{\pi}, \tag{1.21}$$

$$\sum_{k=0}^{\infty}\frac{(8118k^2+9443k+1241)\binom{2k}{k}^2\binom{3k}{k}}{(k+1)(2k-1)(3k-1)(-48)^{3k}}=\frac{2250\sqrt{3}}{\pi}, \tag{1.22}$$

$$\sum_{k=0}^{\infty}\frac{(978k^2-131k-17)\binom{2k}{k}^2\binom{3k}{k}}{(2k-1)(3k-1)(-326592)^k}=-\frac{990\sqrt{7}}{49\pi}, \tag{1.23}$$

$$\sum_{k=0}^{\infty}\frac{(592212k^2+671387k^2+77219)\binom{2k}{k}^2\binom{3k}{k}}{(k+1)(2k-1)(3k-1)(-326592)^k}=\frac{4492125\sqrt{7}}{49\pi}, \tag{1.24}$$

$$\sum_{k=0}^{\infty}\frac{(116234k^2-17695k-1461)\binom{2k}{k}^2\binom{3k}{k}}{(2k-1)(3k-1)(-300)^{3k}}=-2650\frac{\sqrt{3}}{\pi}, \tag{1.25}$$

$$\sum_{k=0}^{\infty}\frac{(223664832k^2+242140765k+18468097)\binom{2k}{k}^2\binom{3k}{k}}{(k+1)(2k-1)(3k-1)(-300)^{3k}}=33497325\frac{\sqrt{3}}{\pi}, \tag{1.26}$$

$$\sum_{k=0}^{\infty}\frac{(122k^2+3k-5)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)648^k}=-\frac{21}{2\pi}, \tag{1.27}$$

$$\sum_{k=0}^{\infty}\frac{(1903k^2+114k+41)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)648^k}=\frac{343}{2\pi}, \tag{1.28}$$

$$\sum_{k=0}^{\infty}\frac{(40k^2-2k-1)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)(-1024)^k}=-\frac{4}{\pi}, \tag{1.29}$$

$$\sum_{k=0}^{\infty}\frac{(8k^2-2k-1)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)(-1024)^k}=-\frac{16}{5\pi}, \tag{1.30}$$

$$\sum_{k=0}^{\infty}\frac{(176k^2-6k-5)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)48^{2k}}=-8\frac{\sqrt{3}}{\pi}, \tag{1.31}$$

$$\sum_{k=0}^{\infty}\frac{(208k^2+66k+23)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)48^{2k}}=\frac{128}{\sqrt{3}\,\pi},\tag{1.32}$$

$$\sum_{k=0}^{\infty}\frac{(6722k^2-411k-152)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)(-63^2)^k}=-195\frac{\sqrt{7}}{\pi},\tag{1.33}$$

$$\sum_{k=0}^{\infty}\frac{(281591k^2-757041k-231992)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)(-63^2)^k}=-274625\frac{\sqrt{7}}{\pi},\tag{1.34}$$

$$\sum_{k=0}^{\infty}\frac{(560k^2-42k-11)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)12^{4k}}=-24\frac{\sqrt{2}}{\pi},\tag{1.35}$$

$$\sum_{k=0}^{\infty}\frac{(112k^2+114k+23)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)12^{4k}}=\frac{256\sqrt{2}}{5\pi},\tag{1.36}$$

$$\sum_{k=0}^{\infty}\frac{(248k^2-18k-5)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)(-3\times 2^{12})^k}=-\frac{28}{\sqrt{3}\,\pi},\tag{1.37}$$

$$\sum_{k=0}^{\infty}\frac{(680k^2+1482k+337)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)(-3\times 2^{12})^k}=\frac{5488\sqrt{3}}{9\pi},\tag{1.38}$$

$$\sum_{k=0}^{\infty}\frac{(1144k^2-102k-19)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)(-2^{10}3^4)^k}=-\frac{60}{\pi},\tag{1.39}$$

$$\sum_{k=0}^{\infty}\frac{(3224k^2+4026k+637)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)(-2^{10}3^4)^k}=\frac{2000}{\pi},\tag{1.40}$$

$$\sum_{k=0}^{\infty}\frac{(7408k^2-754k-103)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)28^{4k}}=-\frac{560\sqrt{3}}{3\pi},\tag{1.41}$$

$$\sum_{k=0}^{\infty}\frac{(3641424k^2+4114526k+493937)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)28^{4k}}=896000\frac{\sqrt{3}}{\pi},\tag{1.42}$$

$$\sum_{k=0}^{\infty}\frac{(4744k^2-534k-55)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)(-2^{14}3^45)^k}=-\frac{1932\sqrt{5}}{25\pi},\tag{1.43}$$

$$\sum_{k=0}^{\infty}\frac{(18446264k^2+20356230k+1901071)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)(-2^{14}3^45)^k}=66772496\frac{\sqrt{5}}{25\pi},\tag{1.44}$$

$$\sum_{k=0}^{\infty}\frac{(413512k^2-50826k-3877)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)(-2^{10}21^4)^k}=-\frac{12180}{\pi},\tag{1.45}$$

$$\sum_{k=0}^{\infty}\frac{(1424799848k^2+1533506502k+108685699)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)(-2^{10}21^4)^k}=\frac{341446000}{\pi},\tag{1.46}$$

$$\sum_{k=0}^{\infty}\frac{(71312k^2-7746k-887)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)1584^{2k}}=-840\frac{\sqrt{11}}{\pi},\tag{1.47}$$

$$\sum_{k=0}^{\infty}\frac{(50678512k^2+56405238k+5793581)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)1584^{2k}}=5488000\frac{\sqrt{11}}{\pi},\tag{1.48}$$

$$\sum_{k=0}^{\infty}\frac{(7329808k^2-969294k-54073)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)396^{4k}}=-120120\frac{\sqrt{2}}{\pi},\tag{1.49}$$

$$\sum_{k=0}^{\infty}\frac{(2140459883152k^2+2259867244398k+119407598201)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)396^{4k}}=44\times 1820^3\frac{\sqrt{2}}{\pi},\tag{1.50}$$

$$\sum_{k=0}^{\infty}\frac{(164k^2-k-3)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)20^{3k}}=-\frac{7\sqrt{5}}{2\pi},\tag{1.51}$$

$$\sum_{k=0}^{\infty}\frac{(2696k^2+206k+93)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)20^{3k}}=\frac{686}{\sqrt{5}\,\pi},\tag{1.52}$$

$$\sum_{k=0}^{\infty}\frac{(220k^2-8k-3)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-2^{15})^k}=-\frac{7\sqrt{2}}{\pi},\tag{1.53}$$

$$\sum_{k=0}^{\infty}\frac{(836k^2-1048k-309)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-2^{15})^k}=-\frac{686\sqrt{2}}{\pi},\tag{1.54}$$

$$\sum_{k=0}^{\infty}\frac{(504k^2-11k-8)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-15)^{3k}}=-\frac{9\sqrt{15}}{\pi},\tag{1.55}$$

$$\sum_{k=0}^{\infty}\frac{(189k^2-11k-8)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-15)^{3k}}=-\frac{243\sqrt{15}}{35\pi},\tag{1.56}$$

$$\sum_{k=0}^{\infty}\frac{(516k^2-19k-7)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(2\times 30^3)^k}=-\frac{11\sqrt{15}}{2\pi},\tag{1.57}$$

$$\sum_{k=0}^{\infty}\frac{(3237k^2+1922k+491)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(2\times 30^3)^k}=\frac{3993\sqrt{15}}{10\pi},\tag{1.58}$$

$$\sum_{k=0}^{\infty}\frac{(684k^2-40k-7)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-96)^{3k}}=-\frac{9\sqrt{6}}{\pi},\tag{1.59}$$

$$\sum_{k=0}^{\infty}\frac{(2052k^2+2536k+379)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-96)^{3k}}=\frac{486\sqrt{6}}{\pi},\tag{1.60}$$

$$\sum_{k=0}^{\infty}\frac{(2556k^2-131k-29)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)66^{3k}}=-\frac{63\sqrt{33}}{4\pi},\tag{1.61}$$

$$\sum_{k=0}^{\infty}\frac{(203985k^2+212248k+38083)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)66^{3k}}=\frac{83349\sqrt{33}}{4\pi},\tag{1.62}$$

$$\sum_{k=0}^{\infty}\frac{(5812k^2-408k-49)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-3\times 160^3)^k}=-\frac{253\sqrt{30}}{9\pi},\tag{1.63}$$

$$\sum_{k=0}^{\infty}\frac{(3471628k^2+3900088k+418289)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-3\times160^3)^k}=\frac{32388554\sqrt{30}}{135\pi}, \tag{1.64}$$

$$\sum_{k=0}^{\infty}\frac{(35604k^2-2936k-233)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-960)^{3k}}=-189\frac{\sqrt{15}}{\pi}, \tag{1.65}$$

$$\sum_{k=0}^{\infty}\frac{(13983084k^2+15093304k+1109737)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-960)^{3k}}=\frac{4500846\sqrt{15}}{5\pi}, \tag{1.66}$$

$$\sum_{k=0}^{\infty}\frac{(157752k^2-11243k-1304)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)255^{3k}}=-\frac{513\sqrt{255}}{2\pi}, \tag{1.67}$$

$$\sum_{k=0}^{\infty}\frac{(28240947k^2+31448587k+3267736)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)255^{3k}}=\frac{45001899\sqrt{255}}{70\pi}, \tag{1.68}$$

$$\sum_{k=0}^{\infty}\frac{(2187684k^2-200056k-11293)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-5280)^{3k}}=-1953\frac{\sqrt{330}}{\pi}, \tag{1.69}$$

$$\begin{aligned}\sum_{k=0}^{\infty}&\frac{(101740699836k^2+107483900696k+5743181813)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-5280)^{3k}}\\&=\frac{4966100118\sqrt{330}}{5\pi},\end{aligned} \tag{1.70}$$

$$\begin{aligned}\sum_{k=0}^{\infty}&\frac{(16444841148k^2-1709536232k-53241371)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-640320)^{3k}}\\&=-1672209\frac{\sqrt{10005}}{\pi},\end{aligned} \tag{1.71}$$

and

$$\sum_{k=0}^{\infty}\frac{P(k)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-640320)^{3k}}=\frac{18\times557403^3\sqrt{10005}}{5\pi}, \tag{1.72}$$

where

$$\begin{aligned}P(k):=&637379600041024803108k^2+657229991696087780968k\\&+19850391655004126179.\end{aligned}$$

Recall that the Catalan numbers are given by

$$C_n:=\frac{\binom{2n}{n}}{n+1}=\binom{2n}{n}-\binom{2n}{n+1}\ \ (n\in\mathbb{N}).$$

For $k\in\mathbb{N}$ it is easy to see that

$$\frac{\binom{2k}{k}}{2k-1}=\begin{cases}-1&\text{if } k=0,\\ 2C_{k-1}&\text{if } k>0.\end{cases}$$

Thus, for any $a,b,c,m\in\mathbb{Z}$ with $|m|\ge 64$, we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(ak^2+bk+c)\binom{2k}{k}^3}{(2k-1)^3m^k}&=-c+\sum_{k=1}^{\infty}\frac{(ak^2+bk+c)(2C_{k-1})^3}{m^k}\\&=-c+\frac{8}{m}\sum_{k=0}^{\infty}\frac{a(k+1)^2+b(k+1)+c}{m^k}C_k^3.\end{aligned}$$

For example, (1.2) has the equivalent form

$$\sum_{k=0}^{\infty}\frac{4k+3}{(-64)^k}C_k^3=8-\frac{16}{\pi}. \tag{1.2$'$}$$

For any odd prime p, the congruence (1.4) of V.J.W. Guo and J.-C. Liu [19] has the equivalent form

$$\sum_{k=0}^{(p+1)/2}\frac{(4k-1)\binom{2k}{k}^3}{(2k-1)^3(-64)^k}\equiv p\left(\frac{-1}{p}\right)+p^3(E_{p-3}-2)\pmod{p^4}$$

(where $E_0,E_1,\ldots$ are the Euler numbers), and we note that this is also equivalent to the congruence

$$\sum_{k=0}^{(p-1)/2}\frac{4k+3}{(-64)^k}C_k^3\equiv 8\left(1-p\left(\frac{-1}{p}\right)-p^3(E_{p-3}-2)\right)\pmod{p^4}.$$

Recently, C. Wang [50] proved that for any prime $p>3$ we have

$$\sum_{k=0}^{(p+1)/2}\frac{(3k-1)\binom{2k}{k}^3}{(2k-1)^2 16^k}\equiv p+2p^3\left(\frac{-1}{p}\right)(E_{p-3}-3)\pmod{p^4}$$

and

$$\sum_{k=0}^{p-1}\frac{(3k-1)\binom{2k}{k}^3}{(2k-1)^2 16^k}\equiv p-2p^3\pmod{p^4}.$$

(Actually, Wang stated his results only in the language of hypergeometric series.) These two congruences extend a conjecture of Guo and M. J. Schlosser [21].

We are also able to prove some other variants of Ramanujan-type series such as

$$\sum_{k=0}^{\infty}\frac{(56k^2+118k+61)\binom{2k}{k}^3}{(k+1)^2 4096^k}=\frac{192}{\pi}$$

and

$$\sum_{k=0}^{\infty}\frac{(420k^2+992k+551)\binom{2k}{k}^3}{(k+1)^2(2k-1)4096^k}=-\frac{1728}{\pi}.$$

Now we state our second theorem.

Theorem 1.2. *We have the identities*

$$\sum_{k=1}^{\infty}\frac{28k^2+31k+8}{(2k+1)^2k^3\binom{2k}{k}^3}=\frac{\pi^2-8}{2}, \tag{1.73}$$

$$\sum_{k=1}^{\infty}\frac{42k^2+39k+8}{(2k+1)^3k^3\binom{2k}{k}^3}=\frac{9\pi^2-88}{2}, \tag{1.74}$$

$$\sum_{k=1}^{\infty}\frac{(8k^2+5k+1)(-8)^k}{(2k+1)^2k^3\binom{2k}{k}^3}=4-6G, \tag{1.75}$$

$$\sum_{k=1}^{\infty}\frac{(30k^2+33k+7)(-8)^k}{(2k+1)^3k^3\binom{2k}{k}^3}=54G-52, \tag{1.76}$$

$$\sum_{k=1}^{\infty}\frac{(3k+1)16^k}{(2k+1)^2k^3\binom{2k}{k}^3}=\frac{\pi^2-8}{2}, \tag{1.77}$$

$$\sum_{k=1}^\infty\frac{(4k+1)(-64)^k}{(2k+1)^2k^2\binom{2k}{k}^3}=4-8G,\tag{1.78}$$

$$\sum_{k=1}^\infty\frac{(4k+1)(-64)^k}{(2k+1)^3k^3\binom{2k}{k}^3}=16G-16,\tag{1.79}$$

$$\sum_{k=1}^\infty\frac{(2k^2-11k-3)8^k}{(2k+1)(3k+1)k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{48-5\pi^2}{2},\tag{1.80}$$

$$\sum_{k=2}^\infty\frac{(178k^2-103k-39)8^k}{(k-1)(2k+1)(3k+1)k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{1125\pi^2-11096}{36},\tag{1.81}$$

$$\sum_{k=1}^\infty\frac{(5k+1)(-27)^k}{(2k+1)(3k+1)k^2\binom{2k}{k}^2\binom{3k}{k}}=6-9K,\tag{1.82}$$

$$\sum_{k=2}^\infty\frac{(45k^2+5k-2)(-27)^{k-1}}{(k-1)(2k+1)(3k+1)k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{37-48K}{16},\tag{1.83}$$

$$\sum_{k=1}^\infty\frac{(98k^2-21k-8)81^k}{(2k+1)(4k+1)k^3\binom{2k}{k}^2\binom{4k}{2k}}=216-20\pi^2,\tag{1.84}$$

$$\sum_{k=2}^\infty\frac{(1967k^2-183k-104)81^k}{(k-1)(2k+1)(4k+1)k^3\binom{2k}{k}^2\binom{4k}{2k}}=\frac{20000\pi^2-190269}{120},\tag{1.85}$$

$$\sum_{k=1}^\infty\frac{(46k^2+3k-1)(-144)^k}{(2k+1)(4k+1)k^3\binom{2k}{k}^2\binom{4k}{2k}}=72-\frac{225}{2}K,\tag{1.86}$$

$$\sum_{k=2}^\infty\frac{(343k^2+18k-16)(-144)^k}{(k-1)(2k+1)(4k+1)k^3\binom{2k}{k}^2\binom{4k}{2k}}=\frac{9375K-7048}{10},\tag{1.87}$$

where

$$G:=\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^2}\quad\text{and}\quad K:=\sum_{k=0}^\infty\frac{(\frac{k}{3})}{k^2}.$$

For $k=j+1\in\mathbb{Z}^+$, it is easy to see that

$$(k-1)k\binom{2k}{k}=2(2j+1)j\binom{2j}{j}.$$

Thus, for any $a,b,c,m\in\mathbb{Z}$ with $0<|m|\le 64$, we have

$$\sum_{j=1}^\infty\frac{(aj^2+bj+c)m^j}{(2j+1)^3j^3\binom{2j}{j}^3}=\frac{8}{m}\sum_{k=2}^\infty\frac{(a(k-1)^2+b(k-1)+c)m^k}{(k-1)^3k^3\binom{2k}{k}^3}.$$

For example, (1.77) has the following equivalent form

$$\sum_{k=2}^\infty\frac{(2k-1)(3k-2)16^k}{(k-1)^3k^3\binom{2k}{k}^3}=\pi^2-8.\tag{1.77'}$$

In contrast with the Domb numbers, we introduce a new kind of numbers

$$S_n:=\sum_{k=0}^n\binom{n}{k}^2T_kT_{n-k}\quad(n=0,1,2,\ldots).$$

The values of S_n $(n = 0, \ldots, 10)$ are

$$1,\ 2,\ 10,\ 68,\ 586,\ 5252,\ 49204,\ 475400,\ 4723786,\ 47937812,\ 494786260$$

respectively. We may extend the numbers S_n $(n \in \mathbb{N})$ further. For $b, c \in \mathbb{Z}$, we define

$$S_n(b,c) := \sum_{k=0}^{n} \binom{n}{k}^2 T_k(b,c)T_{n-k}(b,c) \quad (n = 0, 1, 2, \ldots).$$

Note that $S_n(1,1) = S_n$ and $S_n(2,1) = D_n$ for all $n \in \mathbb{N}$.

Now we state our third theorem.

Theorem 1.3. *We have*

$$\sum_{k=0}^{\infty} \frac{7k+3}{24^k} S_k(1,-6) = \frac{15}{\sqrt{2}\,\pi}, \tag{1.88}$$

$$\sum_{k=0}^{\infty} \frac{12k+5}{(-28)^k} S_k(1,7) = \frac{6\sqrt{7}}{\pi}, \tag{1.89}$$

$$\sum_{k=0}^{\infty} \frac{84k+29}{80^k} S_k(1,-20) = \frac{24\sqrt{15}}{\pi}, \tag{1.90}$$

$$\sum_{k=0}^{\infty} \frac{3k+1}{(-100)^k} S_k(1,25) = \frac{25}{8\pi}, \tag{1.91}$$

$$\sum_{k=0}^{\infty} \frac{228k+67}{224^k} S_k(1,-56) = \frac{80\sqrt{7}}{\pi}, \tag{1.92}$$

$$\sum_{k=0}^{\infty} \frac{399k+101}{(-676)^k} S_k(1,169) = \frac{2535}{8\pi}, \tag{1.93}$$

$$\sum_{k=0}^{\infty} \frac{2604k+563}{2600^k} S_k(1,-650) = \frac{850\sqrt{39}}{3\pi}, \tag{1.94}$$

$$\sum_{k=0}^{\infty} \frac{39468k+7817}{(-6076)^k} S_k(1,1519) = \frac{4410\sqrt{31}}{\pi}, \tag{1.95}$$

$$\sum_{k=0}^{\infty} \frac{41667k+7879}{9800^k} S_k(1,-2450) = \frac{40425\sqrt{6}}{4\pi}, \tag{1.96}$$

$$\sum_{k=0}^{\infty} \frac{74613k+10711}{(-530^2)^k} S_k(1,265^2) = \frac{1615175}{48\pi}. \tag{1.97}$$

Remark 1.1. The author found the 10 series in Theorem 1.3 in Nov. 2019.

We shall prove Theorems 1.1-1.3 in the next section. In Sections 3-10, we propose 117 new conjectural series for powers of π involving generalized central trinomial coefficients. In particular, we will present in Section 3 four conjectural series for $1/\pi$ of the following new type:

`Type VIII.` $\sum_{k=0}^{\infty} \frac{a+dk}{m^k} T_k(b,c)T_k(b_*,c_*)^2$,

where a, b, b_*, c, c_*, d, m are integers with $mbb_*cc_*d(b^2-4c)(b_*^2-4c_*)(b^2c_*-b_*^2c) \neq 0$.

Unlike Ramanujan-type series given by others, all our series for $1/\pi$ of types I-VIII have the general term involving a *product* of three generalized central trinomial coefficients.

Motivated by the author's effective way to find new series for $1/\pi$ (cf. Sun [35]) and congruences and series in [45, Section 3], we formulate the following general characterization of rational Ramanujan-type series for $1/\pi$ via congruences.

Conjecture 1.1 (Criterion for Rational Ramanujan-type Series for $1/\pi$). *Suppose that the series $\sum_{k=0}^\infty \frac{bk+c}{m^k}a_k$ converges, where $(a_k)_{k\ge0}$ is an integer sequence and b,c,m are integers with $bcm\ne0$. Suppose also that there are no $a,x\in\mathbb{Z}$ with $a_n=a\binom{2n}{n}\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}x^{n-k}$ for all $n\in\mathbb{N}$. Let $r\in\{1,2,3\}$ and let $d_1,\ldots,d_r\in\mathbb{Z}^+$ with $\sqrt{d_i/d_j}$ irrational for all distinct $i,j\in\{1,\ldots,r\}$. Then*

$$\sum_{k=0}^\infty\frac{bk+c}{m^k}a_k=\frac{\sum_{i=1}^r\lambda_i\sqrt{d_i}}{\pi} \tag{1.98}$$

for some nonzero rational numbers $\lambda_1,\ldots,\lambda_r$ if and only if there are positive integers d_j $(r<j\le3)$ and rational numbers c_1,c_2,c_3 with $\prod_{i=1}^r c_i\ne0$, such that for any prime $p>3$ with $p\nmid m\prod_{i=1}^r d_i$ and $c_1,c_2,c_3\in\mathbb{Z}_p$ we have

$$\sum_{k=0}^{p-1}\frac{bk+c}{m^k}a_k\equiv p\bigg(\sum_{i=1}^r c_i\left(\frac{\varepsilon_i d_i}{p}\right)+\sum_{r<j\le3}c_j\left(\frac{d_j}{p}\right)\bigg)\pmod{p^2}, \tag{1.99}$$

where $\varepsilon_i\in\{\pm1\}$, $\varepsilon_i=-1$ if d_i is not an integer square, and $c_2=c_3=0$ if $r=1$ and $\varepsilon_1=1$.

