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CONGRUENCES FOR APÉRY NUMBERS $\beta_n = \sum_{k=0}^n \binom{n}{k}^2\binom{n+k}{k}$

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Abstract. In this paper we establish some congruences involving the Apéry numbers $\beta_n = \sum_{k=0}^n \binom{n}{k}^2\binom{n+k}{k}$. For example, we show that

$$\sum_{k=0}^{n-1}(11k^2+13k+4)\beta_k \equiv 0 \pmod{2n^2}$$

for any positive integer n, and

$$\sum_{k=0}^{p-1}(11k^2+13k+4)\beta_k \equiv 4p^2+4p^7B_{p-5} \pmod{p^8}$$

for any prime $p>3$, where B_{p-5} is the $(p-5)$th Bernoulli number. We also present certain relations between congruence properties of the two kinds of Apéry numbers, β_n and $A_n = \sum_{k=0}^n \binom{n}{k}^2\binom{n+k}{k}^2$.

1. Introduction

In 1979 R. Apéry (see [1] and [19]) established the irrationality of $\zeta(3) = \sum_{n=1}^\infty 1/n^3$ by using the Apéry numbers

$$A_n = \sum_{k=0}^n \binom{n}{k}^2\binom{n+k}{k}^2 \quad (n \in \mathbb{N} = \{0,1,2,\ldots\}).$$

His method also allowed him to prove the irrationality of $\zeta(2) = \pi^2/6$ via another kind of Apéry numbers

$$\beta_n := \sum_{k=0}^n \binom{n}{k}^2\binom{n+k}{k} \quad (n \in \mathbb{N}).$$

In 2012, the third author [15] proved that

$$\sum_{k=0}^{n-1}(2k+1)A_k \equiv 0 \pmod{n} \quad \text{for all } n \in \mathbb{Z}^+ = \{1,2,3,\ldots\}$$

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and

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p+\frac{7}{6}p^4B_{p-3} \pmod{p^5} \quad \text{for any prime } p>3,$$

where $B_0, B_1, B_2, \ldots$ are the Bernoulli numbers defined by

$$B_0=1, \text{ and } \sum_{k=0}^{n}\binom{n+1}{k}B_k=0 \text{ for all } n\in\mathbb{Z}^+.$$

(For the basic properties of Bernoulli numbers, see [8, pp. 228-248].) The third author also conjectured that

$$\sum_{k=0}^{n-1}(2k+1)(-1)^kA_k \equiv 0 \pmod{n} \quad \text{for all } n\in\mathbb{Z}^+$$

and

$$\sum_{k=0}^{p-1}(2k+1)(-1)^kA_k \equiv p\left(\frac{p}{3}\right) \pmod{p^3} \text{ for any prime } p>3,$$

which was confirmed by V.J.W. Guo and J. Zeng [5]. In 2016 the third author [17] showed that

$$\sum_{k=0}^{n-1}(6k^3+9k^2+5k+1)(-1)^kA_k \equiv 0 \pmod{n^3} \quad \text{for all } n\in\mathbb{Z}^+.$$

In view of this, it is natural to ask whether the Apéry numbers β_n also have such kind of congruence properties. This is the main motivation of this paper.

Another motivation is due to a theorem of E. Rowland and R. Yassawi [12] about A_n ($n\in\mathbb{N}$) modulo 16. We will extend their result to A_n modulo 32; quite unexpectedly, this requires consideration of β_n modulo 8.

Now we state our main results which were first found via `Mathematica`.

Theorem 1.1. *For any positive integer n, we have*

$$\sum_{k=0}^{n-1}(11k^2+13k+4)\beta_k \equiv 0 \pmod{2n^2}, \tag{1.1}$$

$$\sum_{k=0}^{n-1}(11k^2+9k+2)(-1)^k\beta_k \equiv 0 \pmod{2n^2}, \tag{1.2}$$

$$\sum_{k=0}^{n-1}(11k^3+7k^2-1)(-1)^k\beta_k \equiv 0 \pmod{n^2}. \tag{1.3}$$

Theorem 1.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(11k^2+13k+4)\beta_k\equiv 4p^2+4p^7B_{p-5}\pmod{p^8}, \tag{1.4}$$

$$\sum_{k=0}^{p-1}(11k^2+9k+2)(-1)^k\beta_k\equiv 2p^2+10p^3H_{p-1}-p^7B_{p-5}\pmod{p^8}, \tag{1.5}$$

and

$$\begin{aligned}&\sum_{k=0}^{p-1}(11k^3+7k^2-1)(-1)^k\beta_k\\ \equiv&2p^3-3p^2+(10p-5)p^3H_{p-1}-\frac{3}{2}p^7B_{p-5}\pmod{p^8},\end{aligned} \tag{1.6}$$

where H_{p-1} denotes the harmonic number $\sum_{k=1}^{p-1}\frac{1}{k}$.

Theorem 1.3. *Let n be a nonnegative integer. If*

$$A_n\equiv 8j+4i+1\pmod{16} \tag{1.7}$$

with $\{i,j\}\subseteq\{0,1\}$, then

$$\beta_n\equiv 2(i+j)+1\pmod 4. \tag{1.8}$$

Remark 1.4. For any $n\in\mathbb{N}$, we clearly have

$$A_n=1+4\sum_{0<k\leqslant n}\binom{n+k}{2k}^2\binom{2k-1}{k-1}^2\equiv 1\pmod 4.$$

Theorem 1.5. *Let n be a nonnegative integer. If*

$$A_n\equiv 16k+8j+4i+1\pmod{32}\ \text{and}\ \beta_n\equiv 4m+2l+1\pmod 8$$

with $\{i,j,k,l,m\}\subseteq\{0,1\}$, then

$$L_2(n)\equiv 4(k+m)+2j+i\pmod 8, \tag{1.9}$$

where $L_2(n)$ (the binary run-length of n) is defined as the number of blocks of consecutive 0's and 1's in the binary expansion of n.

We will prove Theorem 1.1 in the next section. We are going to provide some lemmas in Section 3 and show Theorem 1.2 in Section 4. Theorems 1.3 and 1.5 will be proved in Section 5.

