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NEW TYPE SERIES FOR POWERS OF π

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Abstract. Motivated by Ramanujan-type series and Zeilberger-type series, in this paper we investigate two new types of series for powers of π. For example, we prove that

$$\sum_{k=0}^{\infty}(198k^2-425k+210)\frac{k^3\binom{2k}{k}^3}{4096^k}=-\frac{1}{21\pi}$$

and

$$\sum_{k=0}^{\infty}\frac{198k^2-227k+47}{\binom{2k}{k}^3}=\frac{3264-4\pi^2}{63}.$$

We also pose many conjectures in this new direction.

1. Introduction

In 1859, G. Bauer got the following identity:

$$\sum_{k=0}^{\infty}\frac{4k+1}{(-64)^k}\binom{2k}{k}^3=\frac{2}{\pi}.$$

This began the history of series for π^{-1}. In 1914, S. Ramanujan [16] posed 16 conjectural series of the following form:

$$\sum_{k=0}^{\infty}\frac{bk+c}{m^k}a(k)=\frac{\lambda\sqrt{d}}{\pi}$$

where $b,c,m\in\mathbb{Z}$ with $bm\neq 0$, $d\in\mathbb{Z}^+=\{1,2,3,\ldots\}$ is squarefree, $\lambda\in\mathbb{Q}\setminus\{0\}$, and $a(k)\in\mathbb{Z}$ is one of the products

$$\binom{2k}{k}^3,\ \binom{2k}{k}^2\binom{3k}{k},\ \binom{2k}{k}^2\binom{4k}{2k},\ \binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}. \tag{1.1}$$

There are totally 36 series of the above type, which are called rational Ramanujan-type series for π^{-1}. For surveys of such series, one may consult [2, 3, 8] and S. Cooper [11, Chapter 14]). In general, a Ramanujan-type series (of type R) has the form

$$\sum_{k=0}^{\infty}\frac{P_d(k)}{m^k}a(k)=\frac{\alpha}{\pi^d}, \tag{R}$$

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where $P_d(x) \in \mathbb{Z}[x]$ with $\deg P = d > 0$, m is a nonzero integer, α is an algebraic number, and $a(k) \in \mathbb{Z}$ for all $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. For example, B. Gourevich used the PSLQ algorithm to find the conjectural identity

$$\sum_{k=0}^{\infty} \frac{168k^3 + 76k^2 + 14k + 1}{2^{20k}} \binom{2k}{k}^7 = \frac{32}{\pi^3}. \tag{1.2}$$

In this paper we introduce a new type of series different from Ramanujan's type. A series of *type S* has the form

$$\sum_{k=0}^{\infty} \frac{P(k)Q_{2d}(k)}{m^k} a(k) = \frac{\alpha}{\pi^d}, \tag{S}$$

where $P(x), Q_{2d}(x) \in \mathbb{Z}[x]$ with $\deg P > 0$ and $\deg Q_{2d} \le 2d$, m is a nonzero integer, α is an algebraic number, and $a(k) \in \mathbb{Z}$ for all $k \in \mathbb{N}$. We also investigate what happens if we replace $\binom{2k}{k}$ in the general summand of the classical Ramanujan-type series by $\binom{2k}{k+1} = kC_k$, where C_k denotes the Catalan number $\frac{1}{k+1}\binom{2k}{k} = \binom{2k}{k} - \binom{2k}{k+1}$.

Now we state our first theorem which gives variants of the Ramanujan series involving cubes of central binomial coefficients.

Theorem 1.1. *We have the following identities:*

$$\sum_{k=0}^{\infty} \frac{(6k^2 - 19k + 6)k^3\binom{2k}{k}^3}{256^k} = -\frac{1}{12\pi}, \tag{S1}$$

$$\sum_{k=0}^{\infty} \frac{(13608k^2 + 25050k + 10589)\binom{2k}{k+1}^3}{256^k} = 27296 - \frac{84604}{\pi}, \tag{S1$'$}$$

$$\sum_{k=0}^{\infty} \frac{(18k^2 - 29k + 16)k^3\binom{2k}{k}^3}{(-512)^k} = \frac{\sqrt{2}}{24\pi}, \tag{S2}$$

$$\sum_{k=0}^{\infty} \frac{(83592k^2 + 152922k + 64925)\binom{2k}{k+1}^3}{(-512)^k} = 625282\frac{\sqrt{2}}{\pi} - 281920, \tag{S2$'$}$$

$$\sum_{k=0}^{\infty} \frac{(198k^2 - 425k + 210)k^3\binom{2k}{k}^3}{4096^k} = -\frac{1}{21\pi}, \tag{S3}$$

$$\sum_{k=0}^{\infty} \frac{(32473224k^2 + 58012446k + 24235261)\binom{2k}{k+1}^3}{4096^k} = 667628032 - \frac{2097324016}{\pi}. \tag{S3$'$}$$

Remark 1.1. Motivated by [27], we may also look at p-adic congruences corresponding to those identities in Theorem 1.1. For example, inspired by the identity (S1) we conjecture that for any prime $p>3$ we have

$$\sum_{k=1}^{(p-1)/2}\frac{k^3\binom{2k}{k}^3}{256^k}(6k^2-19k+6)\equiv(-1)^{\frac{p+1}{2}}\frac{p+2p^3}{48}\pmod{p^4}.$$

The paper [23] cotains many conjectural p-adic congruences.

For our variants of other classical Ramanujan-type series, the reader may consult Section 4.

In 1993 D. Zeilberger [30] used the WZ method to establish the identity

$$\sum_{k=1}^\infty\frac{21k-8}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{6}. \tag{1.3}$$

More such series were conjectured by the author [19], and confirmed by J. Guillera and M. Rogers [14]. In 1997, T. Amdeberhan and Zeilberger [1] used the WZ method to obtain the identity

$$\sum_{k=1}^\infty\frac{(-1)^k(205k^2-160k+32)}{k^5\binom{2k}{k}^5}=-2\zeta(3). \tag{1.4}$$

In general, a Zeilberger-type series (of type Z) has the form

$$\sum_{k=1}^\infty\frac{P_d(k)m^k}{k^{2d+1}a(k)}=cL(d+1,\chi) \tag{Z}$$

where $P_d(x)\in\mathbb{Z}[x]$ with $\deg P_d=d$, m is a nonzero integer, $a(k)$ is a product of $2d+1$ binomial coefficients, c is a nonzero rational number, and $L(2d,\chi)$ is the Dirichlet L function associated with a Dirichlet character χ. For example, in 2010 the author conjectured that

$$\sum_{k=1}^\infty\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3) \tag{1.5}$$

(cf. [20]) which remains open.

Motivated by the author's work [25] and the two new identities

$$\sum_{k=1}^\infty\frac{(22k^2-7k-3)64^k}{(2k+1)(3k+1)k^3\binom{2k}{k}^2\binom{3k}{k}}=48-4\pi^2$$

and

$$\sum_{k=1}^\infty\frac{(22k^2+5k+1)64^k}{(2k+1)(3k+1)k^2\binom{2k}{k}^2\binom{3k}{k}}=4\pi^2-16$$

missing from [25, Theorem 1.2], we introduce series of *type* T:

$$\sum_{k=1}^\infty\frac{Q_{2d}(k)m^k}{a(k)}=c_0+c_1L(d+1,\chi), \tag{T}$$

where $Q_{2d}(x)\in\mathbb{Z}[x]$ with $\deg Q_{2d}\le 2d$, m is a nonzero integer, $a(k)$ is a product of $2d+1$ binomial coefficients, c_0 and c_1 are algebraic numbers, and χ is a Dirichlet character. We also investigate what happens if we replace

$\binom{2k}{k}$ in the denominator of a general summand of the Zeilberger-type series (Z) by $\binom{2k}{k+1}=kC_k$.

Throught this paper, we adopt the notation

$$G=L\left(2,\left(\frac{-4}{\cdot}\right)\right)=\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^2}$$

and

$$K=L\left(2,\left(\frac{-3}{\cdot}\right)\right)=\sum_{k=1}^\infty\frac{(\frac{k}{3})}{k^2},$$

where $(\frac{d}{\cdot})$ with $d\equiv 0,1 \pmod 4$ is the Kronecker symbol, and $(\frac{k}{3})$ is the Legendre symbol. The number G is usually called the Catalan constant.

Now we state our second theorem.

Theorem 1.2. *We have the following identities:*

$$\sum_{k=0}^\infty\frac{198k^2-227k+47}{\binom{2k}{k}^3}=\frac{3264-4\pi^2}{63},\tag{T1}$$

$$\sum_{k=1}^\infty\frac{4884741k^2-7292783k+2041168}{\binom{2k}{k+1}^3}=\frac{68102148-10093535\pi^2}{126},\tag{T1$'$}$$

$$\sum_{k=0}^\infty\frac{(54k^2-33k+18)(-8)^k}{\binom{2k}{k}^3}=8+2G,\tag{T2}$$

$$\sum_{k=1}^\infty\frac{(11277k^2-13124k+3212)(-8)^k}{\binom{2k}{k+1}^3}=\frac{38954G-24679}{3},\tag{T2$'$}$$

$$\sum_{k=0}^\infty\frac{(850k^2-1133k+69)8^k}{\binom{2k}{k}^2\binom{3k}{k}}=\frac{885-12\pi^2}{5},\tag{T3}$$

$$\sum_{k=1}^\infty\frac{(4621650k^2-7550827k+1785654)8^k}{\binom{2k}{k+1}^2\binom{3k}{k+1}}=\frac{6728880-1468127\pi^2}{5},\tag{T3$'$}$$

$$\sum_{k=0}^\infty\frac{(18k^2-39k-6)16^k}{\binom{2k}{k}^3}=\frac{16-\pi^2}{4},\tag{T4}$$

$$\sum_{k=1}^\infty\frac{(2778k^2-6499k+1679)16^k}{\binom{2k}{k+1}^3}=\frac{89008-20305\pi^2}{12},\tag{T4$'$}$$