For a Ramanujan-type series of the form (1.98), we call r its *rank*. We believe that there are some Ramanujan-type series of rank three but we have not yet found such a series.

Conjecture 1.2. *Let $(a_n)_{n\ge0}$ be an integer sequence with no $a,x\in\mathbb{Z}$ such that $a_n=a\binom{2n}{n}\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}x^{n-k}$ for all $n\in\mathbb{N}$, and let $b,c,m,d_1,d_2,d_3\in\mathbb{Z}$ with $bcm\ne0$. Assume that $\lim_{n\to+\infty}\sqrt[n]{|a_n|}=r<|m|$, and $\pi\sum_{k=0}^\infty\frac{bk+c}{m^k}a_k$ is an algebraic number. Suppose that $c_1,c_2,c_3\in\mathbb{Q}$ with $c_1+c_2+c_3=a_0c$, and*

$$\sum_{k=0}^{p-1}\frac{bk+c}{m^k}a_k\equiv p\left(c_1\left(\frac{d_1}{p}\right)+c_2\left(\frac{d_2}{p}\right)+c_3\left(\frac{d_3}{p}\right)\right)\pmod{p^2} \tag{1.100}$$

for all primes $p>3$ with $p\nmid d_1d_2d_3m$ and $c_1,c_2,c_3\in\mathbb{Z}_p$. Then, for any prime $p>3$ with $p\nmid m$, $c_1,c_2,c_3\in\mathbb{Z}_p$ and $(\frac{d_1}{p})=(\frac{d_2}{p})=(\frac{d_3}{p})=\delta\in\{\pm1\}$, we have

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}\frac{bk+c}{m^k}a_k-p\delta\sum_{k=0}^{n-1}\frac{bk+c}{m^k}a_k\bigg)\in\mathbb{Z}_p\quad\text{for all } n\in\mathbb{Z}^+.$$

Joint with the author's PhD student Chen Wang, we pose the following conjecture.

Conjecture 1.3 (Chen Wang and Z.-W. Sun). *Let $(a_k)_{k\ge0}$ be an integer sequence with $a_0=1$. Let $b,c,m,d_1,d_2,d_3\in\mathbb{Z}$ with $bm\ne0$, and let c_1,c_2,c_3 be rational numbers. If $\pi\sum_{k=0}^\infty\frac{bk+c}{m^k}a_k$ is an algebraic number, and the congruence (1.100) holds for all primes $p>3$ with $p\nmid d_1d_2d_3m$ and $c_1,c_2,c_3\in\mathbb{Z}_p$, then we must have $c_1+c_2+c_3=c$.*

Remark 1.2. The author [39, Conjecture 1.1(i)] conjectured that

$$\sum_{k=0}^{p-1}(8k+5)T_k^2\equiv3p\left(\frac{-3}{p}\right)\pmod{p^2}$$

for any prime $p>3$, which was confirmed by Y.-P. Mu and Z.-W. Sun [26]. This is not a counterexample to Conjecture 1.3 since $\sum_{k=0}^\infty(8k+5)T_k^2$ diverges.

All the new series and related congruences in Sections 3-9 support Conjectures 1.1-1.3. We discover the conjectural series for $1/\pi$ in Sections 3-9 based on the author's previous `Philosophy about Series for` $1/\pi$ stated in [35], the PSLQ algorithm to discover integer relations (cf. [13]), and the following `Duality Principle` based on the author's experience and intuition.

Conjecture 1.4 (Duality Principle). *Let $(a_k)_{k\ge0}$ be an integer sequence such that*

$$a_k\equiv\left(\frac dp\right)D^ka_{p-1-k}\ (\mathrm{mod}\ p)\tag{1.101}$$

for any prime $p\nmid 6dD$ and $k\in\{0,\ldots,p-1\}$, where d and D are fixed nonzero integers. If $a_0,a_1,\ldots$ are not all zero and m is a nonzero integer such that

$$\sum_{k=0}^\infty\frac{bk+c}{m^k}a_k=\frac{\lambda_1\sqrt{d_1}+\lambda_2\sqrt{d_2}+\lambda_3\sqrt{d_3}}{\pi}$$

for some $b,d_1,d_2,d_3\in\mathbb{Z}^+$, $c\in\mathbb{Z}$ and $\lambda_1,\lambda_2,\lambda_3\in\mathbb{Q}$, then m divides D, and

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k}\equiv\left(\frac dp\right)\sum_{k=0}^{p-1}\frac{a_k}{(D/m)^k}\ (\mathrm{mod}\ p^2)\tag{1.102}$$

for any prime $p>3$ with $p\nmid dD$.

Remark 1.3. (i) For any prime $p>3$ with $p\nmid dDm$, the congruence (1.102) holds modulo p by (1.101) and Fermat's little theorem. We call $\sum_{k=0}^{p-1}a_k/(\frac Dm)^k$ the *dual* of the sum $\sum_{k=0}^{p-1}a_k/m^k$.

(ii) For any $b,c\in\mathbb{Z}$ and odd prime $p\nmid b^2-4c$, it is known (see, e.g., [39, Lemma 2.2]) that

$$T_k(b,c)\equiv\left(\frac{b^2-4c}{p}\right)(b^2-4c)^kT_{p-1-k}(b,c)\ (\mathrm{mod}\ p)\tag{1.103}$$

for all $k=0,1,\ldots,p-1$.

For a series $\sum_{k=0}^\infty a_k$ with $a_0,a_1,\ldots$ real numbers, if $\lim_{k\to+\infty}a_{k+1}/a_k=r\in(-1,1)$ then we say that the series converge at a geometric rate with ratio r. Except for (7.1), all other conjectural series in Sections 3-9 converge at geometric rates and thus one can easily check them numerically via a computer.

In Section 10, we pose two curious conjectural series for π involving the central trinomial coefficients.

2. Proofs of Theorems 1.1-1.3

Lemma 2.1. *Let $m\neq0$ and $n\ge0$ be integers. Then*

$$\sum_{k=0}^n\frac{((64-m)k^3-32k^2-16k+8)\binom{2k}k^3}{(2k-1)^2m^k}=\frac{8(2n+1)}{m^n}\binom{2n}n^3,\tag{2.1}$$

$$\sum_{k=0}^n\frac{((64-m)k^3-96k^2+48k-8)\binom{2k}k^3}{(2k-1)^3m^k}=\frac{8}{m^n}\binom{2n}n^3,\tag{2.2}$$

$$\sum_{k=0}^n\frac{((108-m)k^3-54k^2-12k+6)\binom{2k}k^2\binom{3k}k}{(2k-1)(3k-1)m^k}=\frac{6(3n+1)}{m^n}\binom{2n}n^2\binom{3n}n,\tag{2.3}$$

$$\begin{aligned}\sum_{k=0}^{n}&\frac{((108-m)k^3-(54+m)k^2-12k+6)\binom{2k}{k}^2\binom{3k}{k}}{(k+1)(2k-1)(3k-1)m^k}\\&=\frac{6(3n+1)}{(n+1)m^n}\binom{2n}{n}^2\binom{3n}{n},\end{aligned}\tag{2.4}$$

$$\begin{aligned}\sum_{k=0}^{n}&\frac{((256-m)k^3-128k^2-16k+8)\binom{2k}{k}^2\binom{4k}{2k}}{(2k-1)(4k-1)m^k}\\&=\frac{8(4n+1)}{m^n}\binom{2n}{n}^2\binom{4n}{2n},\end{aligned}\tag{2.5}$$

$$\begin{aligned}\sum_{k=0}^{n}&\frac{((256-m)k^3-(128+m)k^2-16k+8)\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)(2k-1)(4k-1)m^k}\\&=\frac{8(4n+1)}{(n+1)m^n}\binom{2n}{n}^2\binom{4n}{2n},\end{aligned}\tag{2.6}$$

$$\begin{aligned}\sum_{k=0}^{n}&\frac{((1728-m)k^3-864k^2-48k+24)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)m^k}\\&=\frac{24(6n+1)}{m^n}\binom{2n}{n}\binom{3n}{n}\binom{6n}{3n},\end{aligned}\tag{2.7}$$

$$\begin{aligned}\sum_{k=0}^{n}&\frac{((1728-m)k^3-(864+m)k^2-48k+24)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)m^k}\\&=\frac{24(6n+1)}{(n+1)m^n}\binom{2n}{n}\binom{3n}{n}\binom{6n}{3n}.\end{aligned}\tag{2.8}$$

Remark 2.1. The eight identities in Lemma 2.1 can be easily proved by induction on n. In light of Stirling's formula, $n!\sim\sqrt{2\pi n}(n/e)^n$ as $n\to+\infty$, we have

$$\binom{2n}{n}\sim\frac{4^n}{\sqrt{n\pi}},\quad \binom{2n}{n}\binom{3n}{n}\sim\frac{\sqrt3\,27^n}{2n\pi},\tag{2.9}$$

$$\binom{2n}{n}\binom{4n}{2n}\sim\frac{64^n}{\sqrt2\,n\pi},\quad \binom{3n}{n}\binom{6n}{n}\sim\frac{432^n}{2n\pi}.\tag{2.10}$$

Proof of Theorem 1.1. Just apply Lemma 2.1 and the 36 known rational Ramanujan-type series listed in [16]. Let us illustrate the proofs by showing (1.1), (1.2), (1.71) and (1.72) in details.

By (2.1) with $m=-64$, we have

$$\sum_{k=0}^{\infty}\frac{(16k^3-4k^2-2k+1)\binom{2k}{k}^3}{(2k-1)^2(-64)^k}=\lim_{n\to+\infty}\frac{2n+1}{(-64)^n}\binom{2n}{n}^3=0.$$

Note that

$$16k^3-4k^2-2k+1=(4k+1)(2k-1)^2+2k(4k-1)$$

and recall Bauer's series

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}.$$

So, we get

$$\sum_{k=0}^{\infty}\frac{k(4k-1)\binom{2k}{k}^3}{(2k-1)^2(-64)^k}=-\frac{1}{2}\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=-\frac{1}{\pi}.$$

This proves (1.1). By (2.2) with $m=-64$, we have

$$\sum_{k=0}^{n}\frac{(4k-1)(4k^2-2k+1)\binom{2k}{k}^3}{(2k-1)^3(-64)^k}=\frac{\binom{2n}{n}^3}{(-64)^n}$$

and hence

$$\sum_{k=0}^{\infty}\frac{(2k(2k-1)(4k-1)+4k-1)\binom{2k}{k}^3}{(2k-1)^3(-64)^k}=\lim_{n\to+\infty}\frac{\binom{2n}{n}^3}{(-64)^n}=0.$$

Combining this with (1.1) we immediately get (1.2).

In view of (2.7) with $m=-640320^3$, we have

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(10939058860032072k^3-36k^2-2k+1)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-640320)^{3k}}\\=&\frac{6n+1}{(-640320)^{3n}}\binom{2n}{n}\binom{3n}{n}\binom{6n}{3n}.\end{aligned}$$

and hence

$$\sum_{k=0}^{\infty}\frac{(10939058860032072k^3-36k^2-2k+1)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-640320)^{3k}}=0.$$

In 1987, D. V. Chudnovsky and G. V. Chudnovsky [8] got the formula

$$\sum_{k=0}^{\infty}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}=\frac{3\times 53360^2}{2\pi\sqrt{10005}},$$

which enabled them to hold the world record for the calculation of π during 1989–1994. Note that

$$\begin{aligned}&10939058860032072k^3-36k^2-2k+1\\=&1672209(2k-1)(6k-1)(545140134k+13591409)\\&+426880(16444841148k^2-1709536232k-53241371)\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(16444841148k^2-1709536232k-53241371)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-640320)^{3k}}\\=&-\frac{1672209}{426880}\times\frac{3\times 53360^2}{2\pi\sqrt{10005}}=-1672209\frac{\sqrt{10005}}{\pi}.\end{aligned}$$

This proves (1.71).

By (2.8) with $m=-640320^3$, we have

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(10939058860032072k^3+10939058860031964k^2-2k+1)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-640320)^{3k}}\\&=\frac{6n+1}{(n+1)(-640320)^{3n}}\binom{2n}{n}\binom{3n}{n}\binom{6n}{3n}\end{aligned}$$

and hence

$$\sum_{k=0}^{\infty}\frac{(10939058860032072k^3+10939058860031964k^2-2k+1)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-640320)^{3k}}=0.$$

Note that

$$\begin{aligned}&2802461(10939058860032072k^3+10939058860031964k^2-2k+1)\\=&1864188626454(k+1)(16444841148k^2-1709536232k-53241371)+5P(k).\end{aligned}$$

Therefore, with the help of (1.71) we get

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{P(k)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-640320)^{3k}}\\=&-\frac{1864188626454}{5}\times(-1672209)\frac{\sqrt{10005}}{\pi}=18\times 557403^3\frac{\sqrt{10005}}{5\pi}.\end{aligned}$$

This proves (1.72).

The identities (1.3)–(1.70) can be proved similarly. □

Lemma 2.2. *Let m and $n>0$ be integers. Then*

$$\sum_{k=1}^{n}\frac{m^k((m-64)k^3-32k^2+16k+8)}{(2k+1)^2k^3\binom{2k}{k}^3}=\frac{m^{n+1}}{(2n+1)^2\binom{2n}{n}^3}-m, \tag{2.11}$$

$$\sum_{k=1}^{n}\frac{m^k((m-64)k^3-96k^2-48k-8)}{(2k+1)^3k^3\binom{2k}{k}^3}=\frac{m^{n+1}}{(2n+1)^3\binom{2n}{n}^3}-m, \tag{2.12}$$

$$\sum_{k=1}^{n}\frac{m^k((m-108)k^3-54k^2+12k+6)}{(2k+1)(3k+1)k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{m^{n+1}}{(2n+1)(3n+1)\binom{2n}{n}^2\binom{3n}{n}}-m, \tag{2.13}$$

$$\begin{aligned}&\sum_{1<k\le n}\frac{m^k((m-108)k^3-(54+m)k^2+12k+6)}{(k-1)(2k+1)(3k+1)k^3\binom{2k}{k}^2\binom{3k}{k}}\\&\quad=\frac{m^{n+1}}{n(2n+1)(3n+1)\binom{2n}{n}^2\binom{3n}{n}}-\frac{m^2}{144},\end{aligned} \tag{2.14}$$

$$\begin{aligned}&\sum_{k=1}^{n}\frac{m^k((m-256)k^3-128k^2+16k+8)}{(2k+1)(4k+1)k^3\binom{2k}{k}^2\binom{4k}{2k}}\\&\quad=\frac{m^{n+1}}{(2n+1)(4n+1)\binom{2n}{n}^2\binom{4n}{2n}}-m,\end{aligned} \tag{2.15}$$

$$\begin{aligned}&\sum_{1<k\le n}\frac{m^k((m-256)k^3-(128+m)k^2+16k+8)}{(k-1)(2k+1)(4k+1)k^3\binom{2k}{k}^2\binom{4k}{2k}}\\&\quad=\frac{m^{n+1}}{n(2n+1)(4n+1)\binom{2n}{n}^2\binom{4n}{2n}}-\frac{m^2}{360}.\end{aligned} \tag{2.16}$$

Remark 2.2. This can be easily proved by induction on n.

Proof of Theorem 1.2. We just apply Lemma 2.2 and use the known identities:

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{6},\quad \sum_{k=1}^{\infty}\frac{(4k-1)(-64)^k}{k^3\binom{2k}{k}^3}=-16G,$$

$$\sum_{k=1}^{\infty}\frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3}=-2G,\quad \sum_{k=1}^{\infty}\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{2},$$

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}}=K,\quad \sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}K,$$

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^2\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2,\ \sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},\ \sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2.$$

Here, the first identity was found and proved by D. Zeilberger [52] in 1993. The second, third and fourth identities were obtained by J. Guillera [17] in 2008. The fifth identity on K was conjectured by Sun [33] and later confirmed by K. Hessami Pilehrood and T. Hessami Pilehrood [22] in 2012. The last four identities were also conjectured by Sun [33], and they were later proved in the paper [18, Theorem 3] by Guillera and M. Rogers.

Let us illustrate our proofs by proving (1.77)-(1.79) and (1.82)-(1.83) in details.

In view of (2.11) with $m=16$, we have

$$\sum_{k=1}^{n}\frac{16^k(-48k^3-32k^2+16k+8)}{(2k+1)^2k^3\binom{2k}{k}^3}=\frac{16^{n+1}}{(2n+1)^2\binom{2n}{n}^3}-16$$

for all $n\in\mathbb{Z}^+$, and hence

$$\sum_{k=1}^{\infty}\frac{16^k(6k^3+4k^2-2k-1)}{(2k+1)^2k^3\binom{2k}{k}^3}=\lim_{n\to+\infty}\left(\frac{-2\times16^n}{(2n+1)^2\binom{2n}{n}^3}+2\right)=2.$$

Notice that

$$2(6k^3+4k^2-2k-1)=(2k+1)^2(3k-1)-(3k+1).$$

So we have

$$-\sum_{k=1}^{\infty}\frac{(3k+1)16^k}{(2k+1)^2k^3\binom{2k}{k}^3}=2\times2-\sum_{k=1}^{\infty}\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}=4-\frac{\pi^2}{2}$$

and hence (1.77) holds.

By (2.11) with $m=-64$, we have

$$\sum_{k=1}^{n}\frac{(-64)^k(-128k^3-32k^2+16k+8)}{(2k+1)^2k^3\binom{2k}{k}^3}=\frac{(-64)^{n+1}}{(2n+1)^2\binom{2n}{n}^3}+64$$

for all $n\in\mathbb{Z}^+$, and hence

$$\sum_{k=1}^{\infty}\frac{(-64)^k(16k^3+4k^2-2k-1)}{(2k+1)^2k^3\binom{2k}{k}^3}=-8+\lim_{n\to+\infty}\frac{8(-64)^n}{(2n+1)^2\binom{2n}{n}^3}=-8.$$

Since $16k^3+4k^2-2k-1=(4k-1)(2k+1)^2-2k(4k+1)$ and

$$\sum_{k=1}^{\infty}\frac{(4k-1)(-64)^k}{k^3\binom{2k}{k}^3}=-16G,$$

we see that

$$-16G-2\sum_{k=1}^\infty\frac{(4k+1)(-64)^k}{(2k+1)^2k^2\binom{2k}{k}^3}=-8$$

and hence (1.78) holds. In light of (2.12) with $m=-64$, we have

$$\sum_{k=1}^n\frac{(-64)^k(-128k^3-96k^2-48k-8)}{(2k+1)^3k^3\binom{2k}{k}^3}=\frac{(-64)^{n+1}}{(2n+1)^3\binom{2n}{n}^3}+64$$

for all $n\in\mathbb{Z}^+$, and hence

$$\sum_{k=1}^\infty\frac{(-64)^k(16k^3+12k^2+6k+1)}{(2k+1)^3k^3\binom{2k}{k}^3}=-8+\lim_{n\to+\infty}\frac{8(-64)^n}{(2n+1)^3\binom{2n}{n}^3}=-8.$$

Since $16k^3+12k^2+6k+1=2k(2k+1)(4k+1)+(4k+1)$, with the aid of (1.78) we obtain

$$\sum_{k=1}^\infty\frac{(4k+1)(-64)^k}{(2k+1)^3k^3\binom{2k}{k}^3}=-8-2(4-8G)=16G-16.$$

This proves (1.79).

By (2.13) with $m=-27$, we have

$$\sum_{k=1}^\infty\frac{(45k^3+18k^2-4k-2)(-27)^k}{(2k+1)(3k+1)k^3\binom{2k}{k}^2\binom{3k}{k}}=-9.$$

As

$$2(45k^3+18k^2-4k-2)=(15k-4)(2k+1)(3k+1)-3k(5k+1)$$

and

$$\sum_{k=1}^\infty\frac{(15k-4)(-27)^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=-27K,$$

we see that (1.82) follows. By (2.14) with $m=-27$, we have

$$-3\sum_{k=2}^\infty\frac{(-27)^k(45k^3+9k^2-4k-2)}{(k-1)(2k+1)(3k+1)k^3\binom{2k}{k}^2\binom{3k}{k}}=-\frac{(-27)^2}{144}$$

and hence

$$\sum_{k=2}^\infty\frac{(-27)^k(45k^3+9k^2-4k-2)}{(k-1)(2k+1)(3k+1)k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{27}{16}.$$

As

$$45k^3+9k^2-4k-2=9(k-1)k(5k+1)+(45k^2+5k-2),$$

with the aid of (1.82) we get

$$\sum_{k=2}^\infty\frac{(-27)^k(45k^2+5k-2)}{(k-1)(2k+1)(3k+1)k^3\binom{2k}{k}^2\binom{3k}{k}}$$
$$=\frac{27}{16}-9\left(6-9K-\frac{6(-27)}{12^2}\right)=\frac{27}{16}(48K-37)$$

and hence (1.83) follows.

Other identities in Theorem 1.2 can be proved similarly. □

For integers $n \geq k \geq 0$, we define

$$s_{n,k} := \frac{1}{\binom{n}{k}} \sum_{i=0}^{k} \binom{n}{2i}\binom{n}{2(k-i)}\binom{2i}{i}\binom{2(k-i)}{k-i}. \tag{2.17}$$

For $n \in \mathbb{N}$ we set

$$t_n := \sum_{0<k\leq n} \binom{n-1}{k-1}(-1)^k 4^{n-k} s_{n+k,k}. \tag{2.18}$$

Lemma 2.3. *For any $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k 4^{n-k} s_{n+k,k} = f_n \tag{2.19}$$

and

$$(2n+1)t_{n+1} + 8nt_n = (2n+1)f_{n+1} - 4(n+1)f_n, \tag{2.20}$$

where f_n denotes the Franel number $\sum_{k=0}^{n}\binom{n}{k}^3$.