2. Proof of Theorem 1.1

Lemma 2.1. *For any $n \in \mathbb{Z}^+$, we have*

$$\sum_{k=0}^{n-1}(11k^2+13k+4)\beta_k = -\sum_{k=0}^{n-1}\binom{n}{k+1}^2\binom{n+k}{k}a_1(n,k), \quad (2.1)$$

$$\sum_{k=0}^{n-1}(11k^2+9k+2)(-1)^{n-1-k}\beta_k = \sum_{k=0}^{n-1}\binom{n}{k+1}^2\binom{n+k}{k}a_2(n,k), \quad (2.2)$$

$$\sum_{k=0}^{n-1}(11k^3+7k^2-1)(-1)^{n-1-k}\beta_k = \sum_{k=0}^{n-1}\binom{n}{k+1}^2\binom{n+k}{k}a_3(n,k), \quad (2.3)$$

where

$$\begin{aligned} a_1(n,k) =& n^3-n^2k-nk^2-2n^2-5nk+k^2-4n+2k+1, \\ a_2(n,k) =& n(3n-k-1), \\ a_3(n,k) =& 4n^3-2n^2k-nk^2-4n^2-3nk+k^2-2n+2k+1. \end{aligned} \quad (2.4)$$

Proof. Note that

$$\sum_{k=0}^{n-1}(11k^2+13k+4)\beta_k = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}F(k,l)$$

is a double sum with

$$F(k,l) = (11k^2+13k+4)\binom{k}{l}^2\binom{k+l}{l}.$$

Using the method in Mu and Sun's paper [9], we find

$$F_1(k,l) = -\frac{(k-l)^2a_1(k,l)}{(l+1)^2(11k^2+13k+4)}F(k,l)$$

and

$$F_2(k,l) = \frac{l(3kl-l^2+2k-2l)}{11k^2+13k+4}F(k,l)$$

so that

$$F(k,l) = (F_1(k+1,l)-F_1(k,l)) + (F_2(k,l+1)-F_2(k,l))$$

which can be verified directly. This allows us to reduce the double sum $\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}F(k,l)$ to the right-hand side of (2.1).

Identities (2.2) and (2.3) can be deduced similarly; in fact,

$$\begin{aligned} &(11k^2+9k+2)(-1)^k\binom{k}{l}^2\binom{k+l}{l} \\ =&(G_1(k+1,l)-G_1(k,l)) + (G_2(k,l+1)-G_2(k,l)) \end{aligned}$$

and

$$\begin{aligned}&(11k^3+7k^2-1)(-1)^k\binom{k}{l}^2\binom{k+l}{l}\\=&(H_1(k+1,l)-H_1(k,l))+(H_2(k,l+1)-H_2(k,l)),\end{aligned}$$

where

$$\begin{aligned}G_1(k,l)=&-\frac{(k-l)^2a_2(k,l)}{(l+1)^2}(-1)^k\binom{k}{l}^2\binom{k+l}{l},\\G_2(k,l)=&-l(6k+l+3)(-1)^k\binom{k}{l}^2\binom{k+l}{l},\\H_1(k,l)=&-\frac{(k-l)^2a_3(k,l)}{(l+1)^2}(-1)^k\binom{k}{l}^2\binom{k+l}{l},\\H_2(k,l)=&-l(8k^2+l^2+4k+2l+2)(-1)^k\binom{k}{l}^2\binom{k+l}{l}.\end{aligned}$$

This ends our proof. □

Lemma 2.2. *For any positive even integer n, we have*

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{2k}\binom{2k}{k+1}\equiv 1 \pmod 2. \tag{2.5}$$

Proof. Clearly,

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{2k}\binom{2k}{k+1}=\sum_{0<k<n}\binom{n-1}{k}\binom{n+k}{k+1}\binom{n-1}{k-1}.$$

For $k\in\{0,\ldots,n-1\}$, if $2\mid k$ then $n+k$ is even and hence

$$\binom{n+k}{k+1}=\frac{n+k}{k+1}\binom{n+k-1}{k}\equiv 0 \pmod 2;$$

if $2\nmid k$ then

$$\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}\equiv 0 \pmod 2.$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{2k}\binom{2k}{k+1}\equiv&\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k+1}\binom{n-1}{k}\\\equiv&\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{k+1}\binom{n+k}{k+1}\\=&(-1)^n \pmod 2,\end{aligned}$$

where we have applied (in the last step) the Chu-Vandermonde identity (cf. [4, (3.1)]). This proves (2.5). □

Proof of Theorem 1.1. For each $k=0,\ldots,n-1$, clearly

$$\binom{n}{k+1}^2(k+1)^2=n^2\binom{n-1}{k}^2$$

and

$$\binom{n}{k+1}^2\binom{n+k}{k}n(k+1)=n^2\binom{n-1}{k}\binom{n+k}{2k}(n-k)C_k,$$

where C_k is the Catalan number $\binom{2k}{k}/(k+1)=\binom{2k}{k}-\binom{2k}{k+1}$.

(i) As

$$a_1(n,k)=n^2(n-k-2)-n(k+1)(k+4)+(k+1)^2,$$

we have

$$\begin{aligned}&\frac{1}{n^2}\sum_{k=0}^{n-1}\binom{n}{k+1}^2\binom{n+k}{k}a_1(n,k)\\ =&\sum_{k=0}^{n-1}\binom{n}{k+1}^2\binom{n+k}{k}(n-k-2)\\ &-\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{2k}(n-k)(k+4)C_k+\sum_{k=0}^{n-1}\binom{n-1}{k}^2\binom{n+k}{k}.\end{aligned}$$

Clearly $11k^2+13k+4\equiv 0 \pmod 2$ for all $k=0,\ldots,n-1$. So, by Lemma 2.1 and the above, (1.1) holds when $2\nmid n$.

Now suppose that n is even. By Lemma 2.1 and the above, we have

$$\begin{aligned}&\frac{1}{n^2}\sum_{k=0}^{n-1}(11k^2+13k+4)\beta_k\\ \equiv&\sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n+k}{k}k+\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{2k}kC_k\\ &+\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k} \pmod 2.\end{aligned}$$

With the help of the Chu–Vandermonde identity,

$$\begin{aligned}\sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n+k}{k}k =&\sum_{k=0}^{n-1}\binom{n}{n-1-k}\binom{-n-1}{k}(-1)^k k\\ \equiv&(-n-1)\sum_{k=1}^{n-1}\binom{n}{n-1-k}\binom{-n-2}{k-1}\\ \equiv&\binom{-2}{n-2}=(-1)^n(n-1)\equiv 1 \pmod 2\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k} =&\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{-n-1}{k}(-1)^k\\ \equiv&\sum_{k=0}^{n-1}\binom{n-1}{n-1-k}\binom{-n-1}{k}\\ =&\binom{-2}{n-1}=(-1)^n n\equiv 0 \pmod 2.\end{aligned}$$

In view of (2.5), we also have

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{2k}kC_k \equiv 1 \pmod 2 \tag{2.6}$$

since $kC_k = \binom{2k}{k+1}$. Therefore (1.1) holds.