$$\sum_{k=0}^{\infty}\frac{(315k^2-54k+119)(-27)^k}{\binom{2k}{k}^2\binom{3k}{k}}=\frac{10+216K}{5}, \tag{T5}$$

$$\sum_{k=1}^{\infty}\frac{(3391155k^2-3249747k+646792)(-27)^{k-1}}{\binom{2k}{k+1}^2\binom{3k}{k+1}}=\frac{2344210-5115501K}{30}, \tag{T5$'$}$$

$$\sum_{k=0}^{\infty}\frac{(1298k^2-7807k-3165)64^k}{\binom{2k}{k}^2\binom{3k}{k}}=195-192\pi^2, \tag{T6}$$

$$\sum_{k=1}^{\infty}\frac{(3470742k^2-20527615k+4631490)64^{k-1}}{\binom{2k}{k+1}^2\binom{3k}{k+1}}=387807-198173\pi^2, \tag{T6$'$}$$

$$\sum_{k=0}^{\infty}\frac{(2050k^2-5331k-1357)81^k}{\binom{2k}{k}^2\binom{4k}{2k}}=\frac{8800-2592\pi^2}{35}, \tag{T7}$$

$$\sum_{k=1}^{\infty}\frac{(17439700k^2-47250409k+8929776)81^{k-1}}{\binom{2k}{k+1}^2\binom{4k}{2k+1}} =\frac{43153685-44613764\pi^2}{210}, \tag{T7$'$}$$

$$\sum_{k=0}^{\infty}\frac{(50k^2+93k+22)(-144)^k}{\binom{2k}{k}^2\binom{4k}{2k}}=\frac{243K-200}{4}, \tag{T8}$$

$$\sum_{k=1}^{\infty}\frac{(53725k^2-27494k+3675)(-144)^{k-1}}{\binom{2k}{k+1}^2\binom{4k}{2k+1}}=\frac{78296-122235K}{128}. \tag{T8$'$}$$

Our third theorem is motivated by the Amdeberhan-Zeilberger identity (1.4).

Theorem 1.3. *We have*

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{\binom{2k}{k}^5}(684700k^4-1418358k^3+1100639k^2-365392k+47327) =\frac{231168-64\zeta(3)}{5}. \tag{T9}$$

Recall that the Domb numbers are given by

$$\mathrm{Domb}(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}\quad (n=0,1,2,\ldots).$$

By H. H. Chan, S. H. Chan and Z. Liu [4] and M. D. Rogers [17],

$$\sum_{k=0}^{\infty}\frac{5k+1}{64^k}\mathrm{Domb}(k)=\frac{8}{\sqrt{3}\,\pi}\quad\text{and}\quad\sum_{k=0}^{\infty}\frac{3k+1}{(-32)^k}\mathrm{Domb}(k)=\frac{2}{\pi}. \tag{1.6}$$

Franel numbers of order 4 are those integers

$$f_n^{(4)}=\sum_{k=0}^n\binom nk^4\ \ (n\in\mathbb N).$$

In 2005 Yifan Yang discovered the first $\frac1\pi$-series involving Franel numbers of order four:

$$\sum_{k=0}^\infty\frac{4k+1}{36^k}f_k^{(4)}=\frac{18}{\sqrt{15}\,\pi}. \tag{1.7}$$

For more such series deduced via modular forms, see S. Cooper [9]. Apéry numbers of the second kind are those integers

$$\beta_n=\sum_{k=0}^n\binom nk^2\binom{n+k}k\ \ (n\in\mathbb N).$$

It is known (cf. Table 14.7 of [11, p. 653]) that

$$\sum_{k=0}^\infty\frac{5k+1}{72^k}\binom{2k}k\beta_k=\frac{9\sqrt2}{2\pi},\ \ \sum_{k=0}^\infty\frac{11k+2}{147^k}\binom{2k}k\beta_k=\frac{49\sqrt3}{10\pi},$$
$$\sum_{k=0}^\infty\frac{190k+29}{(-828)^k}\binom{2k}k\beta_k=\frac{18\sqrt{23}}{\pi},\ \ \sum_{k=0}^\infty\frac{682k+71}{(-15228)^k}\binom{2k}k\beta_k=\frac{162\sqrt{47}}{5\pi}.$$

For $n\in\mathbb N$ define the Cooper number

$$\mathrm{Co}(n)=\sum_{k=0}^n\binom nk^2\binom{n+k}k\binom{2k}n.$$

S. Cooper [10] deduced the following three identities for $1/\pi$:

$$\sum_{k=0}^\infty\frac{39k+10}{(-64)^k}\mathrm{Co}(k)=\frac{64}{\sqrt7\,\pi}, \tag{1.8}$$
$$\sum_{k=0}^\infty\frac{21k+4}{125^k}\mathrm{Co}(k)=\frac{125}{8\pi}, \tag{1.9}$$
$$\sum_{k=0}^\infty\frac{11895k+1286}{(-22)^{3k}}\mathrm{Co}(k)=\frac{22^3}{\sqrt7\,\pi}. \tag{1.10}$$

Now we state our fourth theorem.

Theorem 1.4. (i) *We have*

$$\sum_{k=1}^\infty\frac{k(3k+1)^2}{(-32)^k}\mathrm{Domb}(k)=-\frac2\pi, \tag{1.11}$$
$$\sum_{k=1}^\infty\frac{k^2(5k-3)}{64^k}\mathrm{Domb}(k)=\frac{8\sqrt3}{9\pi}. \tag{1.12}$$

(ii) *We have the following identities:*

$$\sum_{k=1}^{\infty}\frac{kf_k^{(4)}}{36^k}(40k^2-42k-1)=\frac{198\sqrt{15}}{25\pi}, \tag{1.13}$$

$$\sum_{k=1}^{\infty}\frac{kf_k^{(4)}}{(-64)^k}(80k^2+16k+3)=-\frac{544\sqrt{15}}{675\pi}, \tag{1.14}$$

$$\sum_{k=1}^{\infty}\frac{kf_k^{(4)}}{196^k}(33000k^2-3410k+291)=\frac{1582\sqrt{7}}{3\pi}, \tag{1.15}$$

$$\sum_{k=1}^{\infty}\frac{kf_k^{(4)}}{(-324)^k}(2720k^2+141k-7)=-\frac{3969\sqrt{5}}{200\pi}, \tag{1.16}$$

$$\sum_{k=1}^{\infty}\frac{kf_k^{(4)}}{1296^k}(156000k^2-2205k+563)=\frac{17901\sqrt{2}}{32\pi}, \tag{1.17}$$

$$\sum_{k=1}^{\infty}\frac{kf_k^{(4)}}{5776^k}(26079360k^2-81592k+33569)=\frac{220628\sqrt{95}}{75\pi}. \tag{1.18}$$

(iii) *We have*

$$\sum_{k=1}^{\infty}\frac{k\binom{2k}{k}\beta_k}{72^k}(125k^2-300k-17)=\frac{999\sqrt{2}}{8\pi}, \tag{1.19}$$

$$\sum_{k=1}^{\infty}\frac{k\binom{2k}{k}\beta_k}{147^k}(121k^2-78k-1)=\frac{21903\sqrt{3}}{1250\pi}, \tag{1.20}$$

$$\sum_{k=1}^{\infty}\frac{k\binom{2k}{k}\beta_k}{(-828)^k}(5234500k^2+395850k-31627)=\frac{22302\sqrt{23}}{\pi}, \tag{1.21}$$

and

$$\sum_{k=1}^{\infty}\frac{k\binom{2k}{k}\beta_k}{(-15228)^k}(4127975500k^2-17838750k+6386881)=-\frac{3699918\sqrt{47}}{5\pi}. \tag{1.22}$$

(iv) *We have*

$$\sum_{k=1}^{\infty}\frac{k\mathrm{Co}(k)}{(-64)^k}(1365k^2+575k+86)=-\frac{7424\sqrt{7}}{147\pi}, \tag{1.23}$$

$$\sum_{k=1}^{\infty}\frac{k\mathrm{Co}(k)}{125^k}(1029k^2-413k+4)=\frac{16375}{96\pi}, \tag{1.24}$$

and

$$\sum_{k=1}^{\infty}\frac{k\mathrm{Co}(k)}{(-22)^{3k}}(3480060675k^2+12712753k-5211590)=\frac{226632032\sqrt{7}}{147\pi}. \tag{1.25}$$

Remark 1.2. Actually our method to show Theorem 1.4 allows us to deduce many more similar identities from all the known Ramanujan-type series for $1/\pi$ listed in [11, pp. 647–658].

We will prove Theorem 1.1 in Section 2, and Theorems 1.2-1.3 in Section 3. Section 4 contains our variants of classical Ramanujan-type series not included in Theorem 1.1. Section 5 is devoted to our proof of Theorem 1.4.

Surprisingly, it seems that each non-classical Ramanujan-type series also has related series of type S. This enables us to pose many conjectural series of type S in Sections 6-8 motivated by known series of type R.

Now we present our general conjecture relating series of type R to series of type S.

Conjecture 1.1. *Suppose that we have a Ramanujan-type identity* (R) *with* $\limsup_{k\to+\infty}\sqrt[k]{|a(k)|}<|m|$. *For any non-constant polynomial* $P(x)\in\mathbb{Z}[x]$, *there is a polynomial* $Q_{2d}(x)\in\mathbb{Z}$ *of degree at most* $2d$ *such that*

$$\sum_{k=0}^{\infty}\frac{P(k)Q_{2d}(k)}{m^k}a(k)=\frac{\alpha'}{\pi^d}$$

for some $\alpha'\in\mathbb{Q}(\alpha)$.

Now we describe our algorithm to deduce series of type S from a Ramanujan-type series (R) with $a(k)$ a product of $2d+1$ binomial coefficients.