Proof. For $n, i, k \in \mathbb{N}$ with $i \leq k$, we set

$$F(n,i,k) = \binom{n}{k}\frac{(-1)^k 4^{n-k}}{\binom{n+k}{k}}\binom{n+k}{2i}\binom{2i}{i}\binom{n+k}{2(k-i)}\binom{2(k-i)}{k-i}.$$

By the telescoping method for double summation [7], for

$$\mathcal{F}(n,i,k) := F(n,i,k) + \frac{7n^2+21n+16}{8(n+1)^2}F(n+1,i,k) - \frac{(n+2)^2}{8(n+1)^2}F(n+2,i,k)$$

with $0 \leq i \leq k$, we find that

$$\mathcal{F}(n,i,k) = (G_1(n,i+1,k) - G_1(n,i,k)) + (G_2(n,i,k+1) - G_2(n,i,k)),$$

where

$$G_1(n,i,k) := \frac{i^2(-k+i-1)(-1)^{k+1}4^{n-k}n!^2(n+k)!p(n,i,k)}{(2n+3)(n-k+2)!(n+k+2-2i)!(n-k+2i)!(i!(k-i+1)!)^2}$$

and

$$G_2(n,i,k) := \frac{2(k-i)(-1)^k 4^{n-k}n!^2(n+k)!q(n,i,k)}{(2n+3)(n-k+2)!(n+k-2i+1)!(n-k+2i+2)!(i!(k-i)!)^2},$$

with $(-1)!, (-2)!, \ldots$ regarded as $+\infty$, and $p(n,i,k)$ and $q(n,i,k)$ given by

$$\begin{aligned}
&-10n^4 + (i-10k-68)n^3 + (-24i^2 + (32k+31)i + 2k^2 - 67k - 172)n^2 \\
&+ (36i^3 + (-68k-124)i^2 + (39k^2+149k+104)i + 2k^3 - 8k^2 - 145k - 192)n \\
&+ 60i^3 + (-114k-140)i^2 + (66k^2+160k+92)i + 3k^3 - 19k^2 - 102k - 80
\end{aligned}$$

and

$$\begin{aligned}
&10(i-k)n^4 + (-20i^2 + (46k+47)i - 6k^2 - 47k)n^3 \\
&+(72i^3 + (-60k-38)i^2 + (22k^2+145k+90)i + 4k^3 - 11k^2 - 90k)n^2 \\
&+(72k+156)i^3 n + (-72k^2 - 60k - 10)i^2 n + (18k^3 + 4k^2 + 165k + 85)in \\
&+(22k^3 - 5k^2 - 85k)n + (120k+60)i^3 + (-120k^2 + 68k - 4)i^2 \\
&+(30k^3 - 56k^2 + 86k + 32)i + 26k^3 - 6k^2 - 32k
\end{aligned}$$

respectively. Therefore

$$\begin{aligned}&\sum_{k=0}^{n+2}\sum_{i=0}^{k}\mathcal{F}(n,i,k)\\=&\sum_{k=0}^{n+2}(G_1(n,k+1,k)-G_1(n,0,k))+\sum_{i=0}^{n+2}(G_2(n,i,n+3)-G_2(n,i,i))\\=&\sum_{k=0}^{n+2}(0-0)+\sum_{i=0}^{n+2}(0-0)=0,\end{aligned}$$

and hence

$$u(n):=\sum_{k=0}^{n}\binom{n}{k}(-1)^k4^{n-k}s_{n+k,k}$$

satisfies the recurrence relation

$$8(n+1)^2u(n)+(7n^2+21n+16)u(n+1)-(n+2)^2u(n+2)=0.$$

As pointed out by J. Franel [14], the Franel numbers satisfy the same recurrence. Note also that $u(0)=f_0=1$ and $u(1)=f_1=2$. So we always have $u(n)=f_n$. This proves (2.19).

The identity (2.20) can be proved similarly. In fact, if we use $v(n)$ denote the left-hand side or the right-hand side of (2.20), then we have the recurrence

$$\begin{aligned}&8(n+1)(n+2)(18n^3+117n^2+249n+172)v(n)\\&+(126n^5+1197n^4+4452n^3+8131n^2+7350n+2656)v(n+1)\\=&(n+3)^2(18n^3+63n^2+69n+22)v(n+2).\end{aligned}$$

In view of the above, we have completed the proof of Lemma 2.3. □

Lemma 2.4. *For any $c\in\mathbb{Z}$ and $n\in\mathbb{N}$, we have*

$$S_n(4,c)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\binom{2(n-k)}{n-k}c^k4^{n-2k}s_{n,k}. \tag{2.21}$$

Proof. For each $k=0,\ldots,n$, we have

$$\begin{aligned}T_k(4,c)T_{n-k}(4,c)&=\sum_{i=0}^{\lfloor k/2\rfloor}\binom{k}{2i}\binom{2i}{i}4^{k-2i}c^i\sum_{j=0}^{\lfloor (n-k)/2\rfloor}\binom{n-k}{2j}\binom{2j}{j}4^{n-k-2j}c^j\\&=\sum_{r=0}^{\lfloor n/2\rfloor}c^r4^{n-2r}\sum_{\substack{i,j\in\mathbb{N}\\ i+j=r}}\binom{k}{2i}\binom{n-k}{2j}\binom{2i}{i}\binom{2j}{j}.\end{aligned}$$

If $i,j\in\mathbb{N}$ and $i+j=r\le n/2$, then

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}^2\binom{k}{2i}\binom{n-k}{2j}&=\binom{n}{2i}\binom{n}{2j}\sum_{k=2i}^{n-2j}\binom{n-2i}{k-2i}\binom{n-2j}{n-k-2j}\\&=\binom{n}{2i}\binom{n}{2j}\binom{2n-2(i+j)}{n-2(i+j)}=\binom{2n-2r}{n}\binom{n}{2i}\binom{n}{2j}\end{aligned}$$

with the aid of the Chu-Vandermonde identity. Therefore

$$\begin{aligned} S_n(4,c) &= \sum_{k=0}^{\lfloor n/2\rfloor} c^k 4^{n-2k}\binom{2n-2k}{n}\binom{n}{k}s_{n,k} \\ &= \sum_{k=0}^{\lfloor n/2\rfloor} c^k 4^{n-2k}\binom{2n-2k}{n-k}\binom{n-k}{k}s_{n,k}. \end{aligned}$$

This proves (2.21). □

Lemma 2.5. *For $k\in\mathbb{N}$ and $l\in\mathbb{Z}^+$, we have*

$$s_{k+l,k} \le (2k+1)4^k l\binom{k+l}{l}. \tag{2.22}$$

Proof. Let $n=k+l$. Then

$$\begin{aligned} \binom{n}{k}s_{n,k} &\le \sum_{\substack{i,j\in\mathbb{N}\\ i+j=k}}\binom{n}{2i}\binom{n}{2j}\sum_{\substack{i,j\in\mathbb{N}\\ i+j=k}}\binom{2i}{i}\binom{2j}{j} \\ &\le \sum_{\substack{s,t\in\mathbb{N}\\ s+t=2k}}\binom{n}{s}\binom{n}{t}\sum_{\substack{i,j\in\mathbb{N}\\ i+j=k}}4^i4^j = \binom{2n}{2k}(k+1)4^k \end{aligned}$$

and

$$\begin{aligned} \frac{\binom{2n}{2k}}{\binom{n}{k}} =& \frac{\binom{2n}{2l}}{\binom{n}{l}} = \prod_{j=0}^{l-1}\frac{2(j+k)+1}{2j+1} \\ \le&(2k+1)\prod_{0<j<l}\frac{2(j+k)}{2j} \\ =&(2k+1)\binom{k+l-1}{l-1}. \end{aligned}$$

Hence

$$s_{k+l,k} \le (k+1)4^k(2k+1)\frac{l}{k+l}\binom{k+l}{l} \le (2k+1)4^k l\binom{k+l}{l}.$$

This proves (2.22). □

To prove Theorem 1.3, we need an auxiliary theorem.

Theorem 2.6. *Let a and b be real numbers. For any integer m with $|m|\ge 94$, we have*

$$\sum_{n=0}^{\infty}(an+b)\frac{S_n(4,-m)}{m^n} = \frac{1}{m+16}\sum_{n=0}^{\infty}(2a(m+4)n-8a+b(m+16))\frac{\binom{2n}{n}f_n}{m^n}. \tag{2.23}$$

Proof. Let $N\in\mathbb{N}$. In view of (2.21),

$$\sum_{n=0}^{N}\frac{S_n(4,-m)}{m^n} = \sum_{n=0}^{N}\frac{1}{m^n}\sum_{k=0}^{\lfloor n/2\rfloor}(-m)^k4^{n-2k}\binom{2n-2k}{n-k}\binom{n-k}{k}s_{n,k}$$

$$=\sum_{l=0}^{N}\frac{\binom{2l}{l}}{m^l}\sum_{k=0}^{N-l}\binom{l}{k}(-1)^k4^{l-k}s_{l+k,k}$$
$$=\sum_{l=0}^{\lfloor N/2\rfloor}\frac{\binom{2l}{l}}{m^l}\sum_{k=0}^{l}\binom{l}{k}(-1)^k4^{l-k}s_{l+k,k}$$
$$+\sum_{N/2<l\le N}\frac{\binom{2l}{l}}{m^l}\sum_{k=0}^{N-l}\binom{l}{k}(-1)^k4^{l-k}s_{l+k,k}$$

and similarly

$$\sum_{n=0}^{N}\frac{nS_n(4,-m)}{m^n}=\sum_{l=0}^{N}\frac{\binom{2l}{l}}{m^l}\sum_{k=0}^{N-l}\binom{l}{k}(-1)^k4^{l-k}(k+l)s_{l+k,k}$$
$$=\sum_{l=0}^{N}\frac{l\binom{2l}{l}}{m^l}\sum_{k=0}^{N-l}\left(\binom{l}{k}+\binom{l-1}{k-1}\right)(-1)^k4^{l-k}s_{l+k,k}$$
$$=\sum_{l=0}^{\lfloor N/2\rfloor}\frac{l\binom{2l}{l}}{m^l}\sum_{k=0}^{l}\left(\binom{l}{k}+\binom{l-1}{k-1}\right)(-1)^k4^{l-k}s_{l+k,k}$$
$$+\sum_{N/2<l\le N}\frac{l\binom{2l}{l}}{m^l}\sum_{k=0}^{N-l}\left(\binom{l}{k}+\binom{l-1}{k-1}\right)(-1)^k4^{l-k}s_{l+k,k},$$

where we consider $\binom{x}{-1}$ as 0.

If l is an integer in the interval $(N/2,N]$, then by Lemma 2.5 we have

$$\Big|\sum_{k=0}^{N-l}\binom{l}{k}(-1)^k4^{l-k}s_{l+k,k}\Big|$$
$$\le\sum_{k=0}^{l}\binom{l}{k}4^{l-k}s_{l+k,k}\le\sum_{k=0}^{l}\binom{l}{k}4^{l-k}(2k+1)4^kl\binom{k+l}{l}$$
$$\le l(2l+1)4^l\sum_{k=0}^{l}\binom{l}{k}\binom{l+k}{k}=l(2l+1)4^lP_l(3),$$

where $P_l(x)$ is the Legendre polynomial of degree l. Thus

$$\Big|\sum_{N/2<l\le N}\frac{\binom{2l}{l}}{m^l}\sum_{k=0}^{N-l}\binom{l}{k}(-1)^k4^{l-k}s_{l+k,k}\Big|$$
$$\le\sum_{N/2<l\le N}l(2l+1)\left(\frac{16}{m}\right)^lP_l(3)\le\sum_{l>N/2}l(2l+1)P_l(3)\left(\frac{16}{m}\right)^l$$

and

$$\Big|\sum_{N/2<l\le N}\frac{l\binom{2l}{l}}{m^l}\sum_{k=0}^{N-l}\left(\binom{l}{k}+\binom{l-1}{k-1}\right)(-1)^k4^{l-k}s_{l+k,k}\Big|$$
$$\le\sum_{N/2<l\le N}\frac{l4^l}{m^l}\sum_{k=0}^{l}2\binom{l}{k}4^{l-k}s_{l+k,k}$$

$$\leq \sum_{N/2<l\leq N} 2l^2(2l+1)\left(\frac{16}{m}\right)^l P_l(3)$$
$$\leq 2\sum_{l>N/2} l^2(2l+1)P_l(3)\left(\frac{16}{m}\right)^l.$$

Recall that

$$P_l(3)=T_l(3,2)\sim\frac{(3+2\sqrt{2})^{l+1/2}}{2\sqrt[4]{2}\sqrt{l\pi}}\quad\text{as } l\to+\infty.$$

As $|m|\geq 94$, we have $|m|>16(3+2\sqrt{2})\approx 93.255$ and hence

$$\sum_{l=0}^{\infty} l^2(2l+1)P_l(3)\left(\frac{16}{m}\right)^l$$

converges. Thus

$$\lim_{N\to+\infty}\sum_{l>N/2} l(2l+1)P_l(3)\left(\frac{16}{m}\right)^l=0=\lim_{N\to+\infty}\sum_{l>N/2} l^2(2l+1)P_l(3)\left(\frac{16}{m}\right)^l$$

and hence by the above we have

$$\sum_{n=0}^{\infty}\frac{S_n(4,-m)}{m^n}=\sum_{l=0}^{\infty}\frac{\binom{2l}{l}}{m^l}\sum_{k=0}^{l}\binom{l}{k}(-1)^k4^{l-k}s_{l+k,k}$$

and

$$\sum_{n=0}^{\infty}\frac{nS_n(4,-m)}{m^n}=\sum_{l=0}^{\infty}\frac{l\binom{2l}{l}}{m^l}\sum_{k=0}^{l}\left(\binom{l}{k}+\binom{l-1}{k-1}\right)(-1)^k4^{l-k}s_{l+k,k}.$$

Therefore, with the aid of (2.19), we obtain

$$\sum_{n=0}^{\infty}\frac{S_n(4,-m)}{m^n}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{m^n}f_n \tag{2.24}$$

and

$$\sum_{n=0}^{\infty}\frac{nS_n(4,-m)}{m^n}=\sum_{n=0}^{\infty}\frac{n\binom{2n}{n}}{m^n}(f_n+t_n). \tag{2.25}$$

In view of (2.25) and (2.20),

$$\begin{aligned}&(m+16)\sum_{n=0}^{\infty}\frac{nS_n(4,-m)}{m^n}\\ =&\sum_{n=1}^{\infty}\frac{n\binom{2n}{n}}{m^{n-1}}(f_n+t_n)+16\sum_{n=0}^{\infty}\frac{n\binom{2n}{n}}{m^n}(f_n+t_n)\\ =&\sum_{n=0}^{\infty}\frac{(n+1)\binom{2n+2}{n+1}(f_{n+1}+t_{n+1})+16n\binom{2n}{n}(f_n+t_n)}{m^n}\\ =&2\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{m^n}\left((2n+1)(f_{n+1}+t_{n+1})+8n(f_n+t_n)\right)\end{aligned}$$

$$=2\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{m^n}\left(2(2n+1)f_{n+1}+4(n-1)f_n\right)$$
$$=2\sum_{n=0}^{\infty}\frac{(n+1)\binom{2n+2}{n+1}f_{n+1}}{m^n}+8\sum_{n=0}^{\infty}\frac{(n-1)\binom{2n}{n}f_n}{m^n}$$
$$=2\sum_{n=0}^{\infty}\frac{n\binom{2n}{n}f_n}{m^{n-1}}+8\sum_{n=0}^{\infty}\frac{(n-1)\binom{2n}{n}f_n}{m^n}=2\sum_{n=0}^{\infty}((m+4)n-4)\frac{\binom{2n}{n}f_n}{m^n}.$$

Combining this with (2.24), we immediately obtain the desired (2.23). □

Proof of Theorem 1.3. Let $a,b,m\in\mathbb{Z}$ with $|m|\ge 6$. Since

$$4^nT_n(1,m)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}4^{n-2k}(16m)^k=T_n(4,16m)$$

for any $n\in\mathbb{N}$, we have $4^nS_n(1,m)=S_n(4,16m)$ for all $n\in\mathbb{N}$. Thus, in light of Theorem 2.6,

$$\sum_{n=0}^{\infty}(an+b)\frac{S_n(1,m)}{(-4m)^n}$$
$$=\sum_{n=0}^{\infty}(an+b)\frac{S_n(4,16m)}{(-16m)^n}$$
$$=\frac{1}{16-16m}\sum_{n=0}^{\infty}(2a(4-16m)n-8a+(16-16m)b)\frac{\binom{2n}{n}f_n}{(-16m)^n}$$
$$=\frac{1}{2(m-1)}\sum_{n=0}^{\infty}(a(4m-1)n+a+2b(m-1))\frac{\binom{2n}{n}f_n}{(-16m)^n}.$$

Therefore

$$\sum_{k=0}^{\infty}\frac{7k+3}{24^k}S_k(1,-6)=\frac{5}{2}\sum_{k=0}^{\infty}\frac{5k+1}{96^k}\binom{2k}{k}f_k,$$
$$\sum_{k=0}^{\infty}\frac{12k+5}{(-28)^k}S_k(1,7)=3\sum_{k=0}^{\infty}\frac{9k+2}{(-112)^k}\binom{2k}{k}f_k,$$
$$\sum_{k=0}^{\infty}\frac{84k+29}{80^k}S_k(1,-20)=27\sum_{k=0}^{\infty}\frac{6k+1}{320^k}\binom{2k}{k}f_k,$$
$$\sum_{k=0}^{\infty}\frac{3k+1}{(-100)^k}S_k(1,25)=\frac{1}{16}\sum_{k=0}^{\infty}\frac{99k+17}{(-400)^k}\binom{2k}{k}f_k,$$
$$\sum_{k=0}^{\infty}\frac{228k+67}{224^k}S_k(1,-56)=5\sum_{k=0}^{\infty}\frac{90k+13}{896^k}\binom{2k}{k}f_k,$$
$$\sum_{k=0}^{\infty}\frac{399k+101}{(-676)^k}S_k(1,169)=\frac{15}{16}\sum_{k=0}^{\infty}\frac{855k+109}{(-2704)^k}\binom{2k}{k}f_k,$$
$$\sum_{k=0}^{\infty}\frac{2604k+563}{2600^k}S_k(1,-650)=51\sum_{k=0}^{\infty}\frac{102k+11}{10400^k}\binom{2k}{k}f_k,$$

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{39468k+7817}{(-6076)^k}S_k(1,1519)&=135\sum_{k=0}^{\infty}\frac{585k+58}{(-24304)^k}\binom{2k}{k}f_k,\\ \sum_{k=0}^{\infty}\frac{41667k+7879}{9800^k}S_k(1,-2450)&=\frac{297}{2}\sum_{k=0}^{\infty}\frac{561k+53}{39200^k}\binom{2k}{k}f_k,\\ \sum_{k=0}^{\infty}\frac{74613k+10711}{(-530^2)^k}S_k(1,265^2)&=\frac{23}{32}\sum_{k=0}^{\infty}\frac{207621k+14903}{(-1060^2)^k}\binom{2k}{k}f_k.\end{aligned}$$

It is known (cf. [5, 4]) that

$$\sum_{k=0}^{\infty}\frac{5k+1}{96^k}\binom{2k}{k}f_k=\frac{3\sqrt{2}}{\pi},\quad \sum_{k=0}^{\infty}\frac{9k+2}{(-112)^k}\binom{2k}{k}f_k=\frac{2\sqrt{7}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{6k+1}{320^k}\binom{2k}{k}f_k=\frac{8\sqrt{15}}{9\pi},\quad \sum_{k=0}^{\infty}\frac{99k+17}{(-400)^k}\binom{2k}{k}f_k=\frac{50}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{90k+13}{896^k}\binom{2k}{k}f_k=\frac{16\sqrt{7}}{\pi},\quad \sum_{k=0}^{\infty}\frac{855k+109}{(-2704)^k}\binom{2k}{k}f_k=\frac{338}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{102k+11}{10400^k}\binom{2k}{k}f_k=\frac{50\sqrt{39}}{9\pi},\quad \sum_{k=0}^{\infty}\frac{585k+58}{(-24304)^k}\binom{2k}{k}f_k=\frac{98\sqrt{31}}{3\pi},$$

$$\sum_{k=0}^{\infty}\frac{561k+53}{39200^k}\binom{2k}{k}f_k=\frac{1225\sqrt{6}}{18\pi},\quad \sum_{k=0}^{\infty}\frac{207621k+14903}{(-1060^2)^k}\binom{2k}{k}f_k=\frac{140450}{3\pi}.$$

So we get the identities (1.88)-(1.97) finally. □

3. New series involving $T_n(b,c)$ for $1/\pi$ related to types I-VIII

Now we pose a conjecture related to the series (I1)-(I4) of Sun [34, 40].

Conjecture 3.1. *We have the following identities:*

$$\sum_{k=0}^{\infty}\frac{50k+1}{(-256)^k}\binom{2k}{k}\binom{2k}{k+1}T_k(1,16)=\frac{8}{3\pi},\tag{I1$'$}$$

$$\sum_{k=0}^{\infty}\frac{(100k^2-4k-7)\binom{2k}{k}^2T_k(1,16)}{(2k-1)^2(-256)^k}=-\frac{24}{\pi},\tag{I1$''$}$$

$$\sum_{k=0}^{\infty}\frac{30k+23}{(-1024)^k}\binom{2k}{k}\binom{2k}{k+1}T_k(34,1)=-\frac{20}{3\pi},\tag{I2$'$}$$

$$\sum_{k=0}^{\infty}\frac{(36k^2-12k+1)\binom{2k}{k}^2T_k(34,1)}{(2k-1)^2(-1024)^k}=-\frac{6}{\pi},\tag{I2$''$}$$

$$\sum_{k=0}^{\infty}\frac{110k+103}{4096^k}\binom{2k}{k}\binom{2k}{k+1}T_k(194,1)=\frac{304}{\pi},\tag{I3$'$}$$

$$\sum_{k=0}^{\infty}\frac{(20k^2+28k-11)\binom{2k}{k}^2T_k(194,1)}{(2k-1)^24096^k}=-\frac{6}{\pi},\tag{I3$''$}$$

$$\sum_{k=0}^{\infty}\frac{238k+263}{4096^k}\binom{2k}{k}\binom{2k}{k+1}T_k(62,1)=\frac{112\sqrt{3}}{3\pi},\tag{I4$'$}$$

$$\sum_{k=0}^\infty\frac{(44k^2+4k-5)\binom{2k}k^2T_k(62,1)}{(2k-1)^2 4096^k}=-\frac{4\sqrt3}{\pi},\tag{I4$''$}$$

$$\sum_{k=0}^\infty\frac{6k+1}{256^k}\binom{2k}k^2T_k(8,-2)=\frac2{\pi}\sqrt{8+6\sqrt2},\tag{I5}$$

$$\sum_{k=0}^\infty\frac{2k+3}{256^k}\binom{2k}k\binom{2k}{k+1}T_k(8,-2)=\frac{6\sqrt{8+6\sqrt2}-16\sqrt[4]2}{3\pi},\tag{I5$'$}$$

$$\sum_{k=0}^\infty\frac{(4k^2+2k-1)\binom{2k}k^2T_k(8,-2)}{(2k-1)^2 256^k}=-\frac{3\sqrt[4]2}{4\pi}.\tag{I5$''$}$$

Remark 3.1. For each $k\in\mathbb N$, we have

$$((1+\lambda_0-\lambda_1)k+\lambda_0)C_k=(k+\lambda_0)\binom{2k}k-(k+\lambda_1)\binom{2k}{k+1}$$

since $\binom{2k}k=(k+1)C_k$ and $\binom{2k}{k+1}=kC_k$. Thus, for example, [40, (I1)] and (I1$'$) together imply that

$$\sum_{k=0}^\infty\frac{26k+5}{(-256)^k}\binom{2k}kC_kT_k(1,16)=\frac{16}{\pi},$$

and (I5) and (I5$'$) imply that

$$\sum_{k=0}^\infty\frac{2k-1}{256^k}\binom{2k}kC_kT_k(8,-2)=\frac4{\pi}\left(\sqrt{8+6\sqrt2}-4\sqrt[4]2\right).$$

For the conjectural identities in Conjecture 3.1, we have conjectures for the corresponding p-adic congruences. For example, in contrast with (I2$'$), we conjecture that for any prime $p>3$ we have the congruences

$$\sum_{k=0}^{p-1}\frac{30k+23}{(-1024)^k}\binom{2k}k\binom{2k}{k+1}T_k(34,1)\equiv\frac p3\left(21\left(\frac2p\right)-10\left(\frac{-1}p\right)-11\right)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{2k+1}{(-1024)^k}\binom{2k}kC_kT_k(34,1)\equiv\frac p3\left(2-3\left(\frac2p\right)+4\left(\frac{-1}p\right)\right)\pmod{p^2}.$$

Concerning (I5) and (I5$''$), we conjecture that

$$\frac1{2^{\lfloor n/2\rfloor+1}n\binom{2n}n}\sum_{k=0}^{n-1}(6k+1)\binom{2k}k^2T_k(8,-2)256^{n-1-k}\in\mathbb Z^+$$

and

$$\frac1{\binom{2n-2}{n-1}}\sum_{k=0}^{n-1}\frac{(1-2k-4k^2)\binom{2k}k^2T_k(8,-2)}{(2k-1)^2256^k}\in\mathbb Z^+$$

for each $n=2,3,\ldots$, and that for any prime $p\equiv1\pmod 4$ with $p=x^2+4y^2$ ($x,y\in\mathbb Z$) we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}k^2T_k(8,-2)}{256^k}\equiv\begin{cases}(-1)^{y/2}(4x^2-2p)\pmod{p^2}&\text{if }p\equiv1\pmod 8,\\(-1)^{(xy-1)/2}8xy\pmod{p^2}&\text{if }p\equiv5\pmod 8,\end{cases}$$

and
$$\sum_{k=0}^{p-1}\frac{(4k^2+2k-1)\binom{2k}{k}^2T_k(8,-2)}{(2k-1)^2256^k}\equiv 0 \pmod{p^2}.$$
By [40, Theorem 5.1], we have
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(8,-2)}{256^k}\equiv 0 \pmod{p^2}$$
for any prime $p\equiv 3 \pmod 4$. The identities (I5), (I5′) and (I5″) were formulated by the author on Dec. 9, 2019.