(ii) As $a_2(n,k) = 3n^2 - (k+1)n$, we have

$$\begin{aligned}&\frac{1}{n^2}\sum_{k=0}^{n-1}\binom{n}{k+1}^2\binom{n+k}{k}a_2(n,k)\\ =&3\sum_{k=0}^{n-1}\binom{n}{k+1}^2\binom{n+k}{k} - \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{2k}(n-k)C_k.\end{aligned}$$

Clearly $11k^2+9k+2 \equiv 0 \pmod 2$ for all $k = 0,\ldots,n-1$. So, by Lemma 2.1 and the above, (1.2) holds when $2 \nmid n$.

Now suppose that n is even. By Lemma 2.1 and the above, we have

$$\begin{aligned}&\frac{1}{n^2}\sum_{k=0}^{n-1}(11k^2+9k+2)(-1)^k\beta_k\\ \equiv&\sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n+k}{k} + \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{2k}kC_k \pmod 2.\end{aligned}$$

By the Chu–Vandermonde identity,

$$\sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n+k}{k}\equiv\sum_{k=0}^{n-1}\binom{n}{n-1-k}\binom{-n-1}{k}=\binom{-1}{n-1}\equiv 1 \pmod 2.$$

Combining this with (2.6) we obtain (1.2).

(iii) Since

$$a_3(n,k)=2n^2(2n-k-2)-n(k+1)(k+2)+(k+1)^2,$$

we have

$$\begin{aligned}&\frac{1}{n^2}\sum_{k=0}^{n-1}\binom{n}{k+1}^2\binom{n+k}{k}a_3(n,k)\\ =&2\sum_{k=0}^{n-1}\binom{n}{k+1}^2\binom{n+k}{k}(2n-k-2)\\ &-\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{2k}(n-k)(k+2)C_k+\sum_{k=0}^{n-1}\binom{n-1}{k}^2\binom{n+k}{k}\end{aligned}$$

and hence (1.3) follows from (2.3).

In view of the above, we have completed the proof of Theorem 1.1. □

3. Some lemmas

For each $s\in\mathbb{Z}^+$, we define

$$H_{n,s}:=\sum_{0<k\leqslant n}\frac{1}{k^s}\qquad\text{for } n=0,1,2,\ldots,$$

and call such numbers *harmonic numbers of order* s. Those $H_n:=H_{n,1}$ ($n\in\mathbb{N}$) are usually called *harmonic numbers*.

Lemma 3.1. *For any* $n\in\mathbb{Z}^+$, *we have*

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k}=H_n,\tag{3.1}$$

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k-1}}{k}H_k=H_{n,2},\tag{3.2}$$

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}H_{k,2}=\frac{H_n}{n}.\tag{3.3}$$

Remark 3.2. The identities (3.1)-(3.3) are known. See [4, (1.45)] for (3.1), [6] for (3.2), and the proof of [16, Lemma 3.1] for simple proofs of (3.2) and (3.3).

Let $p>3$ be a prime. In 1900 J.W.L. Glaisher [2, 3] refined Wolstenholme's work [20] on congruences by showing that

$$H_{p-1}\equiv-\frac{p^2}{3}B_{p-3}\pmod{p^3}\tag{3.4}$$

and

$$H_{p-1,s}\equiv\frac{s}{s+1}pB_{p-1-s}\pmod{p^2}\ \text{ for all } s=1,\dots,p-2.\tag{3.5}$$

Lemma 3.3. (X. Zhou and T. Cai [22, p. 1332]) *Let $r,s\in\mathbb{Z}^+$. For any prime $p>rs+2$, we have*

$$\sum_{1\leqslant i_1<i_2<\ldots<i_r\leqslant p-1}\frac{1}{i_1^si_2^s\ldots i_r^s}\equiv\begin{cases}(-1)^r\frac{s(rs+1)}{2(rs+2)}p^2B_{p-rs-2}\pmod{p^3} & \text{if } 2\nmid rs,\\ (-1)^{r-1}\frac{s}{rs+1}pB_{p-rs-1}\pmod{p^2} & \text{if } 2\mid rs.\end{cases}$$

Lemma 3.4. (i) (J. Zhao [21, Theorems 3.1 and 3.2]) *Let $s,t\in\mathbb{Z}^+$ and let $p\geqslant s+t$ be a prime. Then*

$$\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^sk^t}\equiv\frac{(-1)^t}{s+t}\binom{s+t}{s}B_{p-s-t}\pmod p.$$

If $s\equiv t\pmod 2$ and $p>s+t+1$, then

$$\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^sk^t}\equiv\left((-1)^st\binom{s+t+1}{s}-(-1)^ts\binom{s+t+1}{t}-s-t\right)\times\frac{pB_{p-s-t-1}}{2(s+t+1)}\pmod{p^2}.$$

(ii) (J. Zhao [21, Theorem 3.5]) *Let $r,s,t\in\mathbb{Z}^+$ with $r+s+t$ odd. For any prime $p>r+s+t$, we have*

$$\sum_{1\leqslant i<j<k\leqslant p-1}\frac{1}{i^rj^sk^t}\equiv\left((-1)^r\binom{r+s+t}{r}-(-1)^t\binom{r+s+t}{t}\right)\times\frac{B_{p-(r+s+t)}}{2(r+s+t)}\pmod p.$$

(iii) (J. Zhao [21, Corollary 3.6]) *For any prime $p>5$, we have*

$$\sum_{1\leqslant i<j\leqslant k\leqslant p-1}\frac{1}{i^2jk}\equiv\sum_{1\leqslant i<j\leqslant k\leqslant p-1}\frac{1}{ij^2k}\equiv\sum_{1\leqslant i<j\leqslant k\leqslant p-1}\frac{1}{ijk^2}\equiv 0\pmod p.$$

Lemma 3.5. *For any prime $p>5$, we have*

$$\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k-1}\frac{1}{i^2j^2}\equiv\frac{2}{5}pB_{p-5}-3\frac{H_{p-1}}{p^2}\pmod{p^2}\tag{3.6}$$

and

$$\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k-1}\frac{H_{k-1}}{i^2j^2}\equiv\frac{3H_{p-1}}{p^2}\pmod p. \tag{3.7}$$

Proof. (i) Clearly,

$$\begin{aligned}&\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k-1}\frac{1}{i^2j^2}+\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^2j^2}\\=&\sum_{1\leqslant i<j\leqslant p-1}\frac{\sum_{j<k\leqslant p}1}{i^2j^2}=\sum_{1\leqslant i<j\leqslant p-1}\frac{p-j}{i^2j^2}\end{aligned}$$

and hence

$$\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k-1}\frac{1}{i^2j^2}=(p-1)\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^2j^2}-\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^2j}. \tag{3.8}$$