General Algorithm. Suppose that we have a Ramanujan-type series (R) with $a(k)$ a product of $2d+1$ binomial coefficients. Given a monic polynomial $P(x)\in\mathbb{Z}[x]$ with $\deg P>0$, we first write $P(x)=(x-\alpha_1)\cdots(x-\alpha_r)$ with $\alpha_1,\ldots,\alpha_r$ complex numbers. Using the Gosper algorithm (cf. [15]) we find a polynomial Q_{2d+1} of degree at most $2d+1$ such that $\sum_{k=0}^{n}\frac{a(k)}{m^k}Q_{2d+1}(k)$ has a closed form which tends to 0, so that we get

$$\sum_{k=0}^{\infty}\frac{a(k)}{m^k}Q_{2d+1}(k)=0. \tag{1.26}$$

Note that $(1.26)\times P_d(\alpha_1)-(R)\times Q_{2d+1}(\alpha_1)$ yields an identity of the form

$$\sum_{k=0}^{\infty}\frac{(k-\alpha_1)a(k)}{m^k}R_{2d}(k)=-Q_{2d+1}(\alpha_1)\frac{\alpha}{\pi^d}, \tag{1.27}$$

where R_{2d} is a polynomial of degree at most $2d$. Using the Gosper algorithm, we find

$$\sum_{k=0}^{\infty}\frac{(k-\alpha_1)a(k)}{m^k}\tilde{Q}_{2d+1}(k)=0, \tag{1.28}$$

where $\tilde{Q}_{2d+1}$ is a polynomial of degree at most $2d+1$. Then $(1.28)\times R_{2d}(\alpha_2)-\tilde{Q}_{2d+1}(\alpha_2)\times(1.27)$ yields an identity of the form

$$\sum_{k=0}^{\infty}\frac{(k-\alpha_1)(k-\alpha_2)a(k)}{m^k}\tilde{R}_{2d}(k)=Q_{2d+1}(\alpha_1)\tilde{Q}_{2d+1}(\alpha_2)\frac{\alpha}{\pi^d}$$

with $\tilde{R}_{2d}$ a polynomial of degree at most $2d$. Continue this process, we finally obtain the exact value of

$$\sum_{k=0}^{\infty}\frac{P(k)a(k)}{m^k}Q_{2d}(k)$$

with Q_{2d} a suitable polynomial of degree at most $2d$.

Remark 1.3. In practice this algorithm works well. Actually it could also be used to deduce series of type T from series of type Z.

Example 1.1. Let's implement the algorithm to deduce a series of type S with $P(k)=k^2+1=(k+i)(k-i)$ from Ramanujan's identity

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{256^k}(6k+1)=\frac{4}{\pi}. \tag{1.29}$$

By Gosper's algorithm we find that

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^3}{256^k}(24k^3-12k^2-6k-1)=-(2n+1)^2\frac{\binom{2n}{n}^3}{256^n}.$$

Letting $n\to+\infty$ we get

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{256^k}(24k^3-12k^2-6k-1)=0. \tag{1.30}$$

Note that $6k+1=6(k+i)+1-6i$ and

$$24k^3-12k^2-6k-1=6(k+i)(4k^2-(2+4i)k+2i-5)+30i+11.$$

Via (1.29)$\times(30i+11)+(6i-1)\times$(1.30) we obtain

$$\sum_{k=0}^{\infty}\frac{(k+i)\binom{2k}{k}^3}{256^k}((k-i)(6i-1)(2k-1)+8)=\frac{11+30i}{3\pi}. \tag{1.31}$$

Via the Gosper algorithm, we find that

$$\sum_{k=0}^{\infty}\frac{(k+i)\binom{2k}{k}^3}{256^k}(2(k-i)((132+360i)k^2-(602+144i)k+15-708i)+1373)=0. \tag{1.32}$$

Observe that (1.31)$\times1373-8\times$(1.32) yields

$$\sum_{k=0}^{\infty}\frac{(k^2+1)\binom{2k}{k}^3}{256^k}(192k^2-626k-103)=-\frac{1373}{3\pi}. \tag{1.33}$$

We can also deduce other identities similar to (1.33) such as

$$\sum_{k=0}^{\infty}\frac{(k^2+1)\binom{2k}{k}^3}{(-512)^k}(6k+1)=\frac{11\sqrt{2}}{6\pi},\tag{1.34}$$

$$\sum_{k=0}^{\infty}\frac{(k^2+1)\binom{2k}{k}^3}{4096^k}(126504k^2-921334k-109205)=-\frac{1063412}{3\pi}.\tag{1.35}$$

2. Proof of Theorem 1.1

Lemma 2.1. *Let m be any nonzero integer, and let $n\in\mathbb{N}=\{0,1,2,\ldots\}$.*

(i) *We have*

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^3}{m^k}\big((64-m)k^3+96k^2+48k+8\big)=8(2n+1)^3\frac{\binom{2n}{n}^3}{m^n},$$

$$\sum_{k=0}^{n}\frac{k\binom{2k}{k}^3}{m^k}\big((64-m)k^3+(96+m)k^2+48k+8\big)=8n(2n+1)^3\frac{\binom{2n}{n}^3}{m^n},$$

and

$$\sum_{k=0}^{n}\frac{k^2\binom{2k}{k}^3}{m^k}\big((64-m)k^3+(96+2m)k^2+(48-m)k+8\big)$$
$$=8n^2(2n+1)^3\frac{\binom{2n}{n}^3}{m^n}.$$

(ii) *We have*

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^3}{(k+1)^3m^k}\big((64-m)k^3+(96-3m)k^2+(48-3m)k+8-m\big)$$
$$=-m+8(2n+1)^3\frac{\binom{2n}{n}^3}{(n+1)^3m^n},$$

and

$$\sum_{k=0}^{n}\frac{k\binom{2k}{k}^3}{(k+1)^3m^k}P(k,m)=8m-8(2n+1)^3(m-8n+mn)\frac{\binom{2n}{n}^3}{(n+1)^3m^n},$$

where $P(k,m)$ denotes

$$(512-72m+m^2)k^3+(768-176m+3m^2)k^2+(384-144m+3m^2)k+64-40m+m^2.$$

Also,

$$\sum_{k=0}^{n}\frac{k^2\binom{2k}{k}^3}{(k+1)^3m^k}Q(k,m)+8m(m+8)$$
$$=8(2n+1)^3(m(m+8)-40mn(n+1)+2m^2n+64n^2+m^2n^2)\frac{\binom{2n}{n}^3}{(n+1)^3m^n},$$

where $Q(k,m)$ *denotes*

$$(4096-2624m+1024m^2-m^3)k^3+(6144-6464m+304m^2-3m^3)k^2$$
$$+(3072-5120m+296m^2-3m^3)k+512-1344m+96m^2-m^3.$$

Proof. It is easy to prove all the six identities by induction on n. We find them via the Gosper algorithm (cf. [15]). □

Lemma 2.2 (Sun [25]). *We have*

$$\sum_{k=0}^{\infty}\frac{k(6k-1)\binom{2k}{k}^3}{(2k-1)^3 256^k}=\frac{1}{2\pi}, \tag{2.1}$$

$$\sum_{k=0}^{\infty}\frac{(30k^2+3k-2)\binom{2k}{k}^3}{(2k-1)^3(-512)^k}=\frac{27\sqrt{2}}{8\pi}, \tag{2.2}$$

$$\sum_{k=0}^{\infty}\frac{(42k^2-3k-1)\binom{2k}{k}^3}{(2k-1)^3 4096^k}=\frac{27}{8\pi}. \tag{2.3}$$

Proof of Theorem 1.1. By Stirling's formula,

$$\binom{2n}{n}\sim\frac{4^n}{\sqrt{n\pi}}\quad\text{as } n\to+\infty.$$

Thus, for any integer m with $|m|>64$, by Lemma 2.1 we have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{m^k}\big((64-m)k^3+96k^2+48k+8\big)=0, \tag{2.4}$$

$$\sum_{k=0}^{\infty}\frac{k\binom{2k}{k}^3}{m^k}\big((64-m)k^3+(96+m)k^2+48k+8\big)=0, \tag{2.5}$$

$$\sum_{k=0}^{n}\frac{k^2\binom{2k}{k}^3}{m^k}\big((64-m)k^3+(96+2m)k^2+(48-m)k+8\big)=0; \tag{2.6}$$

also

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^3}{(k+1)^3m^k}\big((64-m)k^3+(96-3m)k^2+(48-3m)k+8\big)=-m, \tag{2.7}$$

$$\sum_{k=0}^{n}\frac{k\binom{2k}{k}^3}{(k+1)^3m^k}P(k,m)=8m \tag{2.8}$$

and

$$\sum_{k=0}^{n}\frac{k^2\binom{2k}{k}^3}{(k+1)^3m^k}Q(k,m),=-8m(m+8) \tag{2.9}$$

where $P(k,m)$ and $Q(k,m)$ are as in Lemma 2.1(ii).

Now we prove (S3) in details. (The identities (S1) and (S2) can be proved similarly.) Taking $m = 4096$ in (2.4)-(2.6), we get the identities

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3}{4096^k} \left(504k^3 - 12k^2 - 6k - 1\right) = 0, \tag{2.10}$$

$$\sum_{k=0}^{\infty} \frac{k\binom{2k}{k}^3}{4096^k} \left(504k^3 - 524k^2 - 6k - 1\right) = 0, \tag{2.11}$$

$$\sum_{k=0}^{\infty} \frac{k^2\binom{2k}{k}^3}{4096^k} \left(504k^3 - 1036k^2 + 506k - 1\right) = 0. \tag{2.12}$$

Combining the Ramanujan identity

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3}{4096^k}(42k+5) = \frac{16}{\pi}$$

with (2.10), and noting that

$$12k(210k^2 - 5k + 1) = 60k(42k^2 - k) + 12k = 5 \times 12k(42k^2 - k) + 12k,$$

we obtain

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3}{4096^k} 12k(210k^2 - 5k + 1) = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3}{4096^k}(5(6k+1) + 12k) = \frac{16}{\pi}$$

and hence

$$\sum_{k=0}^{\infty} \frac{k\binom{2k}{k}^3}{4096^k}(210k^2 - 5k + 1) = \frac{4}{3\pi}.$$

Adding this and (2.11), we see that

$$\sum_{k=0}^{\infty} \frac{k^2\binom{2k}{k}^3}{4096^k}(504k^2 - 314k - 11) = \frac{4}{3\pi}.$$

(2.12) times 11 minus the last formula yields

$$\sum_{k=0}^{\infty} \frac{k^2\binom{2k}{k}^3}{4096^k}(11 \times 504k^3 - (11 \times 1036 + 504)k^2 + (11 \times 506 + 314)k) = -\frac{4}{3\pi},$$

which is equivalent to (S3).