Next we pose a conjecture related to the series (II1)-(II7) and (II10)-(II12) of Sun [34, 40].

Conjecture 3.2. *We have the following identities:*
$$\sum_{k=0}^{\infty}\frac{3k+4}{972^k}\binom{2k}{k+1}\binom{3k}{k}T_k(18,6)=\frac{63\sqrt3}{40\pi},\tag{II1′}$$
$$\sum_{k=0}^{\infty}\frac{91k+107}{10^{3k}}\binom{2k}{k+1}\binom{3k}{k}T_k(10,1)=\frac{275\sqrt3}{18\pi},\tag{II2′}$$
$$\sum_{k=0}^{\infty}\frac{195k+83}{18^{3k}}\binom{2k}{k+1}\binom{3k}{k}T_k(198,1)=\frac{9423\sqrt3}{10\pi},\tag{II3′}$$
$$\sum_{k=0}^{\infty}\frac{483k-419}{30^{3k}}\binom{2k}{k+1}\binom{3k}{k}T_k(970,1)=\frac{6550\sqrt3}{\pi},\tag{II4′}$$
$$\sum_{k=0}^{\infty}\frac{666k+757}{30^{3k}}\binom{2k}{k+1}\binom{3k}{k}T_k(730,729)=\frac{3475\sqrt3}{4\pi},\tag{II5′}$$
$$\sum_{k=0}^{\infty}\frac{8427573k+8442107}{102^{3k}}\binom{2k}{k+1}\binom{3k}{k}T_k(102,1)=\frac{125137\sqrt6}{20\pi},\tag{II6′}$$
$$\sum_{k=0}^{\infty}\frac{959982231k+960422503}{198^{3k}}\binom{2k}{k+1}\binom{3k}{k}T_k(198,1)=\frac{5335011\sqrt3}{20\pi},\tag{II7′}$$
$$\sum_{k=0}^{\infty}\frac{99k+1}{24^{3k}}\binom{2k}{k+1}\binom{3k}{k}T_k(26,729)=\frac{16(289\sqrt{15}-645\sqrt3)}{15\pi},\tag{II10′}$$
$$\sum_{k=0}^{\infty}\frac{45k+1}{(-5400)^k}\binom{2k}{k+1}\binom{3k}{k}T_k(70,3645)=\frac{345\sqrt3-157\sqrt{15}}{6\pi},\tag{II11′}$$
$$\sum_{k=0}^{\infty}\frac{252k-1}{(-13500)^k}\binom{2k}{k+1}\binom{3k}{k}T_k(40,1458)=\frac{25(1212\sqrt3-859\sqrt6)}{24\pi},\tag{II12′}$$
$$\sum_{k=0}^{\infty}\frac{9k+2}{(-675)^k}\binom{2k}{k}\binom{3k}{k}T_k(15,-5)=\frac{7\sqrt{15}}{8\pi},\tag{II13}$$
$$\sum_{k=0}^{\infty}\frac{45k+31}{(-675)^k}\binom{2k}{k+1}\binom{3k}{k}T_k(15,-5)=-\frac{19\sqrt{15}}{8\pi},\tag{II13′}$$
$$\sum_{k=0}^{\infty}\frac{39k+7}{(-1944)^k}\binom{2k}{k}\binom{3k}{k}T_k(18,-3)=\frac{9\sqrt3}{\pi},\tag{II14}$$

$$\sum_{k=0}^\infty\frac{312k+263}{(-1944)^k}\binom{2k}{k+1}\binom{3k}{k}T_k(18,-3)=-\frac{45\sqrt3}{2\pi}.\tag{II14$'$}$$

Remark 3.2. We also have conjectures on related congruences. For example, concerning (II14), for any prime $p>3$ we conjecture that

$$\sum_{k=0}^{p-1}\frac{39k+7}{(-1944)^k}\binom{2k}{k}\binom{3k}{k}T_k(18,-3)\equiv\frac p2\left(13\left(\frac p3\right)+1\right)\pmod{p^2}$$

and that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(18,-3)}{(-1944)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac p3)=(\frac p7)=1\ \&\ p=x^2+21y^2,\\ 2p-2x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac p3)=-1,\ (\frac p7)=1\ \&\ 2p=x^2+21y^2,\\ 12x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac p7)=-1,\ (\frac p3)=1\ \&\ p=3x^2+7y^2,\\ 2p-6x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=1,\ (\frac p3)=(\frac p7)=-1\ \&\ 2p=3x^2+7y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-21}{p})=-1,\end{cases}$$

where x and y are integers. The identities (II13), (II13$'$), (II14) and (II14$'$) were found by the author on Dec. 11, 2019.

The following conjecture is related to the series (III1)-(III10) and (III12) of Sun [34, 40].

Conjecture 3.3. *We have the following identities:*

$$\sum_{k=0}^\infty\frac{17k+18}{66^{2k}}\binom{2k}{k+1}\binom{4k}{2k}T_k(52,1)=\frac{77\sqrt{33}}{12\pi},\tag{III1$'$}$$

$$\sum_{k=0}^\infty\frac{4k+3}{(-96^2)^k}\binom{2k}{k+1}\binom{4k}{2k}T_k(110,1)=-\frac{\sqrt6}{3\pi},\tag{III2$'$}$$

$$\sum_{k=0}^\infty\frac{8k+9}{112^{2k}}\binom{2k}{k+1}\binom{4k}{2k}T_k(98,1)=\frac{154\sqrt{21}}{135\pi},\tag{III3$'$}$$

$$\sum_{k=0}^\infty\frac{3568k+4027}{264^{2k}}\binom{2k}{k+1}\binom{4k}{2k}T_k(257,256)=\frac{869\sqrt{66}}{10\pi},\tag{III4$'$}$$

$$\sum_{k=0}^\infty\frac{144k+1}{(-168^2)^k}\binom{2k}{k+1}\binom{4k}{2k}T_k(7,4096)=\frac{7(1745\sqrt{42}-778\sqrt{210})}{120\pi},\tag{III5$'$}$$

$$\sum_{k=0}^\infty\frac{3496k+3709}{336^{2k}}\binom{2k}{k+1}\binom{4k}{2k}T_k(322,1)=\frac{182\sqrt7}{\pi},\tag{III6$'$}$$

$$\sum_{k=0}^\infty\frac{286k+229}{336^{2k}}\binom{2k}{k+1}\binom{4k}{2k}T_k(1442,1)=\frac{1113\sqrt{210}}{20\pi},\tag{III7$'$}$$

$$\sum_{k=0}^\infty\frac{8426k+8633}{912^{2k}}\binom{2k}{k+1}\binom{4k}{2k}T_k(898,1)=\frac{703\sqrt{114}}{20\pi},\tag{III8$'$}$$

$$\sum_{k=0}^\infty\frac{1608k+79}{912^{2k}}\binom{2k}{k+1}\binom{4k}{2k}T_k(12098,1)=\frac{67849\sqrt{399}}{105\pi},\tag{III9$'$}$$

$$\sum_{k=0}^\infty\frac{134328722k+134635283}{10416^{2k}}\binom{2k}{k+1}\binom{4k}{2k}T_k(10402,1)=\frac{93961\sqrt{434}}{4\pi},\tag{III10$'$}$$

and

$$\sum_{k=0}^\infty\frac{39600310408k+39624469807}{39216^{2k}}\binom{2k}{k+1}\binom{4k}{2k}T_k(39202,1)=\frac{1334161\sqrt{817}}{\pi}.\tag{III12$'$}$$

The following conjecture is related to the series (IV1)-(IV21) of Sun [34, 40].

Conjecture 3.4. *We have the following identities:*

$$\sum_{k=0}^\infty\frac{(356k^2+288k+7)\binom{2k}{k}^2T_{2k}(7,1)}{(k+1)(2k-1)(-48^2)^k}=-\frac{304}{3\pi},\tag{IV1$'$}$$

$$\sum_{k=0}^\infty\frac{(172k^2+141k-1)\binom{2k}{k}^2T_{2k}(62,1)}{(k+1)(2k-1)(-480^2)^k}=-\frac{80}{3\pi},\tag{IV2$'$}$$

$$\sum_{k=0}^\infty\frac{(782k^2+771k+19)\binom{2k}{k}^2T_{2k}(322,1)}{(k+1)(2k-1)(-5760^2)^k}=-\frac{90}{\pi},\tag{IV3$'$}$$

$$\sum_{k=0}^\infty\frac{(34k^2+45k+5)\binom{2k}{k}^2T_{2k}(10,1)}{(k+1)(2k-1)96^{2k}}=-\frac{20\sqrt{2}}{3\pi},\tag{IV4$'$}$$

$$\sum_{k=0}^\infty\frac{(106k^2+193k+27)\binom{2k}{k}^2T_{2k}(38,1)}{(k+1)(2k-1)240^{2k}}=-\frac{10\sqrt{6}}{\pi},\tag{IV5$'$}$$

$$\sum_{k=0}^\infty\frac{(214166k^2+221463k+7227)\binom{2k}{k}^2T_{2k}(198,1)}{(k+1)(2k-1)39200^{2k}}=-\frac{9240\sqrt{6}}{\pi},\tag{IV6$'$}$$

$$\sum_{k=0}^\infty\frac{(112k^2+126k+9)\binom{2k}{k}^2T_{2k}(18,1)}{(k+1)(2k-1)320^{2k}}=-\frac{6\sqrt{15}}{\pi},\tag{IV7$'$}$$

$$\sum_{k=0}^\infty\frac{(926k^2+995k+55)\binom{2k}{k}^2T_{2k}(30,1)}{(k+1)(2k-1)896^{2k}}=-\frac{60\sqrt{7}}{\pi},\tag{IV8$'$}$$

$$\sum_{k=0}^\infty\frac{(1136k^2+2962k+503)\binom{2k}{k}^2T_{2k}(110,1)}{(k+1)(2k-1)24^{4k}}=-\frac{90\sqrt{7}}{\pi},\tag{IV9$'$}$$

$$\sum_{k=0}^\infty\frac{(5488k^2+8414k+901)\binom{2k}{k}^2T_{2k}(322,1)}{(k+1)(2k-1)48^{4k}}=-\frac{294\sqrt{7}}{\pi},\tag{IV10$'$}$$

$$\sum_{k=0}^\infty\frac{(170k^2+193k+11)\binom{2k}{k}^2T_{2k}(198,1)}{(k+1)(2k-1)2800^{2k}}=-\frac{6\sqrt{14}}{\pi},\tag{IV11$'$}$$

$$\sum_{k=0}^\infty\frac{(104386k^2+108613k+4097)\binom{2k}{k}^2T_{2k}(102,1)}{(k+1)(2k-1)10400^{2k}}=-\frac{2040\sqrt{39}}{\pi},\tag{IV12$'$}$$

$$\sum_{k=0}^\infty\frac{(7880k^2+8217k+259)\binom{2k}{k}^2T_{2k}(1298,1)}{(k+1)(2k-1)46800^{2k}}=-\frac{144\sqrt{26}}{\pi},\tag{IV13$'$}$$

$$\sum_{k=0}^\infty\frac{(6152k^2+45391k+9989)\binom{2k}k^2T_{2k}(1298,1)}{(k+1)(2k-1)5616^{2k}}=-\frac{663\sqrt3}{\pi},\tag{IV14$'$}$$

$$\sum_{k=0}^\infty\frac{(147178k^2+2018049k+471431)\binom{2k}k^2T_{2k}(4898,1)}{(k+1)(2k-1)20400^{2k}}=-3740\frac{\sqrt{51}}{\pi},\tag{IV15$'$}$$

$$\sum_{k=0}^\infty\frac{(1979224k^2+5771627k+991993)\binom{2k}k^2T_{2k}(5778,1)}{(k+1)(2k-1)28880^{2k}}=-73872\frac{\sqrt{10}}{\pi},\tag{IV16$'$}$$

$$\sum_{k=0}^\infty\frac{(233656k^2+239993k+5827)\binom{2k}k^2T_{2k}(5778,1)}{(k+1)(2k-1)439280^{2k}}=-4080\frac{\sqrt{19}}{\pi},\tag{IV17$'$}$$

$$\sum_{k=0}^\infty\frac{(5890798k^2+32372979k+6727511)\binom{2k}k^2T_{2k}(54758,1)}{(k+1)(2k-1)243360^{2k}}=-600704\frac{\sqrt{95}}{9\pi},\tag{IV18$'$}$$

$$\sum_{k=0}^\infty\frac{(148k^2+272k+43)\binom{2k}k^2T_{2k}(10,-2)}{(k+1)(2k-1)4608^{k}}=-28\frac{\sqrt{6}}{\pi},\tag{IV19$'$}$$

$$\sum_{k=0}^\infty\frac{(3332k^2+17056k+3599)\binom{2k}k^2T_{2k}(238,-14)}{(k+1)(2k-1)1161216^{k}}=-744\frac{\sqrt{2}}{\pi},\tag{IV20$'$}$$

$$\sum_{k=0}^\infty\frac{(11511872k^2+10794676k+72929)\binom{2k}k^2T_{2k}(9918,-19)}{(k+1)(2k-1)(-16629048064)^{k}}=-390354\frac{\sqrt{7}}{\pi}.\tag{IV21$'$}$$

For the five open conjectural series (VI1), (VI2), (VI3), (VII2) and (VII7) of Sun [34, 40], we make the following conjecture on related supercongruences.

Conjecture 3.5. *Let p be an odd prime and let $n\in\mathbb{Z}^+$. If $(\frac3p)=1$, then*

$$\sum_{k=0}^{pn-1}\frac{66k+17}{(2^{11}3^3)^k}T_k(10,11^2)^3-p\left(\frac{-2}{p}\right)\sum_{k=0}^{n-1}\frac{66k+17}{(2^{11}3^3)^k}T_k(10,11^2)^3$$

divided by $(pn)^2$ is a p-adic integer. If $p\neq5$, then

$$\sum_{k=0}^{pn-1}\frac{126k+31}{(-80)^{3k}}T_k(22,21^2)^3-p\left(\frac{-5}{p}\right)\sum_{k=0}^{n-1}\frac{126k+31}{(-80)^{3k}}T_k(22,21^2)^3$$

divided by $(pn)^2$ is a p-adic integer. If $(\frac7p)=1$ but $p\neq3$, then

$$\sum_{k=0}^{pn-1}\frac{3990k+1147}{(-288)^{3k}}T_k(62,95^2)^3-p\left(\frac{-2}{p}\right)\sum_{k=0}^{n-1}\frac{3990k+1147}{(-288)^{3k}}T_k(62,95^2)^3$$

divided by $(pn)^2$ is a p-adic integer. If $p\equiv\pm1\pmod 8$ but $p\neq7$, then

$$\sum_{k=0}^{pn-1}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k(4,9)^2-p\left(\frac p3\right)\sum_{k=0}^{n-1}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k(4,9)^2$$

divided by $(pn)^2$ *is a* p*-adic integer. If* $(\frac{-6}{p})=1$ *but* $p\neq 7,31$*, then*

$$\sum_{k=0}^{pn-1}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k(73,576)^2$$
$$-p\sum_{k=0}^{n-1}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k(73,576)^2$$

divided by $(pn)^2$ *is a* p*-adic integer.*

Now we pose four conjectural series for $1/\pi$ of type VIII.

Conjecture 3.6. *We have*

$$\sum_{k=0}^{\infty}\frac{40k+13}{(-50)^k}T_k(4,1)T_k(1,-1)^2=\frac{55\sqrt{15}}{9\pi}, \tag{VIII1}$$

$$\sum_{k=0}^{\infty}\frac{1435k+113}{3240^k}T_k(7,1)T_k(10,10)^2=\frac{1452\sqrt{5}}{\pi}, \tag{VIII2}$$

$$\sum_{k=0}^{\infty}\frac{840k+197}{(-2430)^k}T_k(8,1)T_k(5,-5)^2=\frac{189\sqrt{15}}{2\pi}, \tag{VIII3}$$

$$\sum_{k=0}^{\infty}\frac{39480k+7321}{(-29700)^k}T_k(14,1)T_k(11,-11)^2=\frac{6795\sqrt{5}}{\pi}. \tag{VIII4}$$

Remark 3.3. The author found the identity (VIII1) on Nov. 3, 2019. The identities (VIII2), (VIII3) and (VIII4) were formulated on Nov. 4, 2019.

Below we present some conjectures on congruences related to Conjecture 3.6.

Conjecture 3.7. (i) *For each* $n\in\mathbb{Z}^+$*, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(40k+13)(-1)^k50^{n-1-k}T_k(4,1)T_k(1,-1)^2\in\mathbb{Z}^+, \tag{3.1}$$

and this number is odd if and only if n *is a power of two (i.e.,* $n\in\{2^a:\ a\in\mathbb{N}\}$*).*

(ii) *Let* $p\neq 2,5$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{40k+13}{(-50)^k}T_k(4,1)T_k(1,-1)^2\equiv\frac{p}{3}\left(12+5\left(\frac{3}{p}\right)+22\left(\frac{-15}{p}\right)\right)\pmod{p^2}. \tag{3.2}$$

If $(\frac{3}{p})=(\frac{-5}{p})=1$*, then*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{40k+13}{(-50)^k}T_k(4,1)T_k(1,-1)^2-p\sum_{k=0}^{n-1}\frac{40k+13}{(-50)^k}T_k(4,1)T_k(1,-1)^2\right)\in\mathbb{Z}_p \tag{3.3}$$

for all $n\in\mathbb{Z}^+$*.*

(iii) *Let $p\neq 2,5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{T_k(4,1)T_k(1,-1)^2}{(-50)^k}$$
$$\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2)&\text{if } p\equiv 1,9\ (\mathrm{mod}\ 20)\ \&\ p=x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 2x^2-2p\ (\mathrm{mod}\ p^2)&\text{if } p\equiv 3,7\ (\mathrm{mod}\ 20)\ \&\ 2p=x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\ (\mathrm{mod}\ p^2)&\text{if } (\frac{-5}{p})=-1.\end{cases}\tag{3.4}$$

Remark 3.4. The imaginary quadratic field $\mathbb{Q}(\sqrt{-5})$ has class number two.

Conjecture 3.8. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(40k+27)(-6)^{n-1-k}T_k(4,1)T_k(1,-1)^2\in\mathbb{Z},\tag{3.5}$$

and the number is odd if and only if n is a power of two.

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{40k+27}{(-6)^k}T_k(4,1)T_k(1,-1)^2\equiv\frac{p}{9}\left(55\left(\frac{-5}{p}\right)+198\left(\frac{3}{p}\right)-10\right)\ (\mathrm{mod}\ p^2).\tag{3.6}$$

If $(\frac{3}{p})=(\frac{-5}{p})=1$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{40k+27}{(-6)^k}T_k(4,1)T_k(1,-1)^2-p\sum_{k=0}^{n-1}\frac{40k+27}{(-6)^k}T_k(4,1)T_k(1,-1)^2\right)\in\mathbb{Z}_p\tag{3.7}$$

for all $n\in\mathbb{Z}^+$.

(iii) *Let $p>5$ be a prime. Then*

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{T_k(4,1)T_k(1,-1)^2}{(-6)^k}$$
$$\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2)&\text{if } p\equiv 1,9\ (\mathrm{mod}\ 20)\ \&\ p=x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 2p-2x^2\ (\mathrm{mod}\ p^2)&\text{if } p\equiv 3,7\ (\mathrm{mod}\ 20)\ \&\ 2p=x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\ (\mathrm{mod}\ p^2)&\text{if } (\frac{-5}{p})=-1.\end{cases}\tag{3.8}$$

Remark 3.5. This conjecture can be viewed as the dual of Conjecture 3.7. Note that the series $\sum_{k=0}^\infty\frac{(40k+27}{(-6)^k}T_k(4,1)T_k(1,-1)^2$ diverges.

Conjecture 3.9. (i) *For each $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n10^{n-1}}\sum_{k=0}^{n-1}(1435k+113)3240^{n-1-k}T_k(7,1)T_k(10,10)^2\in\mathbb{Z}^+.\tag{3.9}$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{1435k+113}{3240^k}T_k(7,1)T_k(10,10)^2$$
$$\equiv\frac{p}{9}\left(2420\left(\frac{-5}{p}\right)+105\left(\frac{5}{p}\right)-1508\right)\ (\mathrm{mod}\ p^2).\tag{3.10}$$

If $p\equiv 1,9 \pmod{20}$*, then*

$$\sum_{k=0}^{pn-1}\frac{1435k+113}{3240^k}T_k(7,1)T_k(10,10)^2-p\sum_{k=0}^{n-1}\frac{1435k+113}{3240^k}T_k(7,1)T_k(10,10)^2 \quad (3.11)$$

divided by $(pn)^2$ *is a* p*-adic integer for each* $n\in\mathbb{Z}^+$.

(iii) *Let* $p>5$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{T_k(7,1)T_k(10,10)^2}{3240^k} \equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 12x^2-2p \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-15}{p})=-1.\end{cases} \quad (3.12)$$

Remark 3.6. The imaginary quadratic field $\mathbb{Q}(\sqrt{-15})$ has class number two.