Note that $H_{p-1,3}\equiv 0 \pmod{p^2}$ by (3.5) or Lemma 3.3, and

$$\sum_{k=1}^{p-1}\frac{H_{k-1}}{k^2}\equiv -3\frac{H_{p-1}}{p^2}\pmod{p^2} \tag{3.9}$$

by R. Tauraso [18, Theorem 1.3]. Thus

$$\begin{aligned}\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^2j}=&\sum_{i=1}^{p-1}\frac{H_{p-1}-H_i}{i^2}\\=&H_{p-1}H_{p-1,2}-H_{p-1,3}-\sum_{i=1}^{p-1}\frac{H_{i-1}}{i^2}\equiv 3\frac{H_{p-1}}{p^2}\pmod{p^2}.\end{aligned}$$

By Lemma 3.2,

$$\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^2j^2}\equiv -\frac{2}{5}pB_{p-5}\pmod{p^2}. \tag{3.10}$$

Combining these with (3.8), we immediately obtain (3.6).

(ii) As $H_{p-1}\equiv 0 \pmod p$ and

$$\sum_{s=1}^{p-1}H_s=\sum_{1\leqslant r\leqslant s\leqslant p-1}\frac{1}{r}=\sum_{r=1}^{p-1}\frac{p-r}{r}=pH_{p-1}-(p-1)\equiv 1\pmod p,$$

we have

$$\begin{aligned}
&\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k-1}\frac{H_{k-1}}{i^2j^2}\\
\equiv&\sum_{1\leqslant i<j\leqslant p-1}\frac{\sum_{s=1}^{p-1}H_s-\sum_{0<s<j}H_s}{i^2j^2}\\
\equiv&\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^2j^2}\left(1-\sum_{1\leqslant r\leqslant s<j}\frac{1}{r}\right)\\
=&\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^2j^2}\left(1-\sum_{0<r<j}\frac{j-r}{r}\right)=\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^2j}\left(1-\sum_{0<r<j}\frac{1}{r}\right)\\
=&\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^2j}-\sum_{1\leqslant i<j\leqslant p-1}\frac{1}{i^3j}-\sum_{1\leqslant r<i<j\leqslant p-1}\frac{1}{ri^2j}-\sum_{1\leqslant i<r<j\leqslant p-1}\frac{1}{i^2rj}\\
\equiv&\frac{(-1)^1}{3}\binom{3}{2}B_{p-3}-\frac{(-1)^1}{4}\binom{4}{3}B_{p-4}-0-0\equiv\frac{3H_{p-1}}{p^2}\pmod p
\end{aligned}$$

with the help of Lemma 3.4 and (3.4). This proves (3.7).

The proof of Lemma 3.5 is now complete. □

Lemma 3.6. *Let $p>5$ be a prime. Then*

$$\sum_{k=1}^{p-1}\binom{p}{k}\frac{(-1)^k}{k}H_{k-1,2}\equiv\frac{9}{10}p^2B_{p-5}\pmod{p^3}.\tag{3.11}$$

Proof. As

$$(-1)^{k-1}\binom{p-1}{k-1}=\prod_{0\leqslant j<k}\left(1-\frac{p}{j}\right)\equiv1-pH_{k-1}\pmod{p^2}$$

for all $k=1,\ldots,p-1$, we have

$$\begin{aligned}
\sum_{k=1}^{p-1}\binom{p}{k}\frac{(-1)^k}{k}H_{k-1,2}=&p\sum_{k=1}^{p-1}\binom{p-1}{k-1}\frac{(-1)^k}{k^2}H_{k-1,2}\\
\equiv&p\sum_{k=1}^{p-1}\frac{pH_{k-1}-1}{k^2}H_{k-1,2}\pmod{p^3}.
\end{aligned}$$

By (3.10),

$$\sum_{k=1}^{p-1}\frac{H_{k-1,2}}{k^2}\equiv-\frac{2}{5}pB_{p-5}\pmod{p^2}.$$

Note also that

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_{k-1}H_{k-1,2}}{k^2} &= \sum_{\substack{1\leqslant j<k\leqslant p-1\\ 1\leqslant i<k}}\frac{1}{ij^2k^2}\\ &= \sum_{1\leqslant j<k\leqslant p-1}\frac{1}{j^3k^2}+\sum_{1\leqslant i<j<k\leqslant p-1}\frac{1}{ij^2k^2}+\sum_{1\leqslant j<i<k\leqslant p-1}\frac{1}{j^2ik^2}\\ &\equiv \frac{(-1)^2}{3+2}\binom{3+2}{3}B_{p-5}+\left((-1)^1\binom{5}{1}-(-1)^2\binom{5}{2}\right)\frac{B_{p-5}}{2\times 5}+0\\ &\equiv 2B_{p-5}-\frac{3}{2}B_{p-5}=\frac{B_{p-5}}{2}\pmod p\end{aligned}$$

with the help of parts (i) and (iii) of Lemma 3.4. Combining the above, we get the desired congruence (3.11). □

Lemma 3.7. *For any prime $p>5$, we have*

$$\binom{2p-1}{p-1}\equiv 1+2pH_{p-1}-\frac{4}{5}p^5B_{p-5}\pmod{p^6}.$$

Proof. It is known (cf. [18, Theorem 2.4]) that

$$\binom{2p-1}{p-1}\equiv 1+2pH_{p-1}+\frac{2}{3}p^3H_{p-1,3}\pmod{p^6}.$$

By Lemma 3.3,

$$H_{p-1,3}\equiv -\frac{3\times 4}{2\times 5}p^2B_{p-5}=-\frac{6}{5}p^2B_{p-5}\pmod{p^3}. \tag{3.12}$$

So, the desired congruence follows. □

Lemma 3.8. *For any prime $p>7$, we have*

$$H_{p-1}+\frac{p}{2}H_{p-1,2}+\frac{p^2}{6}H_{p-1,3}\equiv 0\pmod{p^6}. \tag{3.13}$$