Next we prove (S3′) in details. (The identities (S1′) and (S2′) can be proved similarly.) For any positive integer k, clearly

$$\frac{\binom{2k}{k}}{2k-1} = \frac{2}{k}\binom{2(k-1)}{k-1}.$$

Thus, by (2.3) we have

$$\begin{aligned}\frac{27}{8\pi}-1=&\sum_{k=1}^{\infty}\frac{42k^2-3k-1}{4096^k}\left(\frac{2}{k}\binom{2(k-1)}{k-1}\right)^3\\ =&\frac{8}{4096}\sum_{k=0}^{\infty}\frac{42(k+1)^2-3(k+1)-1}{4096^k}\cdot\frac{\binom{2k}{k}^3}{(k+1)^3}\end{aligned}$$

and hence

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{(k+1)^3 4096^k}(42k^2+81k+38)=\frac{1728}{\pi}-512. \tag{2.13}$$

On the other hand, applying (2.7) with $m=4096$, we get

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{(k+1)^3 4096^k}(504k^3+1524k^2+1530k+511)=512. \tag{2.14}$$

Note that (2.14)×38−(2.13)×511 yields

$$\sum_{k=0}^{\infty}\frac{k\binom{2k}{k}^3}{(k+1)^3 4096^k}(2128k^2+4050k+1861)=31232-\frac{98112}{\pi}. \tag{2.15}$$

Taking $m=4096$ in (2.8), we obtain

$$\sum_{k=0}^{\infty}\frac{k\binom{2k}{k}^3}{(k+1)^3 4096^k}(257544k^3+775180k^2+777222k+259585)=512. \tag{2.16}$$

Observe that (2.16)×1861−(2.15)×259585 gives

$$\sum_{k=0}^{\infty}\frac{k^2\binom{2k}{k}^3}{(k+1)^3 4096^k}(78162k^2+145175k+64431)=\frac{4153360}{\pi}-1321984. \tag{2.17}$$

Putting $m=4096$ in (2.9), we get

$$\begin{aligned}\sum_{k=0}^{\infty}&\frac{k^2\binom{2k}{k}^3}{(k+1)^3 4096^k}(130830840k^3+392743412k^2+392994810k+131082751)\\ &=262656.\end{aligned} \tag{2.18}$$

Note that (2.18)×64431−(2.17)×131082751 yields

$$\begin{aligned}\sum_{k=0}^{\infty}&\frac{k^3\binom{2k}{k}^3}{(k+1)^3 4096^k}(32473224k^2+58012446k+24235261)\\ &=667628032-\frac{2097324016}{\pi},\end{aligned}$$

which is equivalent to the desired (S3') since $\binom{2k}{k+1}=\frac{k}{k+1}\binom{2k}{k}$. □

3. Proofs of Theorems 1.2 and 1.3

Lemma 3.1. *Let $m\in\mathbb{Z}$ and $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$.*

(i) *We have the following identities:*

$$\sum_{k=1}^n\frac{m^k}{k\binom{2k}{k}^3}((m-64)k^3+(2m+96)k^2-48k+8)=-m+\frac{(n+1)^2m^{n+1}}{\binom{2n}{n}^3},$$
$$\sum_{k=1}^n\frac{m^k}{k^2\binom{2k}{k}^3}((m-64)k^3+(m+96)k^2-48k+8)=-m+\frac{(n+1)m^{n+1}}{\binom{2n}{n}^3},$$
$$\sum_{k=1}^n\frac{m^k}{k^3\binom{2k}{k}^3}((m-64)k^3+96k^2-48k+8)=-m+\frac{m^{n+1}}{\binom{2n}{n}^3},$$

(ii) *We have*

$$\sum_{k=1}^n\frac{(k+1)m^k}{k^3\binom{2k}{k}^3}P_1(k,m)=m(m-432)+(216(n+2)-m(n+1))\frac{m^{n+1}}{\binom{2n}{n}^3},$$

$$\begin{aligned}&\sum_{k=1}^n\frac{(k+1)^2m^k}{k^3\binom{2k}{k}^3}P_2(k,m)-m(m^2-216m+186624)\\
=&-\frac{m^{n+1}}{\binom{2n}{n}^3}\\
&\times(186624(n+1)-216m(n^2+1)+m^2(2n+1)-648mn+m^2n^2+46656n^2),\end{aligned}$$

where

$$\begin{aligned}P_1(k,m)=&(280m-m^2-13824)k^3+(56m+20736)k^2\\&-(8m+10368)k+1728\end{aligned}$$

and

$$\begin{aligned}P_2(k,m)=&(2985984-60480m+280m^2-m^3)k^3\\&-(4478976+58752m+8m^2)k^2+(2239488+15552m)k\\&-(1728m+373248).\end{aligned}$$

Proof. It is easy to prove the five identities by induction on n. We find them via the Gosper algorithm (cf. [15]). □

Proof of (T1). Lemma 3.1(i) with $m = 1$ gives the identities

$$\sum_{k=1}^{n}\frac{63k^3-98k^2+47k-8}{k\binom{2k}{k}^3} =1-\frac{(n+1)^2}{\binom{2n}{n}^3},$$
$$\sum_{k=1}^{n}\frac{63k^3-97k^2+48k-8}{k^2\binom{2k}{k}^3} =1-\frac{n+1}{\binom{2n}{n}^3},$$
$$\sum_{k=1}^{n}\frac{63k^3-96k^2+48k-8}{k^3\binom{2k}{k}^3} =1-\frac{1}{\binom{2n}{n}^3}.$$

Letting $n \to +\infty$ we get

$$\sum_{k=1}^{\infty}\frac{63k^3-98k^2+47k-8}{k\binom{2k}{k}^3} =1, \tag{3.1}$$
$$\sum_{k=1}^{\infty}\frac{63k^3-97k^2+48k-8}{k^2\binom{2k}{k}^3} =1, \tag{3.2}$$
$$\sum_{k=1}^{\infty}\frac{63k^3-96k^2+48k-8}{k^3\binom{2k}{k}^3} =1. \tag{3.3}$$

Note that (3.3) minus (1.3) yields the identity

$$\sum_{k=1}^{\infty}\frac{21k^2-32k+9}{k^2\binom{2k}{k}^3} = \frac{1}{3}-\frac{\pi^2}{18}. \tag{3.4}$$

Via (3.2)$\times 9 + 8\times$(3.4), we obtain

$$\sum_{k=1}^{\infty}\frac{567k^2-705k+176}{k\binom{2k}{k}} = \frac{35}{3}-\frac{4}{9}\pi^2. \tag{3.5}$$

Observe that (3.1)$\times$22+(3.5) yields the identity

$$\sum_{k=1}^{\infty}\frac{198k^2-227k+47}{\binom{2k}{k}^3} = \frac{303-4\pi^2}{63},$$

which is equivalent to (T1) since $303 + 47 \times 63 = 3264$. □

Proof of (T1′). Lemma 3.1(ii) with $m = 1$ gives the identities

$$\sum_{k=1}^{n}\frac{k+1}{k^3\binom{2k}{k}^3}(13545k^3-20792k^2+10376k-1728)$$
$$= 431-\frac{215n+431}{\binom{2n}{n}^3}$$

and

$$\sum_{k=1}^n\frac{(k+1)^2}{k^3\binom{2k}{k}^3}(2925783k^2-4537736k^2+2255040k-374976)$$
$$=186409-\frac{46441n^2+185978n+186409}{\binom{2n}{n}^3}.$$

Letting $n\to+\infty$ we get

$$\sum_{k=1}^n\frac{k+1}{k^3\binom{2k}{k}^3}(13545k^3-20792k^2+10376k-1728)=431,$$
$$\sum_{k=1}^n\frac{(k+1)^2}{k^3\binom{2k}{k}^3}(2925783k^3-4537736k^2+2255040k-374976)=186409.$$

They can be rewritten as follows:

$$\sum_{k=1}^n\frac{k+1}{k^3\binom{2k}{k}^3}((k+1)(13545k^2-34337k+44713)-46441)=431, \tag{3.6}$$

$$\sum_{k=1}^n\frac{(k+1)^2}{k^3\binom{2k}{k}^3}((k+1)(2925783k^2-7463519k+9718559)-10093535)=186409. \tag{3.7}$$

We may rewrite (3.3) in the form

$$\sum_{k=1}^\infty\frac{(k+1)(63k^2-159k+207)-215}{k^3\binom{2k}{k}^3}=1, \tag{3.8}$$

and rewrite (1.3) in the form

$$\sum_{k=1}^\infty\frac{21(k+1)-29}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{6}. \tag{3.9}$$

Note that (3.8)×29 − 215×(3.9) yields

$$\sum_{k=1}^\infty\frac{k+1}{k^3\binom{2k}{k}^3}(609k^2-1537k+496)=\frac{29}{3}-\frac{215}{18}\pi^2,$$

which has the equivalent form

$$\sum_{k=1}^\infty\frac{k+1}{k^3\binom{2k}{k}^3}((k+1)(609k-2146)+2642)=\frac{29}{3}-\frac{215}{18}\pi^2. \tag{3.10}$$

Via (3.6)×2642 + 46441×(3.10) we obtain

$$\sum_{k=1}^\infty\frac{(k+1)^2}{k^3\binom{2k}{k}^3}(166446k^2-290399k+85904)=\frac{22153}{3}-\frac{46441}{18}\pi^2,$$

which has the equivalent form

$$\sum_{k=1}^{\infty}\frac{(k+1)^2}{k^3\binom{2k}{k}^3}((k+1)(166446k-456845)+542749)=\frac{22153}{3}-\frac{46441}{18}\pi^2. \tag{3.11}$$

Observe that (3.7)×542749 + 10093535×(3.11) yields

$$\sum_{k=1}^{\infty}\frac{(k+1)^3}{k^3\binom{2k}{k}^3}(2041168-7292783k+4884741k^2)=\frac{68102148-10093535\pi^2}{126},$$

which is equivalent to the desired (T1′) since $\binom{2k}{k+1}=\frac{k}{k+1}\binom{2k}{k}$. This concludes the proof. □

Remark 3.1. In the spirit of our proofs of (T1) and (T1′), we can prove (T2),(T2′),(T4),(T4′) similarly by using Lemma 3.1 and the known identities

$$\sum_{k=1}^{\infty}\frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3}=-2G \text{ and } \sum_{k=1}^{\infty}\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}=\pi^2$$

(cf. [14]).