Conjecture 3.10. (i) *For each* $n\in\mathbb{Z}^+$*, we have*

$$\frac{3}{2n10^{n-1}}\sum_{k=0}^{n-1}(1435k+1322)50^{n-1-k}T_k(7,1)T_k(10,10)^2\in\mathbb{Z}^+. \quad (3.13)$$

(ii) *Let* $p>5$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{1435k+1322}{50^k}T_k(7,1)T_k(10,10)^2 \equiv\frac{p}{3}\left(3432\left(\frac{5}{p}\right)+968\left(\frac{-1}{p}\right)-434\right) \pmod{p^2}. \quad (3.14)$$

If $p\equiv 1,9 \pmod{20}$*, then*

$$\sum_{k=0}^{pn-1}\frac{1435k+1322}{50^k}T_k(7,1)T_k(10,10)^2-p\sum_{k=0}^{n-1}\frac{1435k+1322}{50^k}T_k(7,1)T_k(10,10)^2 \quad (3.15)$$

divided by $(pn)^2$ *is a* p*-adic integer for each* $n\in\mathbb{Z}^+$.

(iii) *Let* $p>5$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{T_k(7,1)T_k(10,10)^2}{50^k} \equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-15}{p})=-1.\end{cases} \quad (3.16)$$

Remark 3.7. This conjecture can be viewed as the dual of Conjecture 3.9. Note that the series

$$\sum_{k=0}^{\infty}\frac{1435k+1322}{50^k}T_k(7,1)T_k(10,10)^2$$

diverges.

Conjecture 3.11. (i) *For each $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n5^{n-1}}\sum_{k=0}^{n-1}(840k+197)(-1)^k2430^{n-1-k}T_k(8,1)T_k(5,-5)^2\in\mathbb{Z}^+. \tag{3.17}$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{840k+197}{(-2430)^k}T_k(8,1)T_k(5,-5)^2\equiv p\left(140\left(\frac{-15}{p}\right)+5\left(\frac{15}{p}\right)+52\right)\pmod{p^2}. \tag{3.18}$$

If $(\frac{-1}{p})=(\frac{15}{p})=1$, then

$$\sum_{k=0}^{pn-1}\frac{840k+197}{(-2430)^k}T_k(8,1)T_k(5,-5)^2-p\sum_{k=0}^{n-1}\frac{840k+197}{(-2430)^k}T_k(8,1)T_k(5,-5)^2 \tag{3.19}$$

divided by $(pn)^2$ is an p-adic integer for any $n\in\mathbb{Z}^+$.

(iii) *Let $p>7$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{T_k(8,1)T_k(5,-5)^2}{(-2430)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{7})=1,\ p=x^2+105y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{7})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ 2p=x^2+105y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{7})=-1,\ p=3x^2+35y^2,\\ 6x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=(\frac{p}{5})=1,\ 2p=3x^2+35y^2,\\ 2p-20x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1,\ p=5x^2+21y^2,\\ 2p-10x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1,\ 2p=5x^2+21y^2,\\ 28x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=-1,\ (\frac{p}{3})=(\frac{p}{7})=1,\ p=7x^2+15y^2,\\ 14x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{5})=(\frac{p}{7})=1,\ 2p=7x^2+15y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-105}{p})=-1,\end{cases} \tag{3.20}$$

where x and y are integers.

Remark 3.8. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number 8.

Conjecture 3.12. (i) *For each $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(39480k+7321)(-1)^k29700^{n-1-k}T_k(14,1)T_k(11,-11)^2\in\mathbb{Z}^+, \tag{3.21}$$

and this number is odd if and only if n is a power of two.

(ii) *Let $p>5$ be a prime. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{39480k+7321}{(-29700)^k}T_k(14,1)T_k(11,-11)^2\\ \equiv&\,p\left(5738\left(\frac{-5}{p}\right)+70\left(\frac{3}{p}\right)+1513\right)\pmod{p^2}.\end{aligned} \tag{3.22}$$

If $(\frac{3}{p}) = (\frac{-5}{p}) = 1$, *then*

$$\begin{aligned}&\sum_{k=0}^{pn-1}\frac{39480k+7321}{(-29700)^k}T_k(14,1)T_k(11,-11)^2\\&-p\sum_{k=0}^{n-1}\frac{39480k+7321}{(-29700)^k}T_k(14,1)T_k(11,-11)^2\end{aligned}\tag{3.23}$$

divided by $(pn)^2$ *is a* p*-adic integer for each* $n\in\mathbb{Z}^+$.

(iii) *Let* $p>5$ *be a prime with* $p\neq 11$. *Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{T_k(14,1)T_k(11,-11)^2}{(-29700)^k}\\&\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{11})=1,\ p=x^2+165y^2,\\ 2x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{11})=-1,\ 2p=x^2+165y^2,\\ 12x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=-1,\ (\frac{p}{3})=(\frac{p}{11})=1,\ p=3x^2+55y^2,\\ 6x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=1,\ (\frac{p}{3})=(\frac{p}{11})=-1,\ 2p=3x^2+55y^2,\\ 2p-20x^2 \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{11})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ p=5x^2+33y^2,\\ 2p-10x^2 \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{11})=-1,\ (\frac{p}{3})=(\frac{p}{5})=1,\ 2p=5x^2+33y^2,\\ 44x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{5})=(\frac{p}{11})=1,\ p=11x^2+15y^2,\\ 22x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=1,\ (\frac{p}{5})=(\frac{p}{11})=-1,\ 2p=11x^2+15y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-165}{p})=-1,\end{cases}\end{aligned}\tag{3.24}$$

where x *and* y *are integers.*

Remark 3.9. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-165})$ has class number 8.

4. Congruences related to Theorem 1.3

Conjectures 4.1–4.14 below provide congruences related to (1.88)–(1.97).

Conjecture 4.1. (i) *For any* $n\in\mathbb{Z}^+$, *we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(7k+3)S_k(1,-6)24^{n-1-k}\in\mathbb{Z}^+.\tag{4.1}$$

(ii) *Let* $p>3$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{7k+3}{24^k}S_k(1,-6)\equiv\frac{p}{2}\left(5\left(\frac{-2}{p}\right)+\left(\frac{6}{p}\right)\right)\pmod{p^2}.\tag{4.2}$$

If $p\equiv 1 \pmod 3$, *then*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{7k+3}{24^k}S_k(1,-6)-p\left(\frac{-2}{p}\right)\sum_{k=0}^{n-1}\frac{7k+3}{24^k}S_k(1,-6)\right)\in\mathbb{Z}_p\tag{4.3}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{S_k(1,-6)}{24^k}$$
$$\equiv\begin{cases}(\frac{p}{3})(4x^2-2p)\pmod{p^2} & \text{if } p\equiv 1,3\pmod 8\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8.\end{cases}\tag{4.4}$$

Conjecture 4.2. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(12k+5)S_k(1,7)(-1)^k28^{n-1-k}\in\mathbb{Z}^+,\tag{4.5}$$

and this number is odd if and only if n is a power of two.

(ii) *Let $p\neq 7$ be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{12k+5}{(-28)^k}S_k(1,7)\equiv 5p\left(\frac{p}{7}\right)\pmod{p^2},\tag{4.6}$$

and moreover

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{12k+5}{(-28)^k}S_k(1,7)-p\left(\frac{p}{7}\right)\sum_{k=0}^{n-1}\frac{12k+5}{(-28)^k}S_k(1,7)\right)\in\mathbb{Z}_p\tag{4.7}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime $p\neq 2,7$, we have*

$$\sum_{k=0}^{p-1}\frac{S_k(1,7)}{(-28)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=x^2+21y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{7})=1\ \&\ 2p=x^2+21y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=1\ \&\ p=3x^2+7y^2,\\ 2p-6x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ 2p=3x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-21}{p})=-1,\end{cases}\tag{4.8}$$

where x and y are integers.

Conjecture 4.3. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(84k+29)S_k(1,-20)80^{n-1-k}\in\mathbb{Z}^+,\tag{4.9}$$

and this number is odd if and only if n is a power of two.

(ii) *Let p be an odd prime with $p\neq 5$. Then*

$$\sum_{k=0}^{p-1}\frac{84k+29}{80^k}S_k(1,-20)\equiv p\left(2\left(\frac{5}{p}\right)+27\left(\frac{-15}{p}\right)\right)\pmod{p^2}.\tag{4.10}$$

If $p\equiv 1\pmod 3$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{84k+29}{80^k}S_k(1,-20)-p\left(\frac{p}{5}\right)\sum_{k=0}^{n-1}\frac{84k+29}{80^k}S_k(1,-20)\right)\in\mathbb{Z}_p\tag{4.11}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime* $p\neq 2,5$, *we have*

$$\sum_{k=0}^{p-1}\frac{S_k(1,-20)}{80^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1\ \&\ p=x^2+30y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+15y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=3x^2+10y^2,\\ 20x^2-2p\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=5x^2+6y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases}\tag{4.12}$$

where x *and* y *are integers.*

Conjecture 4.4. (i) *For any* $n\in\mathbb{Z}^+$, *we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(3k+1)(-1)^k100^{n-1-k}S_k(1,25)\in\mathbb{Z}^+.\tag{4.13}$$

(ii) *Let* $p\neq 5$ *be an odd prime. Then*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{3k+1}{(-100)^k}S_k(1,25)-p\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\frac{3k+1}{(-100)^k}S_k(1,25)\right)\in\mathbb{Z}_p\tag{4.14}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime* $p>3$ *with* $p\neq 11$, *we have*

$$\sum_{k=0}^{p-1}\frac{S_k(1,25)}{(-100)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{11})=1\ \&\ p=x^2+33y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{11})=-1\ \&\ 2p=x^2+33y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } (\frac{p}{11})=1,\ (\frac{-1}{p})=(\frac{p}{3})=-1\ \&\ p=3x^2+11y^2,\\ 2p-6x^2\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-1}{p})=(\frac{p}{11})=-1\ \&\ 2p=3x^2+11y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-33}{p})=-1,\end{cases}\tag{4.15}$$

where x *and* y *are integers.*

Conjecture 4.5. (i) *For any* $n\in\mathbb{Z}^+$, *we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(228k+67)S_k(1,-56)224^{n-1-k}\in\mathbb{Z}^+,\tag{4.16}$$

and this number is odd if and only if n *is a power of two.*

(ii) *Let* p *be an odd prime with* $p\neq 7$. *Then*

$$\sum_{k=0}^{p-1}\frac{228k+67}{224^k}S_k(1,-56)\equiv p\left(65\left(\frac{-7}{p}\right)+2\left(\frac{14}{p}\right)\right)\pmod{p^2}.\tag{4.17}$$

If $p\equiv 1,3\pmod 8$, *then*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{228k+67}{224^k}S_k(1,-56)-p\left(\frac{p}{7}\right)\sum_{k=0}^{n-1}\frac{228k+67}{224^k}S_k(1,-56)\right)\in\mathbb{Z}_p\tag{4.18}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime* $p\ne 2,7$, *we have*

$$\sum_{k=0}^{p-1}\frac{S_k(1,-56)}{224^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=x^2+42y^2,\\ 8x^2-2p\pmod{p^2}&\text{if }(\frac{p}{7})=1,\ (\frac{-2}{p})=(\frac{p}{3})=-1\ \&\ p=2x^2+21y^2,\\ 2p-12x^2\pmod{p^2}&\text{if }(\frac{-2}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ p=3x^2+14y^2,\\ 2p-24x^2\pmod{p^2}&\text{if }(\frac{p}{3})=1,\ (\frac{-2}{p})=(\frac{p}{7})=-1\ \&\ p=6x^2+7y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-42}{p})=-1,\end{cases}\tag{4.19}$$

where x *and* y *are integers.*

Conjecture 4.6. (i) *For any* $n\in\mathbb{Z}^+$, *we have*

$$\frac1n\sum_{k=0}^{n-1}(399k+101)(-1)^k676^{n-1-k}S_k(1,169)\in\mathbb{Z}^+.\tag{4.20}$$

(ii) *Let* $p\ne 13$ *be an odd prime. Then*

$$\sum_{k=0}^{pn-1}\frac{399k+101}{(-676)^k}S_k(1,169)-p\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\frac{399k+101}{(-676)^k}S_k(1,169)\tag{4.21}$$

divided by $(pn)^2$ *is a* p*-adic integer for any* $n\in\mathbb{Z}^+$.

(iii) *For any prime* $p>3$ *with* $p\ne 19$, *we have*

$$\sum_{k=0}^{p-1}\frac{S_k(1,169)}{(-676)^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{19})=1\ \&\ p=x^2+57y^2,\\ 2x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{19})=-1\ \&\ 2p=x^2+57y^2,\\ 2p-12x^2\pmod{p^2}&\text{if }(\frac{p}{3})=1,\ (\frac{-1}{p})=(\frac{p}{19})=-1\ \&\ p=3x^2+19y^2,\\ 2p-6x^2\pmod{p^2}&\text{if }(\frac{p}{19})=1,\ (\frac{-1}{p})=(\frac{p}{3})=-1\ \&\ 2p=3x^2+19y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-57}{p})=-1,\end{cases}\tag{4.22}$$

where x *and* y *are integers.*

Conjecture 4.7. (i) *For any* $n\in\mathbb{Z}^+$, *we have*

$$\frac1n\sum_{k=0}^{n-1}(2604k+563)S_k(1,-650)2600^{n-1-k}\in\mathbb{Z}^+,\tag{4.23}$$

and this number is odd if and only if $n\in\{2^a:\ a\in\mathbb{N}\}$.

(ii) *Let* p *be an odd prime with* $p\ne 5,13$. *Then*

$$\sum_{k=0}^{p-1}\frac{2604k+563}{2600^k}S_k(1,-650)\equiv p\left(561\left(\frac{-39}{p}\right)+2\left(\frac{26}{p}\right)\right)\pmod{p^2}.\tag{4.24}$$

If $(\frac{-6}{p})=1$, *then*

$$\sum_{k=0}^{pn-1}\frac{2604k+563}{2600^k}S_k(1,-650)-p\left(\frac{26}{p}\right)\sum_{k=0}^{n-1}\frac{2604k+563}{2600^k}S_k(1,-650)\tag{4.25}$$

divided by $(pn)^2$ *is a* p*-adic integer for any* $n\in\mathbb{Z}^+$.

(iii) *For any odd prime* $p\neq 5,13$, *we have*

$$\sum_{k=0}^{p-1}\frac{S_k(1,-650)}{2600^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{13})=1\ \&\ p=x^2+78y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{13})=-1\ \&\ p=2x^2+39y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } (\frac{p}{13})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=3x^2+26y^2,\\ 2p-24x^2\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{13})=-1\ \&\ p=6x^2+13y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-78}{p})=-1,\end{cases}\tag{4.26}$$

where x *and* y *are integers.*

Conjecture 4.8. (i) *For any* $n\in\mathbb{Z}^+$, *we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(39468k+7817)(-1)^k6076^{n-1-k}S_k(1,1519)\in\mathbb{Z}^+,\tag{4.27}$$

and this number is odd if and only if $n\in\{2^a:\ a\in\mathbb{N}\}$.

(ii) *Let* $p\neq 7,31$ *be an odd prime. Then*

$$\sum_{k=0}^{pn-1}\frac{39468k+7817}{(-6076)^k}S_k(1,1519)-p\left(\frac{-31}{p}\right)\sum_{k=0}^{n-1}\frac{39468k+7817}{(-6076)^k}S_k(1,1519)\tag{4.28}$$

divided by $(pn)^2$ *is a* p*-adic integer for any* $n\in\mathbb{Z}^+$.

(iii) *For any prime* $p>3$ *with* $p\neq 7,31$, *we have*

$$\sum_{k=0}^{p-1}\frac{S_k(1,1519)}{(-6076)^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{31})=1\ \&\ p=x^2+93y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{p}{31})=1,\ (\frac{-1}{p})=(\frac{p}{3})=-1\ \&\ 2p=x^2+93y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-1}{p})=(\frac{p}{31})=-1\ \&\ p=3x^2+31y^2,\\ 2p-6x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{31})=-1\ \&\ 2p=3x^2+31y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-93}{p})=-1,\end{cases}\tag{4.29}$$

where x *and* y *are integers.*

Conjecture 4.9. (i) *For any* $n\in\mathbb{Z}^+$, *we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(41667k+7879)9800^{n-1-k}S_k(1,-2450)\in\mathbb{Z}^+.\tag{4.30}$$

(ii) *Let* $p\neq 5,7$ *be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{41667k+7879}{9800^k}S_k(1,-2450)\equiv\frac{p}{2}\left(15741\left(\frac{-6}{p}\right)+17\left(\frac{2}{p}\right)\right)\pmod{p^2}.\tag{4.31}$$

If $p\equiv 1\pmod 3$, *then*

$$\sum_{k=0}^{pn-1}\frac{41667k+7879}{9800^k}S_k(1,-2450)-p\left(\frac{2}{p}\right)\sum_{k=0}^{n-1}\frac{41667k+7879}{9800^k}S_k(1,-2450)\tag{4.32}$$

divided by $(pn)^2$ *is a* p*-adic integer for any* $n\in\mathbb{Z}^+$.

(iii) *For any prime* $p>7$ *with* $p\neq 17$, *we have*

$$\sum_{k=0}^{p-1}\frac{S_k(1,-2450)}{9800^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{2}{p})=(\frac{p}{3})=(\frac{p}{17})=1\ \&\ p=x^2+102y^2,\\8x^2-2p\pmod{p^2}&\text{if }(\frac{p}{17})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=2x^2+51y^2,\\2p-12x^2\pmod{p^2}&\text{if }(\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{17})=-1\ \&\ p=3x^2+34y^2,\\2p-24x^2\pmod{p^2}&\text{if }(\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{17})=-1\ \&\ p=6x^2+17y^2,\\0\pmod{p^2}&\text{if }(\frac{-102}{p})=-1,\end{cases}\tag{4.33}$$

where x *and* y *are integers.*

Conjecture 4.10. (i) *For any* $n\in\mathbb{Z}^+$, *we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(74613k+10711)(-1)^k530^{2(n-1-k)}S_k(1,265^2)\in\mathbb{Z}^+.\tag{4.34}$$

(ii) *Let* $p\neq 5,53$ *be an odd prime. Then*

$$\sum_{k=0}^{pn-1}\frac{74613k+10711}{(-530^2)^k}S_k(1,265^2)-p\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\frac{74613k+10711}{(-530^2)^k}S_k(1,265^2)\tag{4.35}$$

divided by $(pn)^2$ *is a* p*-adic integer for any* $n\in\mathbb{Z}^+$.

(iii) *For any prime* $p>5$ *with* $p\neq 59$, *we have*

$$\sum_{k=0}^{p-1}\frac{S_k(1,265^2)}{(-530^2)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{59})=1\ \&\ p=x^2+177y^2,\\2x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{59})=-1\ \&\ 2p=x^2+177y^2,\\2p-12x^2\pmod{p^2}&\text{if }(\frac{p}{59})=1,\ (\frac{-1}{p})=(\frac{p}{3})=-1\ \&\ p=3x^2+59y^2,\\2p-6x^2\pmod{p^2}&\text{if }(\frac{p}{3})=1,\ (\frac{-1}{p})=(\frac{p}{59})=-1\ \&\ 2p=3x^2+59y^2,\\0\pmod{p^2}&\text{if }(\frac{-177}{p})=-1,\end{cases}\tag{4.36}$$

where x *and* y *are integers.*

Conjecture 4.11. *For any odd prime* p,

$$\sum_{k=0}^{p-1}\frac{S_k}{(-4)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }12\mid p-1\ \&\ p=x^2+y^2\ (x,y\in\mathbb{Z}\ \&\ 3\nmid x),\\4xy\pmod{p^2}&\text{if }12\mid p-5\ \&\ p=x^2+y^2\ (x,y\in\mathbb{Z}\ \&\ 3\mid x-y),\\0\pmod{p^2}&\text{if }p\equiv 3\pmod 4.\end{cases}\tag{4.37}$$

Also, for any prime $p\equiv 1\pmod 4$ *we have*

$$\sum_{k=0}^{p-1}(8k+5)\frac{S_k}{(-4)^k}\equiv 4p\pmod{p^2}.\tag{4.38}$$

Conjecture 4.12. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)4^{n-1-k}S_k(1,-1)\in\mathbb{Z},\tag{4.39}$$

and this number is odd if and only if n is a power of two.

(ii) *For any odd prime p and positive integer n, we have*

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}\frac{4k+3}{4^k}S_k(1,-1)-p\sum_{k=0}^{n-1}\frac{4k+3}{4^k}S_k(1,-1)\bigg)\in\mathbb{Z}_p.\tag{4.40}$$

(iii) *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{S_k(1,-1)}{4^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p\equiv 1,9\pmod{20}\ \&\ p=x^2+5y^2\ (x,y\in\mathbb{Z}),\\2x^2-2p\pmod{p^2}&\text{if } p\equiv 3,7\pmod{20}\ \&\ 2p=x^2+5y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if } (\frac{-5}{p})=-1.\end{cases}\tag{4.41}$$

Conjecture 4.13. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(33k+25)S_k(1,-6)(-6)^{n-1-k}\in\mathbb{Z},\tag{4.42}$$

and this number is odd if and only if n is a power of two.

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{33k+25}{(-6)^k}S_k(1,-6)\equiv p\left(35-10\left(\frac{3}{p}\right)\right)\pmod{p^2}.\tag{4.43}$$

If $p\equiv\pm1\pmod{12}$, then

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}\frac{33k+25}{(-6)^k}S_k(1,-6)-p\sum_{k=0}^{n-1}\frac{33k+25}{(-6)^k}S_k(1,-6)\bigg)\in\mathbb{Z}_p\tag{4.44}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{S_k(1,-6)}{(-6)^k}\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2}&\text{if } p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if } p\equiv 2\pmod 3.\end{cases}\tag{4.45}$$

Conjecture 4.14. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$n\ \Bigg|\ \sum_{k=0}^{n-1}(18k+13)S_k(2,9)8^{n-1-k}.\tag{4.46}$$

(ii) *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{18k+13}{8^k}S_k(2,9)\equiv p\left(1+12\left(\frac{p}{3}\right)\right)\pmod{p^2}.\tag{4.47}$$

If $p\equiv 1\pmod 3$, *then*

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}\frac{18k+13}{8^k}S_k(2,9)-p\sum_{k=0}^{n-1}\frac{18k+13}{8^k}S_k(2,9)\bigg)\in\mathbb{Z}_p \tag{4.48}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime* $p>3$, *we have*

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{S_k(1,-2)}{8^k}&\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{S_k(2,9)}{8^k}\\ &\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,7\pmod{24}\ \&\ p=x^2+6y^2\\ 8x^2-2p\pmod{p^2} & \text{if } p\equiv 5,11\pmod{24}\ \&\ p=2x^2+3y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-6}{p})=-1,\end{cases}\end{aligned} \tag{4.49}$$

where x *and* y *are integers.*

Conjecture 4.15. *Let* p *be an odd prime with* $p\neq 5$. *Then*

$$\sum_{k=0}^{p-1}\frac{S_k(3,1)}{4^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,9\pmod{20}\ \&\ p=x^2+5y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } p\equiv 3,7\pmod{20}\ \&\ 2p=x^2+5y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 11,13,17,19\pmod{20},\end{cases} \tag{4.50}$$

where x *and* y *are integers. If* $(\frac{-5}{p})=1$, *then*

$$\sum_{k=0}^{p-1}\frac{40k+29}{4^k}S_k(3,1)\equiv 18p\pmod{p^2}.$$

Remark 4.1. We also have some similar conjectures involving

$$\sum_{k=0}^{p-1}\frac{S_k(5,4)}{4^k},\ \sum_{k=0}^{p-1}\frac{S_k(4,-5)}{4^k},\ \sum_{k=0}^{p-1}\frac{S_k(7,6)}{6^k},$$
$$\sum_{k=0}^{p-1}\frac{S_k(10,-2)}{32^k},\ \sum_{k=0}^{p-1}\frac{S_k(14,9)}{72^k},\ \sum_{k=0}^{p-1}\frac{S_k(19,9)}{36^k}$$

modulo p^2, where p is a prime greater than 3.