Proof. Clearly,

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{1}{p-k} &\equiv \sum_{k=1}^{p-1}\frac{p^6-k^6}{p-k}\left(-\frac{1}{k^6}\right)=-\sum_{k=1}^{p-1}\frac{p^5+p^4k+p^3k^2+p^2k^3+pk^4+k^5}{k^6}\\ &=-p^5H_{p-1,6}-p^4H_{p-1,5}-p^3H_{p-1,4}-p^2H_{p-1,3}-pH_{p-1,2}-H_{p-1}\\ &\equiv -p^3H_{p-1,4}-p^2H_{p-1,3}-pH_{p-1,2}-H_{p-1}\pmod{p^6}\end{aligned}$$

since $H_{p-1,6}\equiv 0\pmod p$ and $H_{p-1,5}\equiv 0\pmod{p^2}$ by (3.5) or Lemma 3.3. Note that

$$H_{p-1,4}\equiv 4p\left(\frac{B_{2p-6}}{2p-6}-2\frac{B_{p-5}}{p-5}\right)\equiv -\frac{2}{3p}H_{p-1,3}\pmod{p^3}$$

by [14, Theorem 5.1 and Remark 5.1]. Therefore

$$2H_{p-1} + pH_{p-1,2} \equiv \frac{2}{3}p^2 H_{p-1,3} - p^2 H_{p-1,3} \pmod{p^6}$$

and hence (3.13) follows. □

Remark 3.9. Let $p > 3$ be a prime. If $p > 7$ then (3.13) has the following equivalent form

$$2H_{p-1} + pH_{p-1,2} \equiv \frac{2}{5}p^4 B_{p-5} \pmod{p^5} \tag{3.14}$$

in view of (3.12). It is easy to see that (3.14) also holds for $p = 5, 7$.

4. Proof of Theorem 1.2

To prove Theorem 1.2, we need an auxiliary theorem.

Theorem 4.1. *Let $p > 5$ be a prime. Then*

$$\sum_{k=1}^{p-1} \binom{p}{k}^2 \binom{p+k-1}{k-1} \equiv pH_{p-1} + \frac{9}{10}p^5 B_{p-5} \pmod{p^6}, \tag{4.1}$$

$$\sum_{k=1}^{p-1} \binom{p}{k}^2 \binom{p+k-1}{k-1} k \equiv -3p^2 H_{p-1} + \frac{21}{10}p^6 B_{p-5} \pmod{p^7}, \tag{4.2}$$

and

$$\begin{aligned} &\sum_{k=1}^{p-1} \binom{p}{k}^2 \binom{p+k-1}{k-1} k^2 \\ \equiv& - p^2 - (4p^5 + 4p^4 + 2p^3)H_{p-1} + \frac{4}{5}p^7 B_{p-5} \pmod{p^8}. \end{aligned} \tag{4.3}$$

Proof. For each $k = 1, \ldots, p-1$, we clearly have

$$\begin{aligned} \frac{1}{p}\binom{p}{k} =& \frac{(-1)^{k-1}}{k} \prod_{0<j<k} \left(1 - \frac{p}{j}\right) \\ \equiv& \frac{(-1)^{k-1}}{k} \left(1 - pH_{k-1} + p^2 \sum_{1\leqslant i<j\leqslant k-1} \frac{1}{ij}\right) \pmod{p^3} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} &(-1)^{k-1}\binom{p-1}{k-1}\binom{p+k-1}{k-1} \\ =& \prod_{0<j<k} \left(1 - \frac{p^2}{j^2}\right) \equiv 1 - p^2 H_{k-1,2} + p^4 \sum_{1\leqslant i<j\leqslant k-1} \frac{1}{i^2 j^2} \pmod{p^6}. \end{aligned} \tag{4.5}$$

Let $r \in \{0,1,2\}$ and

$$\sigma_r := \sum_{k=1}^{p-1} \binom{p}{k}^2 \binom{p+k-1}{k-1} k^r. \tag{4.6}$$

In view of the above, we have

$$\begin{aligned}\sigma_r =& p\sum_{k=1}^{p-1}\binom{p}{k}k^{r-1}\binom{p-1}{k-1}\binom{p+k-1}{k-1}\\ \equiv& p\sum_{k=1}^{p-1}\binom{p}{k}k^{r-1}(-1)^{k-1}\bigg(1-p^2H_{k-1,2}+p^4\sum_{1\leqslant i<j\leqslant k-1}\frac{1}{i^2j^2}\bigg)\\ \equiv& p\sum_{k=1}^{p-1}\binom{p}{k}k^{r-1}(-1)^{k-1}+p^3\sum_{k=1}^{p-1}\binom{p}{k}k^{r-1}(-1)^kH_{k-1,2}\\ &+p^5\sum_{k=1}^{p-1}\frac{p}{k}(1-pH_{k-1})k^{r-1}\sum_{1\leqslant i<j\leqslant k-1}\frac{1}{i^2j^2} \pmod{p^8}.\end{aligned}$$

Case 1. $r=0$.

In this case,

$$p\sum_{k=1}^{p-1}\binom{p}{k}k^{r-1}(-1)^{k-1}=pH_p-1=pH_{p-1}$$

by (3.1), and hence

$$\sigma_0 \equiv pH_{p-1}+p^3\sum_{k=1}^{p-1}\binom{p}{k}\frac{(-1)^k}{k}H_{k-1,2}\equiv pH_{p-1}+\frac{9}{10}p^5B_{p-5} \pmod{p^6}$$

with the help of Lemma 3.6. This proves (4.1).

Case 2. $r=1$.

In this case,

$$p\sum_{k=1}^{p-1}\binom{p}{k}k^{r-1}(-1)^{k-1}=-p\sum_{k=0}^{p}\binom{p}{k}(-1)^k=-p(1-1)^p=0,$$

and also

$$\sum_{1\leqslant i<j<k\leqslant p-1}\frac{1}{i^2j^2k}\equiv\left((-1)^2\binom{5}{2}-(-1)^1\binom{5}{1}\right)\frac{B_{p-5}}{2\times 5}=\frac{3}{2}B_{p-5} \pmod{p}$$

by Lemma 3.4. Thus

$$\sigma_1\equiv p^3\sum_{k=1}^{p-1}\binom{p}{k}(-1)^kH_{k-1,2}+\frac{3}{2}p^6B_{p-5} \pmod{p^7}. \tag{4.7}$$

In view of (3.3) and (4.4), we have

$$\begin{aligned}\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k H_{k-1,2} &= \sum_{k=1}^{p}\binom{p}{k}(-1)^k H_{k,2} + H_{p,2} - \sum_{k=1}^{p-1}\binom{p}{k}\frac{(-1)^k}{k^2}\\ &\equiv -\frac{H_p}{p} + H_{p,2} + p\sum_{k=1}^{p-1}\frac{1-pH_{k-1}+p^2\sum_{1\leqslant i<j\leqslant k-1}\frac{1}{ij}}{k^3}\\ &\equiv -\frac{H_{p-1}}{p} + H_{p-1,2} + pH_{p-1,3} - p^2\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{jk^3}\\ &\quad + p^3\sum_{1\leqslant i<j<k\leqslant p-1}\frac{1}{ijk^3} \pmod{p^4}.\end{aligned}$$