Lemma 3.2. *Let $m\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$.*

(i) *We have the following identities:*

$$\begin{aligned}\sum_{k=1}^{n}&\frac{m^k}{k\binom{2k}{k}^2\binom{3k}{k}}((m-108)k^3+(2m+162)k^2+(m-78)k+12)\\&=-m+\frac{(n+1)^2m^{n+1}}{\binom{2n}{n}^2\binom{3n}{n}},\\ \sum_{k=1}^{n}&\frac{m^k}{k^2\binom{2k}{k}^2\binom{3k}{k}}((m-108)k^3+(m+162)k^2-78k+12)\\&=-m+\frac{(n+1)m^{n+1}}{\binom{2n}{n}^2\binom{3n}{n}},\\ \sum_{k=1}^{n}&\frac{m^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}((m-108)k^3+162k^2-78k+12)\\&=-m+\frac{m^{n+1}}{\binom{2n}{n}^2\binom{3n}{n}}.\end{aligned}$$

(ii) *We have*

$$\sum_{k=1}^{n}\frac{(k+1)m^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}P_1(k,m)=m(m-720)+(360(n+2)-m(n+1))\frac{m^{n+1}}{\binom{2n}{n}^2\binom{3n}{n}}$$

and

$$\begin{aligned}&\sum_{k=1}^n\frac{(k+1)^2m^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}P_2(k,m)-m(m^2-372m+518400)\\ =&-\frac{m^{n+1}}{\binom{2n}{n}^2\binom{3n}{n}}\\ &\times(m^2(n+1)^2-366mn(n+3)+129600n^2+518400(n+1)-372m),\end{aligned}$$

where

$$\begin{aligned}P_1(k,m)=&(468m-m^2-38880)k^3+(90m+58320)k^2\\ &-(12m+28080)k+4320\end{aligned}$$

and

$$\begin{aligned}P_2(k,m)=&(13996800-169128m+474m^2-m^3)k^3\\ &-(20995200+160380m+12m^2)k^2\\ &+(10108800+41112m)k-1555200\end{aligned}$$

Lemma 3.3. *Let $m\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$.*

(i) *We have the following identities:*

$$\begin{aligned}&\sum_{k=1}^n\frac{m^k}{k\binom{2k}{k}^2\binom{4k}{2k}}((m-256)k^3+(2m+384)k^2+(m-176)k+24)\\ &\qquad=-m+\frac{(n+1)^2m^{n+1}}{\binom{2n}{n}^2\binom{4n}{2n}},\\ &\sum_{k=1}^n\frac{m^k}{k^2\binom{2k}{k}^2\binom{4k}{2k}}((m-256)k^3+(m+384)k^2-176k+24)\\ &\qquad=-m+\frac{(n+1)m^{n+1}}{\binom{2n}{n}^2\binom{4n}{2n}},\\ &\sum_{k=1}^n\frac{m^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}((m-256)k^3+384k^2-176k+24)\\ &\qquad=-m+\frac{m^{n+1}}{\binom{2n}{n}^2\binom{4n}{2n}}.\end{aligned}$$

(ii) *We have*

$$\sum_{k=1}^n\frac{(k+1)m^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}P_1(k,m)=m(m-1680)+(840(n+2)-m(n+1))\frac{m^{n+1}}{\binom{2n}{n}^2\binom{4n}{2n}}$$

and

$$\sum_{k=1}^{n}\frac{(k+1)^2m^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}P_2(k,m)-m(m^2-904m+2822400)$$
$$=-\frac{m^{n+1}}{\binom{2n}{n}^2\binom{4n}{2n}}$$
$$\times(m^2(n+1)^2-872mn(n+3)+705600n(n+4)-904m+2822400),$$

where

$$P_1(k,m)=(1096m-m^2-215040)k^3+(200m+322560)k^2 -(24m+147840)k+20160$$

and

$$P_2(k,m)=(180633600-928832m+1128m^2-m^3)k^3 -(270950400+853120m+24m^2)k^2 +(124185600+209088m)k-(16934400+20160m).$$

Using Lemmas 3.2 and 3.3 as well as some results in [25], we can prove all the remaining formulas in Theorem 1.2.

Proof of Theorem 1.3. By Gosper's algorithm, we find the identity

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^5\binom{2k}{k}^5}(1025k^5-2560k^4+2560k^3-1280k^2+320k-32)=-1+\frac{(-1)^n}{\binom{2n}{n}}$$

which can be easily proved by induction on n. Letting $n\to+\infty$ we get

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^5\binom{2k}{k}^5}(1025k^5-2560k^4+2560k^3-1280k^2+320k-32)=-1.$$

Adding this to (1.4) we get

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^4\binom{2k}{k}^5}(205k^4-512k^3+512k^2-215k+32)=-\frac{1+2\zeta(3)}{5}. \tag{3.12}$$

By Gosper's algorithm, we find that

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^4\binom{2k}{k}^5}(1025k^5-2559k^4+2560k^3-1280k^2+320k-32)=-1+\frac{(-1)^n(n+1)}{\binom{2n}{n}}$$

and hence

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^4\binom{2k}{k}^5}(1025k^5-2559k^4+2560k^3-1280k^2+320k-32)=-1. \tag{3.13}$$

Adding (3.12) and (3.13), we obtain

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3\binom{2k}{k}^5}(1025k^4-2354k^3+2048k^2-768k+105)=-\frac{2}{5}(3+\zeta(3)). \tag{3.14}$$

By Gosper's algorithm, we find that

$$\sum_{k=1}^n\frac{(-1)^k}{k^3\binom{2k}{k}^5}(1025k^5-2558k^4+2561k^3-1280k^2+320k-32)=-1+\frac{(-1)^n(n+1)^2}{\binom{2n}{n}}$$

and hence

$$\sum_{k=1}^\infty\frac{(-1)^k}{k^3\binom{2k}{k}^5}(1025k^5-2558k^4+2561k^3-1280k^2+320k-32)=-1. \quad (3.15)$$

Note that (3.14)×32 + 105×(3.15) yields

$$\begin{aligned}&\sum_{k=1}^\infty\frac{(-1)^k}{k^2\binom{2k}{k}^5}(107625k^4-235790k^3+193577k^2-68864k+9024)\\&\quad=-\frac{64}{5}(3+\zeta(3))-105.\end{aligned} \quad (3.16)$$

By Gosper's algorithm, we find that

$$\sum_{k=1}^n\frac{(-1)^k}{k^2\binom{2k}{k}^5}(1025k^5-2557k^4+2563k^3-1279k^2+320k-32)=-1+\frac{(-1)^n(n+1)^3}{\binom{2n}{n}}$$

and hence

$$\sum_{k=1}^\infty\frac{(-1)^k}{k^2\binom{2k}{k}^5}(1025k^5-2557k^4+2563k^3-1279k^2+320k-32)=-1. \quad (3.17)$$

Via (3.16)+282×(3.17) we get

$$\begin{aligned}&\sum_{k=1}^\infty\frac{(-1)^k}{k\binom{2k}{k}^5}(289050k^4-613449k^3+486976k^2-167101k+21376)\\&\quad=-\frac{64}{5}(3+\zeta(3))-105-282.\end{aligned} \quad (3.18)$$

By Gosper's algorithm, we find that

$$\sum_{k=1}^n\frac{(-1)^k}{k\binom{2k}{k}^5}(1025k^5-2556k^4+2566k^3-1276k^2+321k-32)=-1+\frac{(-1)^n(n+1)^4}{\binom{2n}{n}}$$

and hence

$$\sum_{k=1}^\infty\frac{(-1)^k}{k\binom{2k}{k}^5}(1025k^5-2556k^4+2566k^3-1276k^2+321k-32)=-1. \quad (3.19)$$

Observe that (3.18)+668×(3.19) gives

$$\begin{aligned}&\sum_{k=1}^\infty\frac{(-1)^k}{\binom{2k}{k}^5}(684700k^4-1418358k^3+1100639k^2-365392k+47327)\\&\quad=-\frac{5467+64\zeta(3)}{5},\end{aligned}$$

which is equivalent to the desired (T9) since

$$47327-\frac{5467}{5}=\frac{231168}{5}.$$

This concludes our proof of Theorem 1.3. □

4. Other variants of Ramanujan-type series for $\frac{1}{\pi}$

Lemma 4.1. *Let $m\in\mathbb{Z}\setminus\{0\}$ and $n\in\mathbb{N}$.*

(i) *We have*

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}\left((108-m)k^3+162k^2+78k+12\right)=6(2n+1)(3n+1)(3n+2)\frac{\binom{2n}{n}^2\binom{3n}{n}}{m^n},$$

$$\sum_{k=0}^{n}\frac{k\binom{2k}{k}^2\binom{3k}{k}}{m^k}\left((108-m)k^3+(162+m)k^2+78k+12\right)=6n(2n+1)(3n+1)(3n+2)\frac{\binom{2n}{n}^2\binom{3n}{n}}{m^n},$$

and

$$\sum_{k=0}^{n}\frac{k^2\binom{2k}{k}^2\binom{3k}{k}}{m^k}\left((108-m)k^3+(162+2m)k^2+(78-m)k+12\right)=6n^2(2n+1)(3n+1)(3n+2)\frac{\binom{2n}{n}^2\binom{3n}{n}}{m^n}.$$