Motivated by Theorem 2.6, we pose the following general conjecture.

Conjecture 4.16. *For any odd prime* p *and integer* $m\not\equiv 0\pmod p$, *we have*

$$\sum_{k=0}^{p-1}\frac{S_k(4,-m)}{m^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}f_k}{m^k}\pmod{p^2}. \tag{4.51}$$

and

$$\frac{m+16}{2}\sum_{k=0}^{p-1}\frac{kS_k(4,-m)}{m^k}-\sum_{k=0}^{p-1}((m+4)k-4)\frac{\binom{2k}{k}f_k}{m^k}\equiv 4p\left(\frac{m}{p}\right)\pmod{p^2}. \tag{4.52}$$

Remark 4.2. We have checked this conjecture via `Mathematica`. In view of the proof of Theorem 2.6, both (4.51) and (4.52) hold modulo p.

5. Series for $1/\pi$ involving $T_n(b,c)$ and $Z_n=\sum_{k=0}^n\binom nk\binom{2k}k\binom{2(n-k)}{n-k}$

The numbers

$$Z_n:=\sum_{k=0}^n\binom nk\binom{2k}k\binom{2(n-k)}{n-k}\ \ (n=0,1,2,\ldots)$$

were first introduced by D. Zagier in his paper [51] the preprint of which was released in 2002. Thus we name such numbers as *Zagier numbers*. As pointed out by the author [41, Remark 4.3], for any $n\in\mathbb N$ the number 2^nZ_n coincides with the so-called CLF (Catalan-Larcombe-French) number

$$\mathcal P_n:=2^n\sum_{k=1}^{\lfloor n/2\rfloor}\binom n{2k}\binom{2k}k^24^{n-2k}=\sum_{k=0}^n\frac{\binom{2k}k^2\binom{2(n-k)}{n-k}^2}{\binom nk}.$$

Let p be an odd prime. For any $k=0,\ldots,p-1$, we have

$$\mathcal P_k\equiv\left(\frac{-1}p\right)128^k\mathcal P_{p-1-k}\pmod p$$

by F. Jarvis and H.A. Verrill [24, Corollary 2.2], and hence

$$Z_k=\frac{\mathcal P_k}{2^k}\equiv\left(\frac{-1}p\right)64^k(2^{p-1-k}Z_{p-1-k})\equiv\left(\frac{-1}p\right)32^kZ_{p-1-k}\pmod p.$$

Combining this with Remark 1.3(ii), we see that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{Z_kT_k(b,c)}{m^k}&\equiv\left(\frac{4c-b^2}p\right)\sum_{k=0}^{p-1}\left(\frac{32(b^2-4c)}m\right)^kZ_{p-1-k}T_{p-1-k}(b,c)\\&\equiv\left(\frac{4c-b^2}p\right)\sum_{k=0}^{p-1}\frac{Z_kT_k(b,c)}{(32(b^2-4c)/m)^k}\pmod p\end{aligned}$$

for any $b,c,m\in\mathbb Z$ with $p\nmid(b^2-4c)m$.

J. Wan and Zudilin [49] obtained the following irrational series for $1/\pi$ involving the Legendre polynomials and the Zagier numbers:

$$\sum_{k=0}^\infty(15k+4-2\sqrt6)Z_kP_k\left(\frac{24-\sqrt6}{15\sqrt2}\right)\left(\frac{4-\sqrt6}{10\sqrt3}\right)^k=\frac6\pi(7+3\sqrt6).$$

Via our congruence approach (including Conjecture 1.4), we find 24 rational series for $1/\pi$ involving $T_n(b,c)$ and the Zagier numbers. Theorem 1 of [49] might be helpful to solve some of them.

Conjecture 5.1. *We have the following identities for $1/\pi$.*

$$\sum_{k=1}^\infty\frac{5k+1}{32^k}T_kZ_k=\frac{8(2+\sqrt5)}{3\pi},\tag{5.1}$$

$$\sum_{k=0}^\infty\frac{21k+5}{(-252)^k}T_k(1,16)Z_k=\frac{6\sqrt7}\pi,\tag{5.2}$$

$$\sum_{k=0}^\infty\frac{3k+1}{36^k}T_k(1,-2)Z_k=\frac3\pi,\tag{5.3}$$

$$\sum_{k=0}^\infty\frac{k}{192^k}T_k(14,1)Z_k=\frac{8}{3\pi},\tag{5.4}$$

$$\sum_{k=0}^\infty\frac{30k+11}{(-192)^k}T_k(14,1)Z_k=\frac{12}{\pi},\tag{5.5}$$

$$\sum_{k=0}^\infty\frac{15k+1}{480^k}T_k(22,1)Z_k=\frac{6\sqrt{10}}{\pi},\tag{5.6}$$

$$\sum_{k=0}^\infty\frac{7k+2}{(-672)^k}T_k(26,1)Z_k=\frac{2\sqrt{21}}{3\pi},\tag{5.7}$$

$$\sum_{k=0}^\infty\frac{21k+2}{1152^k}T_k(34,1)Z_k=\frac{18}{\pi},\tag{5.8}$$

$$\sum_{k=0}^\infty\frac{30k-7}{640^k}T_k(62,1)Z_k=\frac{160}{\pi},\tag{5.9}$$

$$\sum_{k=0}^\infty\frac{195k+34}{(-9600)^k}T_k(98,1)Z_k=\frac{80}{\pi},\tag{5.10}$$

$$\sum_{k=0}^\infty\frac{195k+22}{11232^k}T_k(106,1)Z_k=\frac{27\sqrt{13}}{\pi},\tag{5.11}$$

$$\sum_{k=0}^\infty\frac{42k+17}{(-1440)^k}T_k(142,1)Z_k=\frac{33}{\sqrt5\,\pi},\tag{5.12}$$

$$\sum_{k=0}^\infty\frac{2k-1}{1792^k}T_k(194,1)Z_k=\frac{56}{3\pi},\tag{5.13}$$

$$\sum_{k=0}^\infty\frac{1785k+254}{(-37632)^k}T_k(194,1)Z_k=\frac{672}{\pi},\tag{5.14}$$

$$\sum_{k=0}^\infty\frac{210k+23}{40800^k}T_k(202,1)Z_k=\frac{15\sqrt{34}}{\pi},\tag{5.15}$$

$$\sum_{k=0}^\infty\frac{210k-1}{4608^k}T_k(254,1)Z_k=\frac{288}{\pi},\tag{5.16}$$

$$\sum_{k=0}^\infty\frac{21k-5}{5600^k}T_k(502,1)Z_k=\frac{105}{\sqrt2\,\pi},\tag{5.17}$$

$$\sum_{k=0}^\infty\frac{7410k+1849}{(-36992)^k}T_k(1154,1)Z_k=\frac{2992}{\pi},\tag{5.18}$$

$$\sum_{k=0}^\infty\frac{1326k+101}{57760^k}T_k(1442,1)Z_k=\frac{2014}{\sqrt5\,\pi},\tag{5.19}$$

$$\sum_{k=0}^\infty\frac{78k-131}{20800^k}T_k(2498,1)Z_k=\frac{2600}{\pi},\tag{5.20}$$

$$\sum_{k=0}^\infty\frac{62985k+11363}{(-394272)^k}T_k(5474,1)Z_k=\frac{7659\sqrt{10}}{\pi},\tag{5.21}$$

$$\sum_{k=0}^{\infty}\frac{358530k+33883}{486720^k}T_k(6082,1)Z_k=\frac{176280}{\pi},\tag{5.22}$$

$$\sum_{k=0}^{\infty}\frac{510k-1523}{78400^k}T_k(9602,1)Z_k=\frac{33320}{\pi},\tag{5.23}$$

$$\sum_{k=0}^{\infty}\frac{570k-457}{93600^k}T_k(10402,1)Z_k=\frac{1590\sqrt{13}}{\pi}.\tag{5.24}$$

Below we present some conjectures on congruences related to (5.1), (5.2), (5.4) and (5.9).

Conjecture 5.2. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$n\ \Big|\ \sum_{k=0}^{n-1}(5k+1)T_kZ_k32^{n-1-k}.\tag{5.25}$$

(ii) *Let p be an odd prime with $p\neq5$. Then*

$$\sum_{k=0}^{p-1}\frac{5k+1}{32^k}T_kZ_k\equiv\frac{p}{3}\left(5\left(\frac{-5}{p}\right)-2\left(\frac{-1}{p}\right)\right)\pmod{p^2}.\tag{5.26}$$

If $p\equiv\pm1\pmod 5$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{5k+1}{32^k}T_kZ_k-p\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\frac{5k+1}{32^k}T_kZ_k\right)\in\mathbb{Z}_p\tag{5.27}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime $p>5$, we have*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{T_kZ_k}{32^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv1,4\pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 12x^2-2p\pmod{p^2} & \text{if } p\equiv2,8\pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-15}{p})=-1.\end{cases}\tag{5.28}$$

Conjecture 5.3. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(21k+5)T_k(1,16)Z_k252^{n-1-k}\in\mathbb{Z}^+.\tag{5.29}$$

(ii) *Let $p>3$ be a prime with $p\neq7$. Then*

$$\sum_{k=0}^{p-1}\frac{21k+5}{(-252)^k}T_k(1,16)Z_k\equiv\frac{p}{3}\left(16\left(\frac{-7}{p}\right)-\left(\frac{-1}{p}\right)\right)\pmod{p^2}.\tag{5.30}$$

If $(\frac{7}{p})=1$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{21k+5}{(-252)^k}T_k(1,16)Z_k-p\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\frac{21k+5}{(-252)^k}T_k(1,16)Z_k\right)\in\mathbb{Z}_p\tag{5.31}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime $p>3$ with $p\ne 7$, we have*

$$\sum_{k=0}^{p-1}\frac{T_k(1,16)Z_k}{(-252)^k}\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 1,2,4\ (\mathrm{mod}\ 7)\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 3,5,6\ (\mathrm{mod}\ 7).\end{cases} \tag{5.32}$$

Conjecture 5.4. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$n\ \Big|\ \sum_{k=0}^{n-1}kT_k(14,1)Z_k192^{n-1-k}. \tag{5.33}$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{k}{192^k}T_k(14,1)Z_k\equiv\frac{p}{9}\left(\left(\frac{-1}{p}\right)-\left(\frac{2}{p}\right)\right)\ (\mathrm{mod}\ p^2). \tag{5.34}$$

If $p\equiv 1,3\ (\mathrm{mod}\ 8)$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{k}{192^k}T_k(14,1)Z_k-p\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\frac{k}{192^k}T_k(14,1)\right)\in\mathbb{Z}_p \tag{5.35}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime $p>3$, we have*

$$\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(14,1)Z_k}{192^k}\equiv\begin{cases}4x^2-2p\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 1,3\ (\mathrm{mod}\ 8)\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\ (\mathrm{mod}\ p^2) & \text{if } p\equiv 5,7\ (\mathrm{mod}\ 8).\end{cases} \tag{5.36}$$

Conjecture 5.5. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$n\ \Big|\ \sum_{k=0}^{n-1}(30k-7)T_k(62,1)Z_k640^{n-1-k}. \tag{5.37}$$

(ii) *Let p be an odd prime with $p\ne 5$. Then*

$$\sum_{k=0}^{p-1}\frac{30k-7}{640^k}T_k(62,1)Z_k\equiv p\left(2\left(\frac{-1}{p}\right)-9\left(\frac{15}{p}\right)\right)\ (\mathrm{mod}\ p^2). \tag{5.38}$$

If $(\frac{-15}{p})=1$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{30k-7}{640^k}T_k(62,1)Z_k-p\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\frac{30k-7}{640^k}T_k(62,1)Z_k\right)\in\mathbb{Z}_p \tag{5.39}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime $p>5$, we have*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(62,1)Z_k}{640^k}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1\ \&\ p=x^2+30y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+15y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=3x^2+10y^2,\\ 20x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=5x^2+6y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases} \tag{5.40}$$

where x and y are integers.

6. Series for $1/\pi$ involving $T_k(b,c)$ and the Franel numbers

Sun [36, 37] obtained some supercongruences involving the Franel numbers $f_n=\sum_{k=0}^n\binom{n}{k}^3$ $(n\in\mathbb{N})$. M. Rogers and A. Straub [30] confirmed the 520-series for $1/\pi$ involving Franel polynomials conjectured by Sun [34].

Let p be an odd prime. By [24, Lemma 2.6], we have $f_k\equiv(-8)^kf_{p-1-k} \pmod p$ for each $k=0,\ldots,p-1$. Combining this with Remark 1.3(ii), we see that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{f_kT_k(b,c)}{m^k}&\equiv\left(\frac{b^2-4c}{p}\right)\sum_{k=0}^{p-1}\left(\frac{-8(b^2-4c)}{m}\right)^kf_{p-1-k}T_{p-1-k}(b,c)\\&\equiv\left(\frac{b^2-4c}{p}\right)\sum_{k=0}^{p-1}\frac{f_kT_k(b,c)}{(8(4c-b^2)/m)^k} \pmod p\end{aligned}$$

for any $b,c,m\in\mathbb{Z}$ with $p\nmid(b^2-4c)m$.

Wan and Zudilin [49] deduced the following irrational series for $1/\pi$ involving the Legendre polynomials and the Franel numbers:

$$\sum_{k=0}^{\infty}(18k+7-2\sqrt3)f_kP_k\left(\frac{1+\sqrt3}{\sqrt6}\right)\left(\frac{2-\sqrt3}{2\sqrt6}\right)^k=\frac{27+11\sqrt3}{\sqrt2\,\pi}.$$

Via our congruence approach (including Conjecture 1.4), we find 12 rational series for $1/\pi$ involving $T_n(b,c)$ and the Franel numbers; Theorem 1 of [49] might be helpful to solve some of them.

Conjecture 6.1. *We have*

$$\sum_{k=0}^{\infty}\frac{3k+1}{(-48)^k}f_kT_k(4,-2)=\frac{4\sqrt2}{3\pi}, \tag{6.1}$$

$$\sum_{k=0}^{\infty}\frac{99k+23}{(-288)^k}f_kT_k(8,-2)=\frac{39\sqrt2}{\pi}, \tag{6.2}$$

$$\sum_{k=0}^{\infty}\frac{105k+17}{480^k}f_kT_k(8,1)=\frac{92\sqrt5}{3\pi}, \tag{6.3}$$

$$\sum_{k=0}^{\infty}\frac{45k-2}{441^k}f_kT_k(47,1)=\frac{483\sqrt5}{4\pi}, \tag{6.4}$$

$$\sum_{k=0}^\infty\frac{165k+46}{(-2352)^k}f_kT_k(194,1)=\frac{112\sqrt5}{3\pi},\tag{6.5}$$

$$\sum_{k=0}^\infty\frac{42k+5}{11616^k}f_kT_k(482,1)=\frac{374\sqrt2}{15\pi},\tag{6.6}$$

$$\sum_{k=0}^\infty\frac{990k+31}{11200^k}f_kT_k(898,1)=\frac{680\sqrt7}{\pi},\tag{6.7}$$

$$\sum_{k=0}^\infty\frac{585k+172}{(-13552)^k}f_kT_k(1454,1)=\frac{110\sqrt7}{\pi},\tag{6.8}$$

$$\sum_{k=0}^\infty\frac{90k+11}{101568^k}f_kT_k(2114,1)=\frac{92\sqrt{15}}{7\pi},\tag{6.9}$$

$$\sum_{k=0}^\infty\frac{94185k+17014}{(-105984)^k}f_kT_k(2302,1)=\frac{8520\sqrt{23}}{\pi},\tag{6.10}$$

$$\sum_{k=0}^\infty\frac{5355k+1381}{(-61952)^k}f_kT_k(4354,1)=\frac{968\sqrt7}{\pi},\tag{6.11}$$

$$\sum_{k=0}^\infty\frac{210k+23}{475904^k}f_kT_k(16898,1)=\frac{2912\sqrt{231}}{297\pi}.\tag{6.12}$$

We now present a conjecture on congruence related to (6.3).

Conjecture 6.2. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac1n\sum_{k=0}^{n-1}(105k+17)480^{n-1-k}f_kT_k(8,1)\in\mathbb{Z}^+.\tag{6.13}$$

(ii) *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{105k+17}{480^k}f_kT_k(8,1)\equiv\frac p9\left(161\left(\frac{-5}{p}\right)-8\right)\pmod{p^2}.\tag{6.14}$$

If $(\frac{-5}{p})=1$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{105k+17}{480^k}f_kT_k(8,1)-p\sum_{k=0}^{n-1}\frac{105k+17}{480^k}f_kT_k(8,1)\right)\in\mathbb{Z}_p\tag{6.15}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime $p>5$, we have*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{f_kT_k(8,1)}{480^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1,4\pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\2p-12x^2\pmod{p^2}&\text{if }p\equiv2,8\pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }(\frac{-15}{p})=-1.\end{cases}\tag{6.16}$$

Remark 6.1. This conjecture was formulated by the author on Oct. 25, 2019.

Conjecture 6.3. *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{4n}\sum_{k=0}^{n-1}(-1)^{n-1-k}(105k+88)f_kT_k(8,1)\in\mathbb{Z}^+. \tag{6.17}$$

(ii) *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}(-1)^k(105k+88)f_kT_k(8,1)\equiv\frac{8}{3}p\left(23\left(\frac{-3}{p}\right)+10\left(\frac{15}{p}\right)\right)\pmod{p^2}. \tag{6.18}$$

If $(\frac{-5}{p})=1$, then

$$\sum_{k=0}^{pn-1}(-1)^k(105k+88)f_kT_k(8,1)-p\left(\frac{p}{3}\right)\sum_{k=0}^{n-1}(-1)^k(105k+88)f_kT_k(8,1) \tag{6.19}$$

divided by $(pn)^2$ is a p-adic integer for any $n\in\mathbb{Z}^+$.

(iii) *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}(-1)^kf_kT_k(8,1)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv1,4\pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2\pmod{p^2} & \text{if } p\equiv2,8\pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-15}{p})=-1.\end{cases} \tag{6.20}$$

Remark 6.2. This conjecture is the dual of Conjecture 6.2.

The following conjecture is related to the identity (6.8).

Conjecture 6.4. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{2n}\sum_{k=0}^{n-1}(-1)^k(585k+172)13552^{n-1-k}f_kT_k(1454,1)\in\mathbb{Z}^+. \tag{6.21}$$

(ii) *Let $p>2$ be a prime with $p\neq7,11$. Then*

$$\sum_{k=0}^{p-1}\frac{585k+172}{(-13552)^k}f_kT_k(1454,1)\equiv\frac{p}{11}\left(1580\left(\frac{-7}{p}\right)+312\left(\frac{273}{p}\right)\right)\pmod{p^2}. \tag{6.22}$$

If $(\frac{-39}{p})=1$, then

$$\sum_{k=0}^{pn-1}\frac{585k+172}{(-13552)^k}f_kT_k(1454,1)-p\left(\frac{p}{7}\right)\sum_{k=0}^{n-1}\frac{585k+172}{(-13552)^k}f_kT_k(1454,1) \tag{6.23}$$

divided by $(pn)^2$ is a p-adic integer for any $n\in\mathbb{Z}^+$.

(iii) *Let $p>3$ be a prime with $p\neq 7,11,13$. Then*

$$\sum_{k=0}^{p-1}\frac{f_kT_k(1454,1)}{(-13552)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{7})=(\frac{p}{13})=1,\ p=x^2+273y^2,\\2x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{7})=1,\ (\frac{p}{3})=(\frac{p}{13})=-1,\ 2p=x^2+273y^2,\\2p-12x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=(\frac{p}{13})=1,\ p=3x^2+91y^2,\\2p-6x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{7})=(\frac{p}{13})=-1,\ 2p=3x^2+91y^2,\\28x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{13})=-1,\ (\frac{p}{3})=(\frac{p}{7})=1,\ p=7x^2+39y^2,\\14x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{7})=(\frac{p}{13})=1,\ 2p=7x^2+39y^2,\\52x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=1,\ (\frac{p}{7})=(\frac{p}{13})=-1,\ p=13x^2+21y^2,\\26x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{13})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1,\ 2p=13x^2+21y^2,\\0\pmod{p^2}&\text{if }(\frac{-273}{p})=-1,\end{cases}\tag{6.24}$$

where x and y are integers.

Remark 6.3. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-273})$ has class number 8.

The following conjecture is related to the identity (6.10).

Conjecture 6.5. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{2n}\sum_{k=0}^{n-1}(-1)^k(94185k+17014)105984^{n-1-k}f_kT_k(2302,1)\in\mathbb{Z}^+.\tag{6.25}$$

(ii) *Let $p>3$ be a prime with $p\neq 23$. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{94185k+17014}{(-105984)^k}f_kT_k(2302,1)\\\equiv&\frac{p}{16}\left(22659+249565\left(\frac{-23}{p}\right)\right)\pmod{p^2}.\end{aligned}\tag{6.26}$$

If $(\frac{p}{23})=1$, then

$$\sum_{k=0}^{pn-1}\frac{94185k+17014}{(-105984)^k}f_kT_k(2302,1)-p\sum_{k=0}^{n-1}\frac{94185k+17014}{(-105984)^k}f_kT_k(2302,1)\tag{6.27}$$

divided by $(pn)^2$ is a p-adic integer for any $n\in\mathbb{Z}^+$.

(iii) *Let $p>3$ be a prime with $p\neq 23$. Then*

$$\sum_{k=0}^{p-1}\frac{f_kT_k(2302,1)}{(-105984)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{23})=1,\ p=x^2+345y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{23})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ 2p=x^2+345y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=-1,\ (\frac{p}{3})=(\frac{p}{23})=1,\ p=3x^2+115y^2,\\ 6x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{5})=(\frac{p}{23})=1,\ 2p=3x^2+115y^2,\\ 2p-20x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=1,\ (\frac{p}{3})=(\frac{p}{23})=-1,\ p=5x^2+69y^2,\\ 2p-10x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=1,\ (\frac{p}{5})=(\frac{p}{23})=-1,\ 2p=5x^2+69y^2,\\ 2p-60x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{23})=-1,\ p=15x^2+23y^2,\\ 2p-30x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{23})=-1,\ (\frac{p}{3})=(\frac{p}{5})=1,\ 2p=15x^2+23y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-345}{p})=-1,\end{cases}\tag{6.28}$$

where x and y are integers.

Remark 6.4. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-345})$ has class number 8.

The following conjecture is related to the identity (6.12).