Note that

$$H_{p-1,2} \equiv \frac{2}{5}p^3B_{p-5} - \frac{2}{p}H_{p-1} \pmod{p^4} \text{ and } H_{p-1,3} \equiv -\frac{6}{5}p^2B_{p-5} \pmod{p^3}$$

by (3.14) and (3.12). In view of Lemma 3.4,

$$\sum_{1\leqslant j<k\leqslant p-1}\frac{1}{jk^3} \equiv \left(-3\binom{5}{1} - (-1)\binom{5}{3} - 1 - 3\right)\frac{pB_{p-5}}{2\times 5} = -\frac{9}{10}pB_{p-5} \pmod{p^2}$$

and

$$\sum_{1\leqslant i<j<k\leqslant p-1}\frac{1}{ijk^3} \equiv \left((-1)^1\binom{5}{1} - (-1)^3\binom{5}{3}\right)\frac{B_{p-5}}{2\times 5} = \frac{B_{p-5}}{2} \pmod{p}.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k H_{k-1,2} &\equiv -3\frac{H_{p-1}}{p} + \frac{2}{5}p^3B_{p-5} - \frac{6}{5}p^3B_{p-5} + \frac{9}{10}p^3B_{p-5} + \frac{p^3}{2}B_{p-5}\\ &\equiv -\frac{3}{p}H_{p-1} + \frac{3}{5}p^3B_{p-5} \pmod{p^4}.\end{aligned}$$

Combining this with (4.7), we see that

$$\sigma_1 \equiv p^3\left(-\frac{3}{p}H_{p-1} + \frac{3}{5}p^3B_{p-5}\right) + \frac{3}{2}p^6B_{p-5} = -3p^2H_{p-1} + \frac{21}{10}p^6B_{p-5} \pmod{p^7}.$$

This proves (4.2).

Case 3. $r = 2$.

In this case,

$$p\sum_{k=1}^{p-1}\binom{p}{k}k^{r-1}(-1)^{k-1} = p^2\sum_{k=1}^{p-1}\binom{p-1}{k-1}(-1)^{k-1} = p^2\left((1-1)^{p-1}-1\right) = -p^2$$

and hence

$$\begin{aligned}\sigma_2 \equiv & -p^2+p^4\sum_{k=1}^{p-1}\binom{p-1}{k-1}(-1)^k H_{k-1,2}\\ & +p^6\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k-1}\frac{1}{i^2j^2}-p^7\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k-1}\frac{H_{k-1}}{i^2j^2}\\ \equiv & -p^2+p^4\left(\frac{H_{p-1}}{p-1}+H_{p-1,2}\right)+p^6\left(\frac{2}{5}pB_{p-5}-3\frac{H_{p-1}}{p^2}\right)-p^7\frac{3H_{p-1}}{p^2}\\ \equiv & -p^2+p^4\left(-(p+1)H_{p-1}+\frac{2}{5}p^3B_{p-5}-\frac{2}{p}H_{p-1}\right)\\ & +\frac{2}{5}p^7B_{p-5}-3p^4H_{p-1}-3p^5H_{p-1} \pmod{p^8}\end{aligned}$$

with the help of (3.3), (3.14) and Lemma 3.5. This yields the desired (4.3).

In view of the above, we have completed the proof of Theorem 4.1. □

Proof of Theorem 1.2. It is easy to see that (1.4)-(1.6) hold for $p=5$. Below we assume $p>5$.

For $i=1,2,3$ let

$$S_i:=\sum_{k=0}^{p-1}\binom{p}{k+1}^2\binom{p+k}{k}a_i(p,k)$$

with $a_i(n,k)$ given by Lemma 2.1. In view of Lemma 2.1, it suffices to show the following three congruences:

$$S_1\equiv -4p^2-4p^7B_{p-5} \pmod{p^8}, \tag{4.8}$$

$$S_2\equiv 2p^2+10p^3H_{p-1}+p^7B_{p-5} \pmod{p^8}, \tag{4.9}$$

$$S_3\equiv 2p^3-3p^2+(10p^4-5p^3)H_{p-1}-\frac{3}{2}p^7B_{p-5} \pmod{p^8}, \tag{4.10}$$

Clearly,

$$\begin{aligned}S_i=&\sum_{j=1}^{p}\binom{p}{j}^2\binom{p+j-1}{j-1}a_i(p,j-1)\\ =&\binom{2p-1}{p-1}a_i(p,p-1)+\sum_{k=1}^{p-1}\binom{p}{k}^2\binom{p+k-1}{k-1}a_i(p,k-1).\end{aligned} \tag{4.11}$$

Note that

$$a_1(p,p-1)=-p^3-3p^2,\ a_2(p,p-1)=2p^2 \text{ and } a_3(p,p-1)=p^3-2p^2.$$

In view of Lemma 3.7, we have

$$\begin{aligned}
&\binom{2p-1}{p-1} a_i(p,p-1)\\
\equiv&\left(1+2pH_{p-1}-\frac{4}{5}p^5B_{p-5}\right)a_i(p,p-1)\\
\equiv&\begin{cases} -p^3-2p^4H_{p-1}-3p^2-6p^3H_{p-1}+\frac{12}{5}p^7B_{p-5} \pmod{p^8} & \text{if } i=1,\\ 2p^2+4p^3H_{p-1}-\frac{8}{5}p^7B_{p-5} \pmod{p^8} & \text{if } i=2,\\ p^3+2p^4H_{p-1}-2p^2-4p^3H_{p-1}+\frac{8}{5}p^7B_{p-5} \pmod{p^8} & \text{if } i=3.\end{cases}
\end{aligned} \tag{4.12}$$

For $r=0,1,2$ let σ_r be defined as in (4.6). Noting

$$a_1(p,k-1)=(p^3-p^2)-(p^2+3p)k-(p-1)k^2$$

and applying Theorem 4.1, we get

$$\begin{aligned}
&\sum_{k=1}^{p-1}\binom{p}{k}^2\binom{p+k-1}{k-1}a_1(p,k-1)\\
=&(p^3-p^2)\sigma_0-(p^2+3p)\sigma_1-(p-1)\sigma_2\\
\equiv&(p^3-p^2)\left(pH_{p-1}+\frac{9}{10}p^5B_{p-5}\right)-(p^2+3p)\left(-3p^2H_{p-1}+\frac{21}{10}p^6B_{p-5}\right)\\
&-(p-1)\left(-p^2-(4p^5+4p^4+2p^3)H_{p-1}+\frac{4}{5}p^7B_{p-5}\right)\\
\equiv&p^3-p^2+(2p^4+6p^3)H_{p-1}-\frac{32}{5}p^7B_{p-5} \pmod{p^8}.
\end{aligned}$$

Combining this with (4.11) and (4.12), we get (4.8).