(ii) *We have*

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(k+1)^2m^k}\left((108-m)k^3+(162-2m)k^2+(78-m)k+12\right)=6(2n+1)(3n+1)(3n+2)\frac{\binom{2n}{n}^2\binom{3n}{n}}{(n+1)^2m^n}$$

and

$$\sum_{k=0}^{n}\frac{k\binom{2k}{k}^2\binom{3k}{k}}{(k+1)^2m^k}\left((108-m)k^3+(162-m)k^2+(78+m)k+m+12\right)=6n(2n+1)(3n+1)(3n+2)\frac{\binom{2n}{n}^2\binom{3n}{n}}{(n+1)^2m^n}.$$

Proof. It is easy to prove the five identities by induction on n. We find them by the Gosper algorithm. □

Theorem 4.1. *We have the following identities:*

$$\sum_{k=0}^{\infty}\frac{(150k^2-51k+89)k^3\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}=\frac{32\sqrt{3}}{15\pi},\tag{S4}$$

$$\sum_{k=0}^{\infty}\frac{(1550k^2+3029k+1481)\binom{2k}{k+1}^2\binom{3k}{k}}{(-192)^k}=-\frac{80\sqrt{3}}{\pi},\tag{S4$'$}$$

$$\sum_{k=0}^{\infty}\frac{(738k^2-4023k+745)k^3\binom{2k}{k}^2\binom{3k}{k}}{216^k}=-\frac{24\sqrt{3}}{\pi},\tag{S5}$$

$$\sum_{k=0}^{\infty}\frac{(702k^2+1491k+787)\binom{2k}{k+1}^2\binom{3k}{k}}{216^k}=\frac{228\sqrt{3}}{\pi},\tag{S5$'$}$$

$$\sum_{k=0}^{\infty}\frac{(159426k^2-292761k+153995)k^3\binom{2k}{k}^2\binom{3k}{k}}{(-12)^{3k}}=\frac{96\sqrt{3}}{\pi},\tag{S6}$$

$$\sum_{k=0}^{\infty}\frac{(604962k^2+1206195k+601183)\binom{2k}{k+1}^2\binom{3k}{k}}{(-12)^{3k}}=-\frac{6864\sqrt{3}}{\pi},\tag{S6$'$}$$

$$\sum_{k=0}^{\infty}\frac{(900k^2-2097k+929)k^3\binom{2k}{k}^2\binom{3k}{k}}{1458^k}=-\frac{27}{20\pi},\tag{S7}$$

$$\sum_{k=0}^{\infty}\frac{(56025k^2+112584k+56551)\binom{2k}{k+1}^2\binom{3k}{k}}{1458^k}=\frac{6615}{4\pi},\tag{S7$'$}$$

$$\sum_{k=0}^{\infty}\frac{(173502k^2-354087k+181205)k^3\binom{2k}{k}^2\binom{3k}{k}}{(-8640)^k}=\frac{32\sqrt{15}}{5\pi},\tag{S8}$$

$$\sum_{k=0}^{\infty}\frac{(2086398k^2+4169997k+2083585)\binom{2k}{k+1}^2\binom{3k}{k}}{(-8640)^k}=-\frac{11504\sqrt{15}}{5\pi},\tag{S8$'$}$$

$$\sum_{k=0}^{\infty}\frac{(216711k^2-473742k+226495)k^3\binom{2k}{k}^2\binom{3k}{k}}{15^{3k}}=-\frac{60\sqrt{3}}{\pi},\tag{S9}$$

$$\sum_{k=0}^{\infty}\frac{(620730k^2+1243839k+623087)\binom{2k}{k+1}^2\binom{3k}{k}}{15^{3k}}=\frac{16935\sqrt{3}}{4\pi},\tag{S9$'$}$$

$$\sum_{k=0}^{\infty}\frac{(36512550k^2-76893003k+39703217)k^3\binom{2k}{k}^2\binom{3k}{k}}{(-48)^{3k}}=\frac{768\sqrt{3}}{5\pi},\tag{S10}$$

$$\sum_{k=0}^{\infty}\frac{Q_{-48}(k)\binom{2k}{k+1}^2\binom{3k}{k}}{(-48)^{3k}}=-3538560\frac{\sqrt{3}}{\pi},\tag{S10$'$}$$

where

$$Q_{-48}(k) := 18009838350k^2 + 36017730237k + 18007890593.$$

Also,

$$\sum_{k=0}^{\infty} \frac{(30437550k^2 - 64346463k + 33372157)k^3\binom{2k}{k}^2\binom{3k}{k}}{(-326592)^k} = \frac{864\sqrt{7}}{35\pi}, \tag{S11}$$

and

$$\sum_{k=0}^{\infty} \frac{Q(k)\binom{2k}{k+1}^2\binom{3k}{k}}{(-326592)^k} = -11756880\frac{\sqrt{7}}{7\pi}, \tag{S11$'$}$$

where

$$Q(k) := 38518093350k^2 + 77034773577k + 38516679853.$$

Moreover,

$$\sum_{k=0}^{\infty} \frac{(1227699410778k^{-}2613341265669k + 1370770039375)k^3\binom{2k}{k}^2\binom{3k}{k}}{(-300)^{3k}} = 12000\frac{\sqrt{3}}{\pi}, \tag{S12}$$

and

$$\sum_{k=0}^{\infty} \frac{Q_{-300}(k)\binom{2k}{k+1}^2\binom{3k}{k}}{(-300)^{3k}} = -13499994000\frac{\sqrt{3}}{\pi}, \tag{S12$'$}$$

where

$$Q_{-300}(k) = 16746723121124538k^2 + 33493438799463639k + 16746715678300691.$$

Proof. By Lemma 4.1(i), for any $m \in \mathbb{Z}$ with $|m| > 108$, we have

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}\big((108-m)k^3 + 162k^2 + 78k + 12\big) = 0, \tag{4.1}$$

$$\sum_{k=0}^{\infty} \frac{k\binom{2k}{k}^2\binom{3k}{k}}{m^k}\big((108-m)k^3 + (162+m)k^2 + 78k + 12\big) = 0, \tag{4.2}$$

and

$$\sum_{k=0}^{n} \frac{k^2\binom{2k}{k}^2\binom{3k}{k}}{m^k}\big((108-m)k^3 + (162+2m)k^2 + (78-m)k + 12\big) = 0. \tag{4.3}$$

We now prove (S4) in details. Taking $m=-192$ in the last three identities, we get

$$\sum_{k=0}^\infty\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}(50k^3+27k^2+13k+2)=0,\tag{4.4}$$

$$\sum_{k=0}^\infty\frac{k\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}(50k^3-5k^2+13k+2)=0,\tag{4.5}$$

$$\sum_{k=0}^\infty\frac{k^2\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}(50k^3-37k^2+45k+2)=0.\tag{4.6}$$

Combining the known identity

$$\sum_{k=0}^\infty\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}(5k+1)=\frac{4}{\sqrt3\pi}\tag{4.7}$$

with (4.4), we get

$$\sum_{k=0}^\infty\frac{k\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}(50k^2+27k+3)=-\frac{8}{\sqrt3\pi}.$$

Combining this with (4.5), we see that

$$\sum_{k=0}^\infty\frac{k\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}(150k^3-(3\times5+2\times50)k^2+(3\times13-2\times27)k)=\frac{16\sqrt3}{\pi},$$

i.e.,

$$\sum_{k=0}^\infty\frac{k^2\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}(30k^2-23k-3)=\frac{16\sqrt3}{5\pi}.\tag{4.8}$$

(4.6)×3 plus (4.8)×2 yields

$$\sum_{k=0}^\infty\frac{k^2\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}(150k^3+(60-111)k^2+(135-46)k)=\frac{32}{5\sqrt3\pi},$$

which is equivalent to (S4).

Next we prove (S4′). Applying 4.1(ii) with $m=-192$, we have

$$\sum_{k=0}^\infty\frac{\binom{2k}{k}^2\binom{3k}{k}}{(k+1)^2(-192)^k}(50k^3+91k^2+45k+2)=0,\tag{4.9}$$

$$\sum_{k=0}^\infty\frac{k\binom{2k}{k}^2\binom{3k}{k}}{(k+1)^2(-192)^k}(50k^3+59k^2-19k-30)=0.\tag{4.10}$$

As

$$50k^3+91k^2=45k+2-10(5k+1)(k+1)^2=-(19k^2+25k+8),$$

from (4.9) and (4.7) we obtain

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(k+1)^2(-192)^k}(19k^2+25k+8)=\frac{40\sqrt{3}}{3\pi}. \tag{4.11}$$

Note that (4.9)×4−(4.11) yields

$$\sum_{k=0}^{\infty}\frac{k\binom{2k}{k}^2\binom{3k}{k}}{(k+1)^2(-192)^k}(40k^2+69k+31)=-\frac{8\sqrt{3}}{3\pi}. \tag{4.12}$$

Via (4.10)×31 + 30×(4.12) we obtain

$$\sum_{k=0}^{\infty}\frac{k^2\binom{2k}{k}^2\binom{3k}{k}}{(k+1)^2(-192)^k}(1550k^2+3029k+1481)=-\frac{80\sqrt{3}}{\pi},$$

which is equivalent to the desired (S4′) since $\binom{2k}{k+1}=\frac{k}{k+1}\binom{2k}{k}$.