Conjecture 6.6. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(210k+23)475904^{n-1-k}f_kT_k(16898,1)\in\mathbb{Z}^+.\tag{6.29}$$

(ii) *Let p be an odd prime with $p\neq 11,13$. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{210k+23}{475904^k}f_kT_k(16898,1)\\ \equiv&\frac{p}{1287}\left(40621\left(\frac{-231}{p}\right)-11020\left(\frac{66}{p}\right)\right)\pmod{p^2}.\end{aligned}\tag{6.30}$$

If $(\frac{-14}{p})=1$, then

$$\begin{aligned}&\sum_{k=0}^{pn-1}\frac{210k+23}{475904^k}f_kT_k(16898,1)\\ &-p\left(\frac{66}{p}\right)\sum_{k=0}^{n-1}\frac{210k+23}{475904^k}f_kT_k(16898,1)\end{aligned}\tag{6.31}$$

divided by $(pn)^2$ is a p-adic integer for any $n\in\mathbb{Z}^+$.

(iii) *Let $p>3$ be a prime with $p\ne 7,11,13$. Then*

$$\sum_{k=0}^{p-1}\frac{f_kT_k(16898,1)}{475904^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p3)=(\frac p7)=(\frac p{11})=1\ \&\ p=x^2+462y^2,\\ 8x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p7)=1,\ (\frac p3)=(\frac p{11})=-1\ \&\ p=2x^2+231y^2,\\ 2p-12x^2\pmod{p^2}&\text{if }(\frac 2p)=(\frac p7)=-1,\ (\frac p3)=(\frac p{11})=1\ \&\ p=3x^2+154y^2,\\ 2p-24x^2\pmod{p^2}&\text{if }(\frac 2p)=(\frac p3)=(\frac p7)=(\frac p{11})=-1\ \&\ p=6x^2+77y^2,\\ 28x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p3)=1,\ (\frac p7)=(\frac p{11})=-1\ \&\ p=7x^2+66y^2,\\ 44x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p3)=-1,\ (\frac p7)=(\frac p{11})=1\ \&\ p=11x^2+42y^2,\\ 56x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p{11})=1,\ (\frac p3)=(\frac p7)=-1\ \&\ p=14x^2+33y^2,\\ 2p-84x^2\pmod{p^2}&\text{if }(\frac 2p)=(\frac p{11})=-1,\ (\frac p3)=(\frac p7)=1\ \&\ p=21x^2+22y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-462}p)=-1,\end{cases}\tag{6.32}$$

where x and y are integers.

Remark 6.5. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-462})$ has class number 8.

The identities (6.5), (6.6), (6.7), (6.9), (6.11) are related to the quadratic fields

$$\mathbb{Q}(\sqrt{-165}),\ \mathbb{Q}(\sqrt{-210}),\ \mathbb{Q}(\sqrt{-210}),\ \mathbb{Q}(\sqrt{-330}),\ \mathbb{Q}(\sqrt{-357})$$

(with class number 8) respectively. We also have conjectures on related congruences similar to Conjectures 6.4, 6.5 and 6.6.

7. Series for $1/\pi$ involving $T_n(b,c)$ and $g_n=\sum_{k=0}^n\binom nk^2\binom{2k}k$

For $n\in\mathbb{N}$ let

$$g_n:=\sum_{k=0}^n\binom nk^2\binom{2k}k.$$

It is known that $g_n=\sum_{k=0}^n\binom nk f_k$ for all $n\in\mathbb{N}$. See [43, 20, 26] for some congruences on polynomials related to these numbers.

Let $p>3$ be a prime. For any $k=0,\ldots,p-1$, we have

$$g_k\equiv\left(\frac{-3}{p}\right)9^kg_{p-1-k}\pmod p$$

by [24, Lemma 2.7(ii)]. Combining this with Remark 1.3(ii), we see that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{g_kT_k(b,c)}{m^k}&\equiv\left(\frac{-3(b^2-4c)}{p}\right)\sum_{k=0}^{p-1}\left(\frac{9(b^2-4c)}{m}\right)^kg_{p-1-k}T_{p-1-k}(b,c)\\&\equiv\left(\frac{3(4c-b^2)}{p}\right)\sum_{k=0}^{p-1}\frac{g_kT_k(b,c)}{(9(b^2-4c)/m)^k}\pmod p\end{aligned}$$

for any $b,c,m\in\mathbb{Z}$ with $p\nmid(b^2-4c)m$.

Wan and Zudilin [49] obtained the following irrational series for $1/\pi$ involving the Legendre polynomials and the sequence $(g_n)_{n\ge0}$:

$$\sum_{k=0}^{\infty}(22k+7-3\sqrt{3})g_kP_k\left(\frac{\sqrt{14\sqrt{3}-15}}{3}\right)\left(\frac{\sqrt{2\sqrt{3}-3}}{9}\right)^k=\frac{9}{2\pi}(9+4\sqrt{3}).$$

Using our congruence approach (including Conjecture 1.4), we find 12 rational series for $1/\pi$ involving $T_n(b,c)$ and g_n; Theorem 1 of [49] might be helpful to solve some of them.

Conjecture 7.1. *We have the following identities.*

$$\sum_{k=0}^{\infty}\frac{8k+3}{(-81)^k}g_kT_k(7,-8)=\frac{9\sqrt{3}}{4\pi},\tag{7.1}$$

$$\sum_{k=0}^{\infty}\frac{4k+1}{(-1089)^k}g_kT_k(31,-32)=\frac{33}{16\pi},\tag{7.2}$$

$$\sum_{k=0}^{\infty}\frac{7k-1}{540^k}g_kT_k(52,1)=\frac{30\sqrt{3}}{\pi},\tag{7.3}$$

$$\sum_{k=0}^{\infty}\frac{20k+3}{3969^k}g_kT_k(65,64)=\frac{63\sqrt{3}}{8\pi},\tag{7.4}$$

$$\sum_{k=0}^{\infty}\frac{280k+93}{(-1980)^k}g_kT_k(178,1)=\frac{20\sqrt{33}}{\pi},\tag{7.5}$$

$$\sum_{k=0}^{\infty}\frac{176k+15}{12600^k}g_kT_k(502,1)=\frac{25\sqrt{42}}{\pi},\tag{7.6}$$

$$\sum_{k=0}^{\infty}\frac{560k-23}{13068^k}g_kT_k(970,1)=\frac{693\sqrt{3}}{\pi},\tag{7.7}$$

$$\sum_{k=0}^{\infty}\frac{12880k+1353}{105840^k}g_kT_k(2158,1)=\frac{4410\sqrt{3}}{\pi},\tag{7.8}$$

$$\sum_{k=0}^{\infty}\frac{299k+59}{(-101430)^k}g_kT_k(2252,1)=\frac{735\sqrt{115}}{64\pi},\tag{7.9}$$

$$\sum_{k=0}^{\infty}\frac{385k+118}{(-53550)^k}g_kT_k(4048,1)=\frac{2415\sqrt{17}}{64\pi},\tag{7.10}$$

$$\sum_{k=0}^{\infty}\frac{385k-114}{114264^k}g_kT_k(10582,1)=\frac{15939\sqrt{3}}{16\pi},\tag{7.11}$$

$$\sum_{k=0}^{\infty}\frac{16016k+1273}{510300^k}g_kT_k(17498,1)=\frac{14175\sqrt{3}}{2\pi}.\tag{7.12}$$

Now we present a conjecture on congruences related to (7.6).

Conjecture 7.2. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{3n}\sum_{k=0}^{n-1}(176k+15)12600^{n-1-k}g_kT_k(502,1)\in\mathbb{Z}^+,\tag{7.13}$$

and this number is odd if and only if $n\in\{2^a:\ a\in\mathbb{N}\}$.

(ii) *Let* $p>7$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{176k+15}{12600^k}g_kT_k(502,1)\equiv p\left(26\left(\frac{-42}{p}\right)-11\left(\frac{21}{p}\right)\right)\pmod{p^2}. \tag{7.14}$$

If $p\equiv 1,3\pmod 8$, *then*

$$\sum_{k=0}^{pn-1}\frac{176k+15}{12600^k}g_kT_k(502,1)-p\left(\frac{21}{p}\right)\sum_{k=0}^{n-1}\frac{176k+15}{12600^k}g_kT_k(502,1) \tag{7.15}$$

divided by $(pn)^2$ *is a p-adic integer for any* $n\in\mathbb{Z}^+$.

(iii) *Let* $p>7$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{g_kT_k(502,1)}{12600^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{7})=1\ \&\ p=x^2+210y^2,\\ 2p-8x^2\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{7})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+105y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{3})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1\ \&\ p=3x^2+70y^2,\\ 2p-20x^2\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{7})=-1\ \&\ p=5x^2+42y^2,\\ 24x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{5})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ p=6x^2+35y^2,\\ 28x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{5})=-1,\ (\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=7x^2+30y^2,\\ 40x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=(\frac{p}{5})=1\ \&\ p=10x^2+21y^2,\\ 56x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{3})=-1,\ (\frac{p}{5})=(\frac{p}{7})=1\ \&\ p=14x^2+15y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-210}{p})=-1,\end{cases} \tag{7.16}$$

where x and y are integers.

Remark 7.1. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-210})$ has class number 8.

The following conjecture is related to the identity (7.8).

Conjecture 7.3. (i) *For any* $n\in\mathbb{Z}^+$, *we have*

$$\frac{1}{3n}\sum_{k=0}^{n-1}(12880k+1353)105840^{n-1-k}g_kT_k(2158,1)\in\mathbb{Z}^+, \tag{7.17}$$

and this number is odd if and only if $n\in\{2^a:\ a\in\mathbb{N}\}$.

(ii) *Let* $p>7$ *be a prime. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{12880k+1353}{105840^k}g_kT_k(2158,1)\\ \equiv&\frac{p}{2}\left(3419\left(\frac{-3}{p}\right)-713\left(\frac{5}{p}\right)\right)\pmod{p^2}.\end{aligned} \tag{7.18}$$

If $(\frac{p}{3})=(\frac{p}{5})$, *then*

$$\sum_{k=0}^{pn-1}\frac{12880k+1353}{105840^k}g_kT_k(2158,1)-p\left(\frac{p}{3}\right)\sum_{k=0}^{n-1}\frac{12880k+1353}{105840^k}g_kT_k(2158,1) \tag{7.19}$$

divided by $(pn)^2$ *is a* p*-adic integer for any* $n\in\mathbb{Z}^+$.

(iii) *Let* $p>11$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{g_kT_k(2158,1)}{105840^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{11})=1,\ p=x^2+330y^2,\\ 2p-8x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{11})=-1,\ p=2x^2+165y^2,\\ 2p-12x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{11})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ p=3x^2+110y^2,\\ 2p-20x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{5})=(\frac{p}{11})=1,\ p=5x^2+66y^2,\\ 24x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{11})=-1,\ (\frac{p}{3})=(\frac{p}{5})=1,\ p=6x^2+55y^2,\\ 40x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=1,\ (\frac{p}{5})=(\frac{p}{11})=-1,\ p=10x^2+33y^2,\\ 44x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{5})=1,\ (\frac{p}{3})=(\frac{p}{11})=-1,\ p=11x^2+30y^2,\\ 60x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{5})=-1,\ (\frac{p}{3})=(\frac{p}{11})=1,\ p=15x^2+22y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-330}{p})=-1,\end{cases}\tag{7.20}$$

where x *and* y *are integers.*

Remark 7.2. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-330})$ has class number 8.

Now we pose a conjecture related to the identity (7.10).

Conjecture 7.4. (i) *For any* $n\in\mathbb{Z}^+$, *we have*

$$\frac{1}{2n}\sum_{k=0}^{n-1}(-1)^k(385k+118)53550^{n-1-k}g_kT_k(4048,1)\in\mathbb{Z}^+.\tag{7.21}$$

(ii) *Let* $p>7$ *be a prime with* $p\neq17$. *Then*

$$\sum_{k=0}^{p-1}\frac{385k+118}{(-53550)^k}g_kT_k(4048,1)\equiv\frac{p}{320}\left(29279\left(\frac{-17}{p}\right)+8481\left(\frac{7}{p}\right)\right)\pmod{p^2}.\tag{7.22}$$

If $(\frac{p}{7})=(\frac{p}{17})$, *then*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{385k+118}{(-53550)^k}g_kT_k(4048,1)-p\left(\frac{7}{p}\right)\sum_{k=0}^{n-1}\frac{385k+118}{(-53550)^k}g_kT_k(4048,1)\right)\tag{7.23}$$

is a p*-adic integer for any* $n\in\mathbb{Z}^+$.

(iii) *Let* $p>7$ *be a prime with* $p\neq17$. *Then*

$$\sum_{k=0}^{p-1}\frac{g_kT_k(4048,1)}{(-53550)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{7})=(\frac{p}{17})=1,\ p=x^2+357y^2,\\ 2p-2x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{7})=(\frac{p}{17})=1,\ 2p=x^2+357y^2,\\ 12x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{7})=-1,\ p=3x^2+119y^2,\\ 2p-6x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=1,\ (\frac{p}{7})=(\frac{p}{17})=-1,\ 2p=3x^2+119y^2,\\ 28x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{17})=-1,\ (\frac{p}{3})=(\frac{p}{7})=1,\ p=7x^2+51y^2,\\ 2p-14x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{7})=1,\ (\frac{p}{3})=(\frac{p}{17})=-1,\ 2p=7x^2+51y^2,\\ 2p-68x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{17})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1,\ p=17x^2+21y^2,\\ 34x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=(\frac{p}{17})=1,\ 2p=17x^2+21y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-357}{p})=-1,\end{cases}\tag{7.24}$$

where x and y are integers.

Remark 7.3. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-357})$ has class number 8.

Now we pose a conjecture related to the identity (7.12).

Conjecture 7.5. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(16016k+1273)510300^{n-1-k}g_kT_k(17498,1)\in\mathbb{Z}^+,\tag{7.25}$$

and this number is odd if and only if $n\in\{2^a:\ a\in\mathbb{N}\}$.

(ii) *Let $p>7$ be a prime. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{16016k+1273}{510300^k}g_kT_k(17498,1)\\ \equiv&\frac{p}{3}\left(6527\left(\frac{-3}{p}\right)-2708\left(\frac{42}{p}\right)\right)\pmod{p^2}.\end{aligned}\tag{7.26}$$

If $(\frac{-14}{p})=1$, then

$$\begin{aligned}&\sum_{k=0}^{pn-1}\frac{16016k+1273}{510300^k}g_kT_k(17498,1)\\ &-p\left(\frac{p}{3}\right)\sum_{k=0}^{n-1}\frac{16016k+1273}{510300^k}g_kT_k(17498,1)\end{aligned}\tag{7.27}$$

divided by $(pn)^2$ is a p-adic integer for each $n\in\mathbb{Z}^+$.

(iii) *Let $p>11$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{g_kT_k(17498,1)}{510300^k}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{7})=(\frac{p}{11})=1\ \&\ p=x^2+462y^2,\\ 2p-8x^2 \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{7})=1,\ (\frac{p}{3})=(\frac{p}{11})=-1\ \&\ p=2x^2+231y^2,\\ 12x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=(\frac{p}{11})=1\ \&\ p=3x^2+154y^2,\\ 2p-24x^2 \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{7})=(\frac{p}{11})=-1\ \&\ p=6x^2+77y^2,\\ 28x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=1,\ (\frac{p}{7})=(\frac{p}{11})=-1\ \&\ p=7x^2+66y^2,\\ 44x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=-1,\ (\frac{p}{7})=(\frac{p}{11})=1\ \&\ p=11x^2+42y^2,\\ 2p-56x^2 \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ p=14x^2+33y^2,\\ 84x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=-1,\ (\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=21x^2+22y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-462}{p})=-1,\end{cases}\tag{7.28}$$

where x and y are integers.

Remark 7.4. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-462})$ has class number 8. We believe that 462 is the largest positive squarefree number d for which the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ can be used to construct a Ramanujan-type series for $1/\pi$.

The identities (7.5), (7.7), (7.9), (7.11) are related to the imaginary quadratic fields $\mathbb{Q}(\sqrt{-165})$, $\mathbb{Q}(\sqrt{-210})$, $\mathbb{Q}(\sqrt{-345})$, $\mathbb{Q}(\sqrt{-330})$ (with class number 8) respectively. We also have conjectures on related congruences similar to Conjectures 7.2, 7.3, 7.4 and 7.5.

To conclude this section, we confirm an open series for $1/\pi$ conjectured by the author (cf. [34, (3.28)] and [35, Conjecture 7.9]) in 2011.

Theorem 7.1. *We have*

$$\sum_{n=0}^\infty\frac{16n+5}{324^n}\binom{2n}{n}g_n(-20)=\frac{189}{25\pi},\tag{7.29}$$

where

$$g_n(x):=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{k}x^k.$$

Proof. The Franel numbers of order 4 are given by $f_n^{(4)}=\sum_{k=0}^n\binom{n}{k}^4$ $(n\in\mathbb{N})$. Note that

$$f_n^{(4)}\le\left(\sum_{k=0}^n\binom{n}{k}^2\right)^2=\binom{2n}{n}^2\le((1+1)^{2n})^2=16^n.$$

By [11, (8.1)], for $|x|<1/16$ and $a,b\in\mathbb{Z}$, we have

$$\begin{aligned}&\sum_{n=0}^\infty\binom{2n}{n}\frac{(an+b)x^n}{(1+2x)^{2n}}\sum_{k=0}^n\binom{n}{k}^2\binom{2(n-k)}{n-k}x^k\\ =&(1+2x)\sum_{n=0}^\infty\left(\frac{4a(1-x)(1+2x)n+6ax(2-x)}{5(1-4x)}+b\right)f_n^{(4)}x^n.\end{aligned}\tag{7.30}$$

Since

$$\frac{x^n}{(1+2x)^{2n}}\sum_{k=0}^n\binom nk^2\binom{2n-2k}{n-k}x^k$$
$$=\frac{x^n}{(1+2x)^{2n}}\sum_{k=0}^n\binom nk^2\binom{2k}{k}x^{n-k}=(2+x^{-1})^{-2n}g_n(x^{-1}),$$

putting $a=16$, $b=5$ and $x=-1/20$ in (7.30) we obtain

$$\sum_{n=0}^\infty\frac{16n+5}{18^{2n}}\binom{2n}{n}g_n(-20)=\frac{378}{125}\sum_{n=0}^\infty\frac{3n+1}{(-20)^n}f_n^{(4)}.$$

As

$$\sum_{n=0}^\infty\frac{3n+1}{(-20)^n}f_n^{(4)}=\frac{5}{2\pi}$$

by Cooper [9], we finally get

$$\sum_{n=0}^\infty\frac{16n+5}{18^{2n}}\binom{2n}{n}g_n(-20)=\frac{378}{125}\times\frac{5}{2\pi}=\frac{189}{25\pi}.$$

This concludes the proof of (7.29). □

8. Series and congruences involving $T_n(b,c)$ and $\beta_n=\sum_{k=0}^n\binom nk^2\binom{n+k}{k}$

Recall that the numbers

$$\beta_n:=\sum_{k=0}^n\binom nk^2\binom{n+k}{k}\quad(n=0,1,2,\ldots)$$

are a kind of Apéry numbers. Let p be an odd prime. For any $k=0,1,\ldots,p-1$, we have

$$\beta_k\equiv(-1)^k\beta_{p-1-k}\ (\mathrm{mod}\ p)$$

by [24, Lemma 2.7(i)]. Combining this with Remark 1.3(ii), we see that

$$\sum_{k=0}^{p-1}\frac{\beta_kT_k(b,c)}{m^k}\equiv\left(\frac{b^2-4c}{p}\right)\sum_{k=0}^{p-1}\left(\frac{-(b^2-4c)}{m}\right)^k\beta_{p-1-k}T_{p-1-k}(b,c)$$
$$\equiv\left(\frac{b^2-4c}{p}\right)\sum_{k=0}^{p-1}\frac{\beta_kT_k(b,c)}{((4c-b^2)/m)^k}\ (\mathrm{mod}\ p)$$

for any $b,c,m\in\mathbb{Z}$ with $p\nmid(b^2-4c)m$.

Wan and Zudilin [49] obtained the following irrational series for $1/\pi$ involving the Legendre polynomials and the numbers β_n:

$$\sum_{k=0}^\infty(60k+16-5\sqrt{10})\beta_kP_k\left(\frac{5\sqrt2+17\sqrt5}{45}\right)\left(\frac{5\sqrt2-3\sqrt5}{5}\right)^k=\frac{135\sqrt2+81\sqrt5}{\sqrt2\,\pi}.$$

Using our congruence approach (including Conjecture 1.4), we find one rational series for $1/\pi$ involving $T_n(b,c)$ and the Apéry numbers β_n (see (8.1) below); Theorem 1 of [49] might be helpful to solve it.

Conjecture 8.1. (i) *We have*

$$\sum_{k=0}^{\infty}\frac{145k+9}{900^k}\beta_kT_k(52,1)=\frac{285}{\pi}. \tag{8.1}$$

Also, for any $n\in\mathbb{Z}^+$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1}(145k+9)900^{n-1-k}\beta_kT_k(52,1)\in\mathbb{Z}^+. \tag{8.2}$$

(ii) *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{145k+9}{900^k}\beta_kT_k(52,1)\equiv\frac{p}{5}\left(133\left(\frac{-1}{p}\right)-88\right)\pmod{p^2}. \tag{8.3}$$

If $p\equiv 1\pmod 4$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{145k+9}{900^k}\beta_kT_k(52,1)-p\sum_{k=0}^{n-1}\frac{145k+9}{900^k}\beta_kT_k(52,1)\right)\in\mathbb{Z}_p \tag{8.4}$$

for all $n\in\mathbb{Z}^+$.

(iii) *Let $p>5$ be a prime. Then*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\beta_kT_k(52,1)}{900^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,4\pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2\pmod{p^2} & \text{if } p\equiv 2,8\pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-15}{p})=-1.\end{cases} \tag{8.5}$$

Remark 8.1. This conjecture was formulated by the author on Oct. 27, 2019.

Conjecture 8.2. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{2n}\sum_{k=0}^{n-1}(-1)^k(15k+8)\beta_kT_k(4,-1)\in\mathbb{Z}, \tag{8.6}$$

and this number is odd if and only if $n\in\{2^a:\ a\in\mathbb{Z}^+\}$.

(ii) *Let p be a prime. Then*

$$\sum_{k=0}^{p-1}(-1)^k(15k+8)\beta_kT_k(4,-1)\equiv\frac{p}{4}\left(27\left(\frac{p}{3}\right)+5\left(\frac{p}{5}\right)\right)\pmod{p^2}. \tag{8.7}$$

If $(\frac{-15}{p})=1$ (i.e., $p\equiv 1,2,4,8\pmod{15}$), then

$$\sum_{k=0}^{pn-1}(-1)^k(15k+8)\beta_kT_k(4,-1)-p\left(\frac{p}{3}\right)\sum_{k=0}^{n-1}(-1)^k(15k+8)\beta_kT_k(2,2) \tag{8.8}$$

divided by $(pn)^2$ is a p-adic integer for any $n\in\mathbb{Z}^+$.

(iii) *For any prime $p>5$, we have*

$$\sum_{k=0}^{p-1}(-1)^k\beta_kT_k(4,-1)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1,4\pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\12x^2-2p\pmod{p^2}&\text{if }p\equiv2,8\pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }(\frac{-15}{p})=-1.\end{cases}\tag{8.9}$$

Remark 8.2. This conjecture was formulated by the author on Nov. 13, 2019.