Similarly,

$$\begin{aligned}
S_2\equiv&2p^2+4p^3H_{p-1}-\frac{8}{5}p^7B_{p-5}+3p^2\sigma_0-p\sigma_1\\
\equiv&2p^2+4p^3H_{p-1}-\frac{8}{5}p^7B_{p-5}+3p^2\left(pH_{p-1}+\frac{9}{10}p^5B_{p-5}\right)\\
&-p\left(-3p^2H_{p-1}+\frac{21}{10}p^6B_{p-5}\right)\\
\equiv&2p^2+10p^3H_{p-1}-p^7B_{p-5} \pmod{p^8}.
\end{aligned}$$

This proves (4.9).

As

$$a_3(p,k-1)=4p^3-2p^2-(2p^2+p)k-(p-1)k^2,$$

we have

$$\begin{aligned}
&\sum_{k=1}^{p-1}\binom{p}{k}^2\binom{p+k-1}{k-1}a_3(p,k-1)\\
=&(4p^3-2p^2)\sigma_0-(2p^2+p)\sigma_1-(p-1)\sigma_2\\
\equiv&(4p^3-2p^2)\left(pH_{p-1}+\frac{9}{10}p^5B_{p-5}\right)-(2p^2+p)\left(-3p^2H_{p-1}+\frac{21}{10}p^6B_{p-5}\right)\\
&-(p-1)\left(-p^2-(4p^5+4p^4+2p^3)H_{p-1}+\frac{4}{5}p^7B_{p-5}\right)\\
\equiv&p^3-p^2+(8p^4-p^3)H_{p-1}-\frac{31}{10}p^7B_{p-5}\pmod{p^8}.
\end{aligned}$$

Combining this with (4.11) and (4.12), we get (4.10).

The proof of Theorem 1.2 is now complete. □

5. Proofs of theorems 1.3 and 1.5

Clearly $L_2(0)=0$, and for each $n\in\mathbb{Z}^+$ we have

$$L_2(n)=L_2\left(\left\lfloor\frac{n}{2}\right\rfloor\right)+\begin{cases}0 & \text{if } n\equiv 0,3 \pmod 4,\\ 1 & \text{if } n\equiv 1,2 \pmod 4.\end{cases}$$

E. Rowland and R. Yassawi discovered that the two least significant binary digits of $L_2(n)$ are completely determined by the fourth and the third least significant digits of the Apéry number A_n. (As $A_n\equiv 1 \pmod 4$, the two least significant digits of A_n are always 0 and 1, and hence these digits contain no information about n.)

Theorem 5.1. [12, Theorem 3.28] *Let* $n\in\mathbb{N}$. *If*

$$A_n\equiv 8j+4i+1 \pmod{16} \tag{5.1}$$

with $\{i,j\}\subseteq\{0,1\}$, *then*

$$L_2(n)\equiv 2j+i \pmod 4. \tag{5.2}$$

Theorem 1.3 shows that the fourth and the third least significant digits of A_n also determine the second least significant digit of β_n (all numbers β_n are odd, so the very least significant digit is always 1).

Theorem 1.5 shows that the knowledge of the five least significant digits of A_n by itself is not sufficient for determining the three least significant digits of $L_2(n)$. This is because the third least significant digit of β_n can be 0 or 1. But the additional knowledge of this digit gives the missing bit of information for determining the three least significant digits of $L_2(n)$ according to Theorem 1.5.

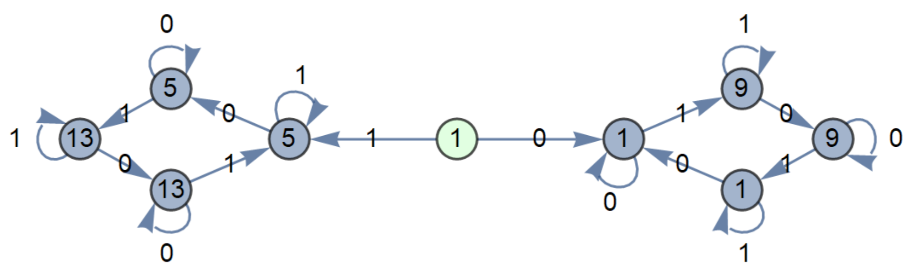

FIGURE 1. Automaton $\mathcal{A}_{16}$ calculating A_n mod 16

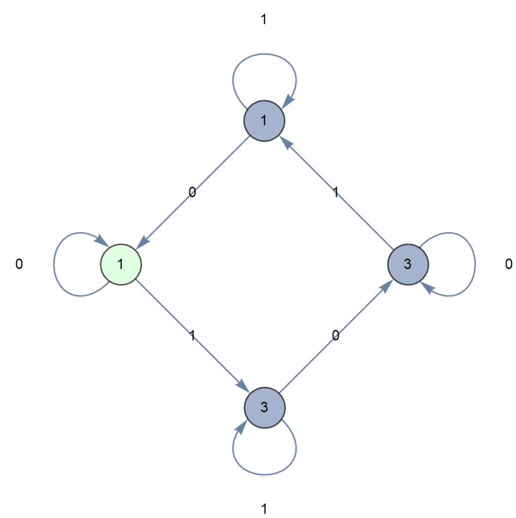

FIGURE 2. Automaton $\mathcal{B}_4$ calculating β_n mod 4

Theorems 1.3 and 1.5 can be proved by the technique used in [12] with the aid of software [10] implemented by E. Rowland.

Namely, Apéry numbers of both kinds are known to be the diagonal sequences of certain rational functions. In particular, A. Straub [13] proved that A_n is the coefficient of $x_1^n x_2^n x_3^n x_4^n$ in the formal Taylor expansion of the function

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}. \tag{5.3}$$

Similar coefficient in the expansion of

$$\frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)-(1-x_1)x_1x_2x_3} \tag{5.4}$$

is, according to [12], equal to β_n.

According to [12, Theorem 2.1], the above representations of Apéry numbers as the diagonal coefficients imply that for every prime p and its power q the sequences A_n mod q and β_n mod q are p-automatic. Moreover, [12] contains two algorithms for constructing corresponding automata; `Mathematica` implementations of these algorithms are given in [10].