All the remaining formulas in Theorem 4.1 can be proved similarly. We omit the details. □ □

Lemma 4.2. *Let $m\in\mathbb{Z}\setminus\{0\}$ and $n\in\mathbb{N}$.*

(i) *We have*

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}\big((256-m)k^3+384k^2+176k+24\big)$$
$$=8(2n+1)(4n+1)(4n+3)\frac{\binom{2n}{n}^2\binom{4n}{2n}}{m^n},$$

$$\sum_{k=0}^{n}\frac{k\binom{2k}{k}^2\binom{4k}{2k}}{m^k}\big((256-m)k^3+(384+m)k^2+176k+24\big)$$
$$=8n(2n+1)(4n+1)(4n+3)\frac{\binom{2n}{n}^2\binom{4n}{2n}}{m^n},$$

and

$$\sum_{k=0}^{n}\frac{k^2\binom{2k}{k}^2\binom{4k}{2k}}{m^k}\big((256-m)k^3+(384+2m)k^2+(176-m)k+24\big)$$
$$=8n^2(2n+1)(4n+1)(4n+3)\frac{\binom{2n}{n}^2\binom{4n}{2n}}{m^n}.$$

(ii) *We have*

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)^2m^k}\big((256-m)k^3+(384-2m)k^2+(176-m)k+24\big)$$
$$=8(2n+1)(4n+1)(4n+3)\frac{\binom{2n}{n}^2\binom{4n}{2n}}{(n+1)^2m^n}$$

and

$$\sum_{k=0}^{n}\frac{k\binom{2k}{k}^2\binom{4k}{2k}}{(k+1)^2m^k}\big((256-m)k^3+(384-m)k^2+(176+m)k+m+24\big)$$
$$=8n(2n+1)(4n+1)(4n+3)\frac{\binom{2n}{n}^2\binom{4n}{2n}}{(n+1)^2m^n}.$$

Proof. It is easy to prove the five identities by induction on n. We find them by the Gosper algorithm. □

Using Lemma 4.2 and the method we prove Theorem 4.1, we obtain the following theorem.

Theorem 4.2. *We have the following identities:*

$$\sum_{k=0}^{\infty}\frac{(854k^2-3639k+910)k^3\binom{2k}{k}^2\binom{4k}{2k}}{648^k}=-\frac{243}{14\pi},\tag{S13}$$

$$\sum_{k=0}^{\infty}\frac{(2107k^2+4359k+2249)\binom{2k}{k+1}^2\binom{4k}{2k}}{648^k}=\frac{567}{\pi},\tag{S13$'$}$$

$$\sum_{k=0}^{\infty}\frac{(400k^2-496k+327)k^3\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k}=\frac{9}{5\pi},\tag{S14}$$

$$\sum_{k=0}^{\infty}\frac{(19840k^2+39324k+19481)\binom{2k}{k+1}^2\binom{4k}{2k}}{(-1024)^k}=-\frac{1000}{\pi},\tag{S14$'$}$$

$$\sum_{k=0}^{\infty}\frac{(6016k^2-14856k+6215)k^3\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}=-\frac{27\sqrt{3}}{4\pi},\tag{S15}$$

$$\sum_{k=0}^{\infty}\frac{(75776k^2+152520k+76729)\binom{2k}{k+1}^2\binom{4k}{2k}}{48^{2k}}=\frac{1746\sqrt{3}}{\pi},\tag{S15$'$}$$

$$\sum_{k=0}^{\infty}\frac{(1326650k^2-2420121k+1281559)k^3\binom{2k}{k}^2\binom{4k}{2k}}{(-63^2)^k}=1944\frac{\sqrt{7}}{5\pi},\tag{S16}$$

and

$$\sum_{k=0}^{\infty}\frac{(8242975k^2+16441878k+8198387)\binom{2k}{k+1}^2\binom{4k}{2k}}{(-63^2)^k}=-\frac{106515\sqrt{7}}{2\pi}.\tag{S16$'$}$$

We also have

$$\sum_{k=0}^{\infty}\frac{(320k^2-678k+337)k^3\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}}=-\frac{243\sqrt{2}}{10240\pi},\tag{S17}$$

$$\sum_{k=0}^{\infty}\frac{(6671360k^2+13350714k+6679321)\binom{2k}{k+1}^2\binom{4k}{2k}}{12^{4k}}=\frac{70065\sqrt{2}}{4\pi}, \tag{S17$'$}$$

$$\sum_{k=0}^{\infty}\frac{(2576k^2-5136k+2635)k^3\binom{2k}{k}^2\binom{4k}{2k}}{(-3\times 2^{12})^k}=\frac{2\sqrt{3}}{7\pi}, \tag{S18}$$

$$\sum_{k=0}^{\infty}\frac{(796544k^2+1591612k+795059)\binom{2k}{k+1}^2\binom{4k}{2k}}{(-3\times 2^{12})^k}=-\frac{8176\sqrt{3}}{3\pi}, \tag{S18$'$}$$

$$\sum_{k=0}^{\infty}\frac{(2428400k^2-5044368k+2584321)k^3\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{10}3^4)^k}=\frac{243}{5\pi}, \tag{S19}$$

$$\sum_{k=0}^{\infty}\frac{(2475740800k^2+4950772932k+2475031103)\binom{2k}{k+1}^2\binom{4k}{2k}}{(-2^{10}3^4)^k}=-\frac{2238840}{\pi}, \tag{S19$'$}$$

$$\sum_{k=0}^{\infty}\frac{(38400k^2-80696k+41609)k^3\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}}=-\frac{49\sqrt{3}}{1080\pi}, \tag{S20}$$

$$\sum_{k=0}^{\infty}\frac{(1967513600k^2+3935104168k+1967590547)\binom{2k}{k+1}^2\binom{4k}{2k}}{28^{4k}}=\frac{3764915\sqrt{3}}{27\pi}, \tag{S20$'$}$$

$$\sum_{k=0}^{\infty}\frac{(423089968k^2-891891888k+463383905)k^3\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{14}3^45)^k}=\frac{972\sqrt{5}}{35\pi}, \tag{S21}$$

$$\sum_{k=0}^{\infty}\frac{q(k)\binom{2k}{k+1}^2\binom{4k}{2k}}{(-2^{14}3^45)^k}=-716633568\frac{\sqrt{5}}{5\pi}, \tag{S21$'$}$$

where

$$q(k):=28206829076608k^2+56413556154372k+28206727074305.$$

We also have

$$\sum_{k=0}^{\infty}\frac{(643835623600k^2-1361740501968k+711617288021)k^3\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{10}21^4)^k}=\frac{11907}{5\pi}, \tag{S22}$$

and

$$\sum_{k=0}^{\infty}\frac{q_k\binom{2k}{k+1}^2\binom{4k}{2k}}{(-2^{10}21^4)^k}=-\frac{263473491960}{\pi}, \tag{S22$'$}$$

where q_k denotes

$$695911303499907200k^2+1391822523134211732k+695911219634175403$$

Moreover,

$$\sum_{k=0}^{\infty}\frac{(13462400k^2-28347528k+14689591)k^3\binom{2k}{k}^2\binom{4k}{2k}}{1584^{2k}}=-\frac{243\sqrt{11}}{140\pi},\quad \text{(S23)}$$

and

$$\sum_{k=0}^{\infty}\frac{u_k\binom{2k}{k+1}^2\binom{4k}{2k}}{1584^{2k}}=16936290\frac{\sqrt{11}}{\pi},\quad \text{(S23}'\text{)}$$

where

$$u_k:=1868912998400k^2+3737843883096k+1868930883259.$$

Also,

$$\sum_{k=0}^{\infty}\frac{(234440315200k^2-497134511862k+260991361673)k^3\binom{2k}{k}^2\binom{4k}{2k}}{396^{4k}}=-\frac{264627\sqrt{2}}{71680\pi},\quad \text{(S24)}$$

and

$$\sum_{k=0}^{\infty}\frac{v_k\binom{2k}{k+1}^2\binom{4k}{2k}}{396^{4k}}=90382094430705\frac{\sqrt{2}}{4\pi},\quad \text{(S24}'\text{)}$$

where

$$v_k:=104221430356652068096000k^2+208442860815019737658620k+104221430458367667816230$$

Lemma 4.3. *Let $m\in\mathbb{Z}\setminus\{0\}$ and $n\in\mathbb{N}$.*

(i) *We have*

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}\big((1728-m)k^3+2592k^2+1104k+120\big)=24(2n+1)(6n+1)(6n+5)\frac{\binom{2n}{n}\binom{3n}{n}\binom{6n}{3n}}{m^n},$$

$$\sum_{k=0}^{n}\frac{k\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}\big((1728-m)k^3+(2592+m)k^2+1104k+120\big)=24n(2n+1)(6n+1)(6n+5)\frac{\binom{2n}{n}\binom{3n}{n}\binom{6n}{3n}}{m^n},$$

and

$$\sum_{k=0}^{n}\frac{k^2\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}\big((1728-m)k^3+(2592+2m)k^2+(1104-m)k+120\big)$$
$$=24n^2(2n+1)(6n+1)(6n+5)\frac{\binom{2n}{n}\binom{3n}{n}\binom{6n}{3n}}{m^n}.$$

(ii) *We have*

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)^2m^k}\big((1728-m)k^3+(2592-2m)k^2+(1104-m)k+120\big)$$
$$=24(2n+1)(6n+1)(6n+5)\frac{\binom{2n}{n}\binom{3n}{n}\binom{6n}{3n}}{(n+1)^2m^n}$$

and

$$\sum_{k=0}^{n}\frac{k\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)^2m^k}\big((1728-m)k^3+(2592-m)k^2+(1104+m)k+m+120\big)$$
$$=24n(2n+1)(6n+1)(6n+5)\frac{\binom{2n}{n}^2\binom{4n}{2n}}{(n+1)^2m^n}.$$

Proof. It is easy to prove the five identities by induction on n. We find them by the Gosper algorithm. □

For any $k\in\mathbb{N}$, we clearly have

$$\binom{3k}{k+1}=\frac{2k}{k+1}\binom{3k}{k}$$

which is similar to $\binom{2k}{k+1}=\frac{k}{k+1}\binom{2k}{k}$. In view of this and Lemma 4.3, by our method to prove Theorem 4.1 we can establish the following theorem.