Conjecture 8.3. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{3}{n2^{\lfloor n/2\rfloor}}\sum_{k=0}^{n-1}(2k+1)(-2)^{n-1-k}\beta_kT_k(2,2)\in\mathbb{Z}^+,\tag{8.10}$$

and this number is odd if and only if n is a power of two.

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{2k+1}{(-2)^k}\beta_kT_k(2,2)\equiv\frac{p}{3}\left(1+2\left(\frac{-1}{p}\right)\right)\pmod{p^2}.\tag{8.11}$$

If $p\equiv1\pmod 4$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{2k+1}{(-2)^k}\beta_kT_k(2,2)-p\sum_{k=0}^{n-1}\frac{2k+1}{(-2)^k}\beta_kT_k(2,2)\right)\in\mathbb{Z}_p\tag{8.12}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any odd prime p, we have*

$$\sum_{k=0}^{p-1}\frac{\beta_kT_k(2,2)}{(-2)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1\pmod4\ \&\ p=x^2+4y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }p\equiv3\pmod4.\end{cases}\tag{8.13}$$

Remark 8.3. This conjecture was formulated by the author on Nov. 13, 2019.

Conjecture 8.4. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n2^{\lfloor(n+1)/2\rfloor}}\sum_{k=0}^{n-1}(3k+2)(-2)^{n-1-k}\beta_kT_k(20,2)\in\mathbb{Z}^+,\tag{8.14}$$

and this number is odd if and only if $n\in\{2^a:\ a=0,2,3,4,\ldots\}$.

(ii) *Let p be any odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{3k+2}{(-2)^k}\beta_kT_k(20,2)\equiv2p\left(\frac{2}{p}\right)\pmod{p^2}.\tag{8.15}$$

If $p\equiv\pm1\pmod 8$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{3k+2}{(-2)^k}\beta_kT_k(20,2)-p\sum_{k=0}^{n-1}\frac{3k+2}{(-2)^k}\beta_kT_k(20,2)\right)\in\mathbb{Z}_p\tag{8.16}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any odd prime p, we have*

$$\sum_{k=0}^{p-1}\frac{\beta_kT_k(20,2)}{(-2)^k} \equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p\equiv1\pmod 4\ \&\ p=x^2+4y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } p\equiv3\pmod 4.\end{cases}\tag{8.17}$$

Conjecture 8.5. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{3}{n}\sum_{k=0}^{n-1}(5k+3)4^{n-1-k}\beta_kT_k(14,-1)\in\mathbb{Z}.\tag{8.18}$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{5k+3}{4^k}\beta_kT_k(14,-1)\equiv\frac{p}{3}\left(4\left(\frac{-2}{p}\right)+5\left(\frac{2}{p}\right)\right)\pmod{p^2}.\tag{8.19}$$

If $p\equiv1\pmod 4$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{5k+3}{4^k}\beta_kT_k(14,-1)-p\left(\frac{2}{p}\right)\sum_{k=0}^{n-1}\frac{5k+3}{4^k}\beta_kT_k(14,-1)\right)\in\mathbb{Z}_p\tag{8.20}$$

for all $n\in\mathbb{Z}^+$.

(iii) *Let $p\neq2,5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\beta_kT_k(14,-1)}{4^k} \equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } (\frac{-2}{p})=(\frac{5}{p})=1\ \&\ p=x^2+10y^2\ (x,y\in\mathbb{Z}),\\ 8x^2-2p\pmod{p^2}&\text{if } (\frac{-2}{p})=(\frac{5}{p})=-1\ \&\ p=2x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } (\frac{-10}{p})=-1.\end{cases}\tag{8.21}$$

Conjecture 8.6. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{3n}\sum_{k=0}^{n-1}(22k+15)(-4)^{n-1-k}\beta_kT_k(46,1)\in\mathbb{Z}^+,\tag{8.22}$$

and this number is odd if and only if n is a power of two.

(ii) *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{22k+15}{(-4)^k}\beta_kT_k(46,1)\equiv\frac{p}{4}\left(357-297\left(\frac{33}{p}\right)\right)\pmod{p^2}.\tag{8.23}$$

If $(\frac{33}{p})=1$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{22k+15}{(-4)^k}\beta_kT_k(46,1)-p\sum_{k=0}^{n-1}\frac{22k+15}{(-4)^k}\beta_kT_k(46,1)\right)\in\mathbb{Z}_p\tag{8.24}$$

for all $n\in\mathbb{Z}^+$.

(iii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\beta_kT_k(46,1)}{(-4)^k}\equiv\begin{cases}x^2-2p\pmod{p^2}&\text{if }(\frac{p}{11})=1\ \&\ 4p=x^2+11y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }(\frac{p}{11})=-1.\end{cases}\tag{8.25}$$

Conjecture 8.7. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(190k+91)(-60)^{n-1-k}\beta_kT_k(82,1)\in\mathbb{Z}^+,\tag{8.26}$$

and this number is odd if and only if n is a power of two.

(ii) *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{190k+91}{(-60)^k}\beta_kT_k(82,1)\equiv\frac{p}{4}\left(111+253\left(\frac{-15}{p}\right)\right)\pmod{p^2}.\tag{8.27}$$

If $(\frac{-15}{p})=1$, then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{190k+91}{(-60)^k}\beta_kT_k(82,1)-p\sum_{k=0}^{n-1}\frac{190k+91}{(-60)^k}\beta_kT_k(82,1)\right)\in\mathbb{Z}_p\tag{8.28}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime $p>7$, we have*

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\beta_kT_k(82,1)}{(-60)^k}\equiv\begin{cases}x^2-2p\pmod{p^2}&\text{if }(\frac{p}{5})=(\frac{p}{7})=1\ \&\ 4p=x^2+35y^2\ (x,y\in\mathbb{Z}),\\2p-5x^2\pmod{p^2}&\text{if }(\frac{p}{5})=(\frac{p}{7})=-1\ \&\ 4p=5x^2+7y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }(\frac{-35}{p})=-1.\end{cases}\tag{8.29}$$

9. Series and congruences involving $w_n=\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k3^{n-3k}\binom{n}{3k}\binom{3k}{k}\binom{2k}{k}$ and $T_n(b,c)$

The numbers

$$w_n:=\sum_{k=0}^{\lfloor n/3\rfloor}(-1)^k3^{n-3k}\binom{n}{3k}\binom{3k}{k}\binom{2k}{k}\quad(n=0,1,2,\ldots)$$

were first introduced by Zagier [51] during his study of Apéry-like integer sequences, who noted the recurrence

$$(n+1)^2w_{n+1}=(9n(n+1)+3)w_n-27n^2w_{n-1}\ (n=1,2,3,\ldots).$$

Lemma 9.1. *Let $p>3$ be a prime. Then*

$$w_k\equiv\left(\frac{-3}{p}\right)27^kw_{p-1-k}\pmod{p}\quad\textit{for all }k=0,\ldots,p-1.$$

Proof. Note that

$$\begin{aligned} w_{p-1} =& \sum_{k=0}^{\lfloor (p-1)/3 \rfloor} (-1)^k 3^{p-1-3k} \binom{p-1}{3k}\binom{3k}{k}\binom{2k}{k} \\ \equiv& \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod p \end{aligned}$$

with the help of the known congruence $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}/27^k \equiv (\frac{p}{3}) \pmod{p^2}$ conjectured by F. Rodriguez-Villegas [28] and proved by E. Mortenson [25]. Similarly,

$$\begin{aligned} w_{p-2} =& \sum_{k=0}^{\lfloor (p-2)/3 \rfloor} (-1)^k 3^{p-2-3k} \binom{p-2}{3k}\binom{3k}{k}\binom{2k}{k} \\ =& \sum_{k=0}^{\lfloor (p-2)/3 \rfloor} (-1)^k 3^{p-2-3k} \frac{3k+1}{p-1}\binom{p-1}{3k+1}\binom{3k}{k}\binom{2k}{k} \\ \equiv& \frac{1}{9}\sum_{k=0}^{p-1}(9k+3)\frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \frac{1}{9}\left(\frac{p}{3}\right) + \frac{1}{9}\sum_{k=0}^{p-1}(9k+2)\frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \pmod p. \end{aligned}$$

By induction,

$$\sum_{k=0}^{n}(9k+2)\frac{\binom{2k}{k}\binom{3k}{k}}{27^k} = (3n+1)(3n+2)\frac{\binom{2n}{n}\binom{3n}{n}}{27^n}$$

for all $n \in \mathbb{N}$. In particular,

$$\sum_{k=0}^{p-1}(9k+2)\frac{\binom{2k}{k}\binom{3k}{k}}{27^k} = \frac{(3p-2)(3p-1)}{27^{p-1}}pC_{p-1}\binom{3p-3}{p-1} \equiv 0 \pmod p.$$

So we have $w_k \equiv (\frac{-3}{p})27^k w_{p-1-k} \pmod p$ for $k = 0, 1$. (Note that $w_0 = 1$ and $w_1 = 3$.)

Now let $k \in \{1, \ldots, p-2\}$ and assume that

$$w_j \equiv \left(\frac{-3}{p}\right) 27^j w_{p-1-j} \quad \text{for all } j = 0, \ldots, k.$$

Then

$$\begin{aligned} &(k+1)^2 w_{k+1} = (9k(k+1)+3)w_k - 27k^2 w_{k-1} \\ \equiv&(9(p-k)(p-k-1)+3)\left(\frac{-3}{p}\right)27^k w_{p-1-k} - 27(p-k)^2\left(\frac{-3}{p}\right)27^{k-1}w_{p-1-(k-1)} \\ =&\left(\frac{-3}{p}\right)27^k \times 27(p-k-1)^2 w_{p-k-2} \pmod p \end{aligned}$$

and hence

$$w_{k+1} \equiv \left(\frac{-3}{p}\right) 27^{k+1} w_{p-1-(k+1)} \pmod p.$$

In view of the above, we have proved the desired result by induction. □

For Lemma 9.1 one may also consult [31, Corollary 3.1]. Let $p>3$ be a prime. In view of Lemma 9.1 and Remark 1.3(ii), we have

$$\sum_{k=0}^{p-1}\frac{w_kT_k(b,c)}{m^k}\equiv\left(\frac{-3(b^2-4c)}{p}\right)\sum_{k=0}^{p-1}\left(\frac{27(b^2-4c)}{m}\right)^k w_{p-1-k}T_{p-1-k}(b,c)$$
$$\equiv\left(\frac{-3(b^2-4c)}{p}\right)\sum_{k=0}^{p-1}\frac{w_kT_k(b,c)}{(27(b^2-4c)/m)^k}\pmod p$$

for any $b,c,m\in\mathbb{Z}$ with $p\nmid(b^2-4c)m$.

Wan and Zudilin [49] obtained the following irrational series for $1/\pi$ involving the Legendre polynomials and the numbers w_n:

$$\sum_{k=0}^{\infty}(14k+7-\sqrt{21})w_kP_k\left(\frac{\sqrt{21}}{5}\right)\left(\frac{7\sqrt{21}-27}{90}\right)^k=\frac{5\sqrt{7(7\sqrt{21}+27)}}{4\sqrt2\,\pi}.$$

Using our congruence approach (including Conjecture 1.4), we find five rational series for $1/\pi$ involving $T_n(b,c)$ and the numbers w_n; Theorem 1 of [49] might be helpful to solve them.

Conjecture 9.1. *We have*

$$\sum_{k=0}^{\infty}\frac{13k+3}{100^k}w_kT_k(14,-1)=\frac{30\sqrt2}{\pi},\tag{9.1}$$
$$\sum_{k=0}^{\infty}\frac{14k+5}{108^k}w_kT_k(18,1)=\frac{27\sqrt3}{\pi},\tag{9.2}$$
$$\sum_{k=0}^{\infty}\frac{19k+2}{486^k}w_kT_k(44,-2)=\frac{81\sqrt3}{4\pi},\tag{9.3}$$
$$\sum_{k=0}^{\infty}\frac{91k+32}{(-675)^k}w_kT_k(52,1)=\frac{45\sqrt3}{2\pi},\tag{9.4}$$
$$\sum_{k=0}^{\infty}\frac{182k+37}{756^k}w_kT_k(110,1)=\frac{315\sqrt3}{\pi}.\tag{9.5}$$

Below we present our conjectures on congruences related to the identities (9.2) and (9.5).

Conjecture 9.2. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac1n\sum_{k=0}^{n-1}(14k+5)108^{n-1-k}w_kT_k(18,1)\in\mathbb{Z}^+,\tag{9.6}$$

and this number is odd if and only if $n\in\{2^a:\ a\in\mathbb{N}\}$.

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{14k+5}{108^k}w_kT_k(18,1)\equiv\frac p4\left(27\left(\frac{-3}{p}\right)-7\left(\frac{21}{p}\right)\right)\pmod{p^2}.\tag{9.7}$$

If $(\frac p7)=1$ (i.e., $p\equiv1,2,4\pmod 7$), then

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{14k+5}{108^k}w_kT_k(18,1)-\left(\frac p3\right)\sum_{k=0}^{n-1}\frac{14k+5}{108^k}w_kT_k(18,1)\right)\in\mathbb{Z}_p\tag{9.8}$$

for all $n \in \mathbb{Z}^+$.

(iii) *For any prime* $p > 7$, *we have*

$$\begin{aligned}&\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{w_kT_k(18,1)}{108^k}\\ &\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{7})=1\ \&\ 4p=x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 5x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{7})=-1\ \&\ 4p=5x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-35}{p})=-1.\end{cases}\end{aligned} \tag{9.9}$$

Conjecture 9.3. (i) *For any* $n \in \mathbb{Z}^+$, *we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(182k+37)756^{n-1-k}w_kT_k(110,1)\in\mathbb{Z}^+, \tag{9.10}$$

and this number is odd if and only if $n \in \{2^a :\ a \in \mathbb{N}\}$.

(ii) *Let* $p > 3$ *be a prime with* $p \neq 7$. *Then*

$$\sum_{k=0}^{p-1}\frac{182k+37}{756^k}w_kT_k(110,1)\equiv\frac{p}{4}\left(265\left(\frac{-3}{p}\right)-117\left(\frac{21}{p}\right)\right) \pmod{p^2}. \tag{9.11}$$

If $(\frac{p}{7}) = 1$ (*i.e.*, $p \equiv 1, 2, 4 \pmod 7$), *then*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{182k+37}{756^k}w_kT_k(110,1)-\left(\frac{p}{3}\right)\sum_{k=0}^{n-1}\frac{182k+37}{756^k}w_kT_k(110,1)\right)\in\mathbb{Z}_p \tag{9.12}$$

for all $n \in \mathbb{Z}^+$.

(iii) *For any prime* $p > 3$ *with* $p \neq 7, 13$, *we have*

$$\begin{aligned}&\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{w_kT_k(110,1)}{756^k}\\ &\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=1\ \&\ 4p=x^2+91y^2\ (x,y\in\mathbb{Z}),\\ 7x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=-1\ \&\ 4p=7x^2+13y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-91}{p})=-1.\end{cases}\end{aligned} \tag{9.13}$$

Now we give one more conjecture in this section.

Conjecture 9.4. (i) *For any integer* $n > 1$, *we have*

$$\frac{1}{3n2^{\lfloor (n+1)/2\rfloor}}\sum_{k=0}^{n-1}(2k+1)54^{n-1-k}w_kT_k(10,-2)\in\mathbb{Z}^+. \tag{9.14}$$

(ii) *Let* $p > 3$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{2k+1}{54^k}w_kT_k(10,-2)\equiv p\left(\frac{p}{3}\right)+\frac{p}{2}(2^{p-1}-1)\left(5\left(\frac{p}{3}\right)+3\left(\frac{3}{p}\right)\right) \pmod{p^3}. \tag{9.15}$$

If $p \equiv 1 \pmod 4$, *then*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}\frac{2k+1}{54^k}w_kT_k(10,-2)-\left(\frac{p}{3}\right)\sum_{k=0}^{n-1}\frac{2k+1}{54^k}w_kT_k(10,-2)\right)\in\mathbb{Z}_p \tag{9.16}$$

for all $n\in\mathbb{Z}^+$.

(iii) *For any prime* $p>3$, *we have*

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{w_kT_k(10,-2)}{54^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }4\mid p-1\ \&\ p=x^2+4y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }p\equiv3\pmod 4.\end{cases}\tag{9.17}$$

Remark 9.1. For primes $p>3$ with $p\equiv3\pmod 4$, in general the congruence (9.16) is not always valid for all $n\in\mathbb{Z}^+$. This does not violate Conjecture 1.2 since $\lim_{k\to+\infty}|w_kT_k(10,-2)|^{1/k}=\sqrt{27}\times\sqrt{10^2-4(-2)}=54$. If the series $\sum_{k=0}^\infty\frac{2k+1}{54^k}w_kT_k(10,-2)$ converges, its value times $\pi/\sqrt3$ should be a rational number.

10. Series for π involving T_n and related congruences

Let p be an odd prime and let $a,b,c,d,m\in\mathbb{Z}$ with $m(b^2-4c)\not\equiv0\pmod p$. Then

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{a+dk}{m^k}\binom{2k}{k}^2T_k(b,c)&\equiv\sum_{k=1}^{(p-1)/2}\frac{a+dk}{k^2m^k}\left(k\binom{2k}{k}\right)^2T_k(b,c)\\&\equiv\sum_{k=1}^{(p-1)/2}\frac{a+dk}{k^2m^k}\left(-\frac{2p}{\binom{2(p-k)}{p-k}}\right)^2T_k(b,c)\pmod p\end{aligned}$$

with the aid of [33, Lemma 2.1]. Thus

$$\begin{aligned}&\sum_{k=1}^{p-1}\frac{a+dk}{m^k}\binom{2k}{k}^2T_k(b,c)\\\equiv&4p^2\sum_{k=1}^{(p-1)/2}\frac{a+dk}{k^2m^k}\times\frac{T_k(b,c)}{\binom{2(p-k)}{p-k}^2}\\\equiv&4p^2\sum_{p/2<k<p}\frac{a+d(p-k)}{(p-k)^2m^{p-k}}\times\frac{T_{p-k}(b,c)}{\binom{2k}{k}^2}\\\equiv&4p^2\sum_{k=1}^{p-1}\frac{(a-dk)m^{k-1}}{k^2\binom{2k}{k}^2}\left(\frac{b^2-4c}{p}\right)(b^2-4c)^{p-k}T_{p-1-(p-k)}(b,c)\\\equiv&\left(\frac{b^2-4c}{p}\right)4p^2\sum_{k=1}^{p-1}\frac{(a-dk)T_{k-1}(b,c)}{k^2\binom{2k}{k}^2}\left(\frac{m}{b^2-4c}\right)^{k-1}\pmod p\end{aligned}$$

in view of Remark 1.3(ii).

Let $p>3$ be a prime. By the above, the author's conjectural congruence (cf. [35, Conjecture 1.3])

$$\sum_{k=0}^{p-1}(105k+44)(-1)^k\binom{2k}{k}^2T_k\equiv p\left(20+24\left(\frac{p}{3}\right)(2-3^{p-1})\right)\pmod{p^3}$$

implies that

$$p^2\sum_{k=1}^{p-1}\frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}}\equiv 11\left(\frac{p}{3}\right)\pmod p.$$

Motivated by this, we pose the following curious conjecture.

Conjecture 10.1. *We have the following identities:*

$$\sum_{k=1}^{\infty}\frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}}=\frac{5\pi}{\sqrt3}+6\log 3, \tag{10.1}$$

$$\sum_{k=2}^{\infty}\frac{(5k-2)T_{k-1}}{(k-1)k^2\binom{2k}{k}^2 3^{k-1}}=\frac{21-2\sqrt3\,\pi-9\log 3}{12}. \tag{10.2}$$

Remark 10.1. The two identities were conjectured by the author on Dec. 7, 2019. One can easily check them numerically via `Mathematica` as the two series converge fast.

Now we state our related conjectures on congruences.

Conjecture 10.2. *For any prime $p>3$, we have*

$$p^2\sum_{k=1}^{p-1}\frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}}\equiv 11\left(\frac{p}{3}\right)+\frac{p}{2}\left(13-35\left(\frac{p}{3}\right)\right)\pmod{p^2} \tag{10.3}$$

and

$$p^2\sum_{k=2}^{p-1}\frac{(5k-2)T_{k-1}}{(k-1)k^2\binom{2k}{k}^2 3^{k-1}}\equiv -\frac12\left(\frac{p}{3}\right)-\frac{p}{8}\left(7+\left(\frac{p}{3}\right)\right)\pmod{p^2}. \tag{10.4}$$

Conjecture 10.3. (i) *We have*

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(-1)^{n-1-k}(5k+2)\binom{2k}{k}C_kT_k\in\mathbb{Z}^+$$

for all $n\in\mathbb{Z}^+$, and also

$$\sum_{k=0}^{p-1}(-1)^k(5k+2)\binom{2k}{k}C_kT_k\equiv 2p\left(1-\left(\frac{p}{3}\right)(3^p-3)\right)\pmod{p^3}$$

for each prime $p>3$.

(ii) *For any prime $p\equiv 1\pmod 3$ and $n\in\mathbb{Z}^+$, we have*

$$\frac{\sum_{k=0}^{pn-1}(-1)^k(5k+2)\binom{2k}{k}C_kT_k-p\sum_{k=0}^{n-1}(-1)^k(5k+2)\binom{2k}{k}C_kT_k}{(pn)^2\binom{2n}{n}^2}$$
$$\equiv\left(\frac{p}{3}\right)\frac{3^p-3}{2p}(-1)^nT_{n-1}\pmod p. \tag{10.5}$$

Remark 10.2. See also [45, Conjecture 67] for a similar conjecture.

Let p be an odd prime. We conjecture that

$$\sum_{k=0}^{p-1}\frac{8k+3}{(-16)^k}\binom{2k}{k}^2T_k(3,-4)\equiv p\left(1+2\left(\frac{-1}{p}\right)\right)\pmod{p^2} \tag{10.6}$$

and

$$\sum_{k=0}^{p-1}\frac{33k+14}{4^k}\binom{2k}{k}^2T_k(8,-2)\equiv p\left(6\left(\frac{-1}{p}\right)+8\left(\frac{2}{p}\right)\right)\pmod{p^2}. \tag{10.7}$$

Though (10.6) implies the congruence

$$p^2\sum_{k=1}^{p-1}\frac{(8k-3)T_{k-1}(3,-4)}{k^2\binom{2k}{k}^2}\left(-\frac{16}{25}\right)^{k-1}\equiv\frac{3}{4}\pmod{p},$$

and (10.7) with $p>3$ implies the congruence

$$p^2\sum_{k=1}^{p-1}\frac{(33k-14)T_{k-1}(8,-2)}{k^2\binom{2k}{k}^2 18^{k-1}}\equiv\frac{7}{2}\left(\frac{2}{p}\right)\pmod{p},$$

we are unable to find the exact values of the two converging series

$$\sum_{k=1}^{\infty}\frac{(8k-3)T_{k-1}(3,-4)}{k^2\binom{2k}{k}^2}\left(-\frac{16}{25}\right)^{k-1}\quad\text{and}\quad\sum_{k=1}^{\infty}\frac{(33k-14)T_{k-1}(8,-2)}{k^2\binom{2k}{k}^2 18^{k-1}}.$$

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