To prove Theorem 1.3, we need to examine the sequence A_n mod 16. Corresponding automaton $\mathcal{A}_{16}$ was calculated in [11] and exhibited in [12], we reproduce it here in Figure 1 (with the initial state marked in light color). This automaton calculates A_n mod 16 in the following way. Let

$$n = \sum_{k=0}^{m} n_k 2^k \tag{5.5}$$

where $\{n_0, \dots, n_m\} \subseteq \{0, 1\}$; start from the initial state and follow the oriented path formed by edges marked $n_0, \dots, n_m$; the final vertex is labeled by A_n mod 16.

Function `AutomaticSequenceReduce` from [10], being applied to (5.4), returns the minimal automaton $\mathcal{B}_4$ calculating β_n mod 4; this automaton is exhibited in Figure 2.

Let

$$f(1) = f(13) = 1 \quad \text{and} \quad f(5) = f(9) = 3.$$

Theorem 1.3 asserts that

$$(\beta_n \mod 4) = f(A_n \mod 16). \tag{5.6}$$

We can construct an automaton calculating $f(A_n$ mod 16) simply by applying function f to the labels of states in automaton $\mathcal{A}_{16}$. The resulting automaton is not minimal, but we can minimize it using, for example, function `AutomatonMinimize` from [10]. The minimal automaton calculating $f(A_n$ mod 16) turns out to be equal to $\mathcal{B}_4$, which proves (5.6) and hence Theorem 1.3 is proved as well.

The minimal automaton $\mathcal{A}_{32}$ calculating A_n mod 32 was constructed in [11], it has 33 states; the minimal automaton $\mathcal{B}_8$ calculating β_n mod 8 has 17 states. Taking into account Theorem 5.1, for proving Theorem 1.5 it is sufficient to work just with the fifth least significant digits of A_n and the third least significant digits of β_n and $L_2(n)$, which allows us to deal with automata having fewer number of states.

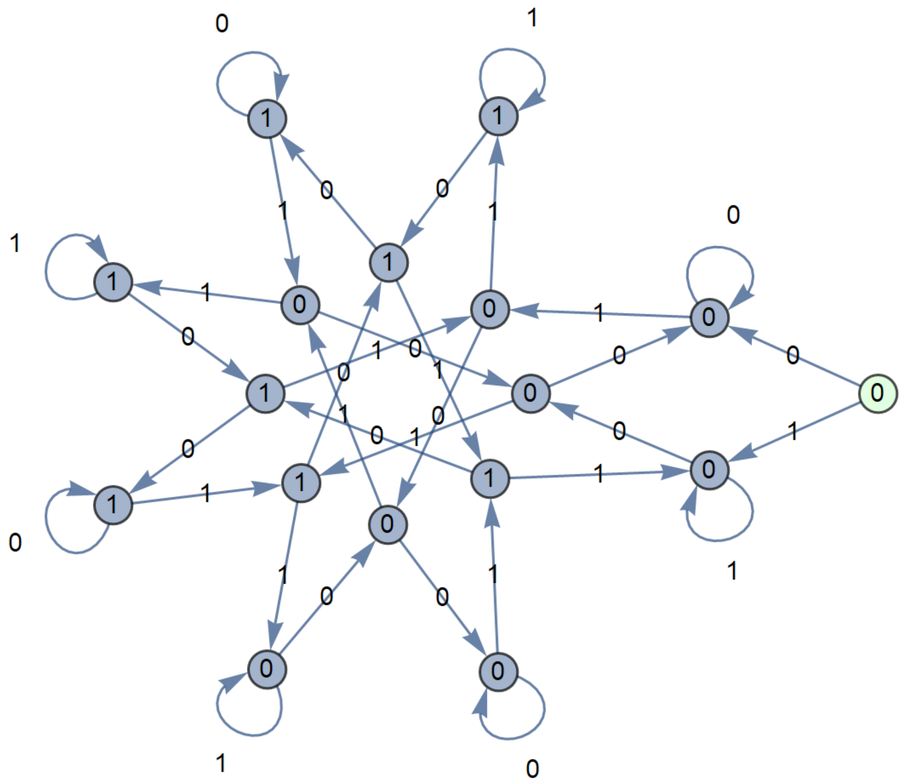

FIGURE 3. Automaton $\mathcal{A}_{32,5}$ calculating the fifth digit of A_n

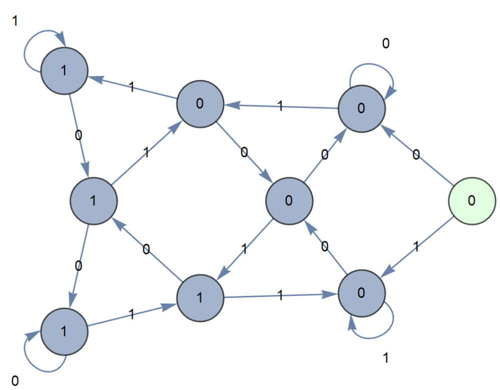

FIGURE 4. Automaton $\mathcal{B}_{8,3}$ calculating the third digit of β_n

Figures 3 and 4 exhibit the minimal automata $\mathcal{A}_{32,5}$ and $\mathcal{B}_{8,3}$ calculating respectively the fifth least significant digit of A_n and the third least significant digit of β_n. These automata are easily constructed from $\mathcal{A}_{32}$ and $\mathcal{B}_8$ by properly relabelling the states and minimizing resulting automata.

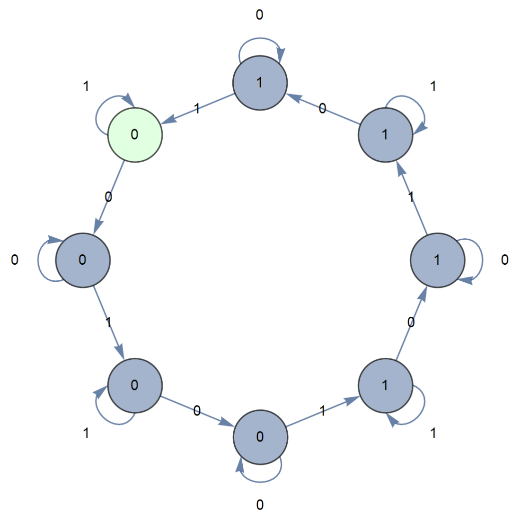

FIGURE 5. Automaton calculating the third digit of $L_2(n)$

With function `AutomatonProduct` from [10] we can easily construct automaton $\mathcal{C}$ which is the product of $\mathcal{A}_{32,5}$ and $\mathcal{B}_{8,3}$. Each state of $\mathcal{C}$ is labeled by two bits (corresponding to the fifth digit of A_n and the third digit of β_n). Replacing each such pair of bits by their sum modulo 2 and performing minimization, we get the automaton exhibited on Figure 5. It is not difficult to see that this automaton calculates the third digit of $L_2(n)$, which proves Theorem 1.5.

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