Theorem 4.3. (i) *We have the following identities:*

$$\sum_{k=0}^{\infty}\frac{(23968k^2-71188k+25539)k^3\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{20^{3k}}=-\frac{1875\sqrt5}{56\pi},\tag{S25}$$

$$\sum_{k=0}^{\infty}\frac{(269696k^2+545164k+275383)\binom{2k}{k+1}\binom{3k}{k+1}\binom{6k}{3k}}{20^{3k}}=\frac{50750\sqrt5}{3\pi},\tag{S25$'$}$$

$$\sum_{k=0}^{\infty}\frac{(90706k^2-168589k+88872)k^3\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-2^{15})^k}=\frac{150\sqrt2}{7\pi},\tag{S26}$$

and

$$\sum_{k=0}^{\infty}\frac{(5821200k^2+11623266k+5801911)\binom{2k}{k+1}\binom{3k}{k+1}\binom{6k}{3k}}{(-2^{15})^k}=-\frac{261184\sqrt2}{3\pi}.\tag{S26$'$}$$

(ii) *We have*

$$\sum_{k=0}^{\infty}\frac{(457326k^2-308241k+342923)k^3\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-15)^{3k}}=\frac{5000\sqrt{15}}{7\pi},\tag{S27}$$

and

$$\sum_{k=0}^{\infty}\frac{(933849k^2+1843866k+910277)\binom{2k}{k+1}\binom{3k}{k+1}\binom{6k}{3k}}{(-15)^{3k}}=-\frac{27125\sqrt{15}}{\pi},\tag{S27$'$}$$

Moreover,

$$\sum_{k=0}^{\infty}\frac{(12826k^2-27741k+13298)k^3\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2\times 30^3)^k}=-\frac{625\sqrt{15}}{972\pi},\tag{S28}$$

$$\sum_{k=0}^{\infty}\frac{(4941882k^2+9895613k+4953661)\binom{2k}{k+1}\binom{3k}{k+1}\binom{6k}{3k}}{(2\times 30^3)^k}=\frac{56375\sqrt{15}}{3\pi}.\tag{S28$'$}$$

(iii) *We have*

$$\sum_{k=0}^{\infty}\frac{(18126342k^2-37421775k+19111480)k^3\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}}=\frac{50\sqrt{6}}{\pi},\tag{S29}$$

and

$$\sum_{k=0}^{\infty}\frac{(6802059888k^2+13603203918k+6801143345)\binom{2k}{k+1}\binom{3k}{k+1}\binom{6k}{3k}}{(-96)^{3k}}=-2358976\frac{\sqrt{6}}{\pi}.\tag{S29$'$}$$

(iv) *We have*

$$\sum_{k=0}^{\infty}\frac{(2248722k^2-4689621k+2357878)k^3\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{66^{3k}}=-\frac{275\sqrt{33}}{28\pi},\tag{S30}$$

$$\sum_{k=0}^{\infty}\frac{(71477721k^2+142985853k+71508077)\binom{2k}{k+1}\binom{3k}{k+1}\binom{6k}{3k}}{66^{3k}}=32956\frac{\sqrt{33}}{\pi},\tag{S30$'$}$$

$$\sum_{k=0}^{\infty}\frac{(2161071858k^2-4497745053k+2312761384)k^3\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-3\times 160^3)^k}=\frac{1250\sqrt{30}}{9\pi},\tag{S31}$$

and

$$\sum_{k=0}^{\infty}\frac{w_k\binom{2k}{k+1}\binom{3k}{k+1}\binom{6k}{3k}}{(-3\times 160^3)^k}=-273064000\frac{\sqrt{30}}{3\pi},\tag{S31$'$}$$

where

$$w_k := 8126882714192k^2 + 16253686120778k + 8126803400291$$

(v) *We have*

$$\sum_{k=0}^{\infty} \frac{(273732850062k^2 - 572136425667k + 296241014776)k^3\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-960)^{3k}} = \frac{5000\sqrt{15}}{21\pi}, \tag{S32}$$

and

$$\sum_{k=0}^{\infty} \frac{x_k\binom{2k}{k+1}\binom{3k}{k+1}\binom{6k}{3k}}{(-960)^{3k}} = -235929568000\frac{\sqrt{15}}{3\pi}, \tag{S32$'$}$$

where

$$x_k := 357405080886027216k^2 + 714810113296569594k + 357405032410460843.$$

Also,

$$\sum_{k=0}^{\infty} \frac{(6491502k^2 - 13521457k + 6955771)k^3\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{255^{3k}} = -\frac{42500\sqrt{255}}{413343\pi}, \tag{S33}$$

and

$$\sum_{k=0}^{\infty} \frac{y_k\binom{2k}{k+1}\binom{3k}{k+1}\binom{6k}{3k}}{255^{3k}} = 2349045125\frac{\sqrt{255}}{54\pi}. \tag{S33$'$}$$

where

$$y_k := 15274005325299k^2 + 30548121249166k + 15274115917127.$$

(vi) *We have*

$$\sum_{k=0}^{\infty} \frac{p(k)k^3\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-5280)^{3k}} = \frac{13750\sqrt{330}}{7\pi} \tag{S34}$$

and

$$\sum_{k=0}^{\infty} \frac{q(k)\binom{2k}{k+1}\binom{3k}{k+1}\binom{6k}{3k}}{(-5280)^{3k}} = -107945164712000\frac{\sqrt{330}}{\pi}, \tag{S34$'$}$$

where

$$p(k) = 2417841826472898k^2 - 5065781116806693k + 2634006768739304$$

and

$$q(k) = 38282547040299680845406 4k^2 + 765650940493903329223926k + 382825470090906516313597.$$

Also,

$$\sum_{k=0}^{\infty}\frac{P(k)k^3\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-640320)^{3k}}=-\frac{83\sqrt{10005}}{\pi}\tag{S35}$$

and

$$\sum_{k=0}^{\infty}\frac{Q(k)\binom{2k}{k+1}\binom{3k}{k+1}\binom{6k}{3k}}{(-640320)^{3k}}=-466966544617045802560000\frac{\sqrt{10005}}{\pi},\tag{S35$'$}$$

where

$$\begin{aligned}P(k):=&37267848198715531945400632877 82k^2\\&-77838607611032940836670213273 91k\\&+40570759412375941952692536264 25\end{aligned}$$

and

$$\begin{aligned}Q(k):=&1626388893999999577578620229159002547888k^2\\&+3252777787999998411771125249563926009942k\\&+1626388893999998834192505020394081197549.\end{aligned}$$

5. Proof of Theorem 1.4

Proof of Theorem 1.4(i). Let $u_n=\mathrm{Domb}(n)/64^n$ and $v_n=\mathrm{Domb}(n)/(-32)^n$ for $n\in\mathbb{N}$. By the Zeilberger algorithm, we find the recurrence

$$(n+1)^3u_n-2(2n+3)(5n^2+15n+12)u_{n+1}+64(n+2)^3u_{n+2}=0$$

and

$$(n+1)^3v_n+(2n+3)(5n^2+15n+12)v_{n+1}+16(n+2)^3v_{n+2}=0.$$

Thus,

$$\begin{aligned}0=&\sum_{n=0}^{\infty}(n+1)^3u_n-2\sum_{n=0}^{\infty}(2(n+1)+1)(5(n+1)^2+5(n+1)+2)u_{n+1}\\&+64\sum_{n=0}^{\infty}(n+2)^3u_{n+2}\\=&\sum_{k=0}^{\infty}(k+1)^3u_k-2\sum_{k=1}^{\infty}(2k+1)(5k^2+5k+2)u_k+64\sum_{k=2}^{\infty}k^3u_k\\=&\sum_{k=0}^{\infty}\left((k+1)^3-2(2k+1)(5k^2+5k+2)+64k^3\right)u_k\\&+2(2\times0+1)(5\times0^2+5\times0+2)u_0-64(0^3u_0+1^3u_1)\\=&3\sum_{k=0}^{\infty}(15k^3-9k^2-5k-1)u_k\end{aligned}$$

and hence

$$\sum_{k=0}^{\infty}(15k^3-9k^2-5k-1)u_k=0. \tag{5.1}$$

Similarly,

$$\begin{aligned}0=&\sum_{k=0}^{\infty}\left((k+1)^3+(2k+1)(5k^2+5k+2)+16k^3\right)v_k\\ =&3\sum_{k=0}^{\infty}(9k^3+6k^2+4k+1)v_k\end{aligned}$$

and hence

$$\sum_{k=0}^{\infty}(9k^3+6k^2+4k+1)v_k=0. \tag{5.2}$$

Combining (5.1) and (5.2) with the two known identities in (1.6), we immediately get the desired (1.12) and (1.11). □

Proof of Theorem 1.4(ii). The second part of Theorem (1.4) can be proved by the same method we prove Theorem (1.4)(i). Here we just provide a proof of (1.13) in details. Let $w_n = f_n^{(4)}/36^n$ for $n \in \mathbb{N}$. By the Zeilberger algorithm, we find the recurrence

$$324(n+2)^3w_{n+2}-18(2n+3)(3n^2+9n+7)w_{n+1}-(n+1)(4n+3)(4n+5)w_n=0.$$

Thus,

$$\begin{aligned}0=&324\sum_{n=0}^{\infty}(n+2)^3w_{n+2}-18\sum_{n=0}^{\infty}(2(n+1)+1)(3(n+1)^2+3(n+1)+1)w_{n+1}\\ &-\sum_{n=0}^{\infty}(n+1)(4n+3)(4n+5)w_n\\ =&324\sum_{k=2}^{\infty}k^3w_k-18\sum_{k=1}^{\infty}(2k+1)(3k^2+3k+1)w_k\\ &-\sum_{k=0}^{\infty}(k+1)(4k+3)(4k+5)w_k\\ =&\sum_{k=0}^{\infty}\left(324k^3-18(2k+1)(3k^2+3k+1)-(k+1)(4k+3)(4k+5)\right)w_k\\ &-324\times 1^3w_1+18(2\times 0+1)(3\times 0^2+3\times 0+1)\end{aligned}$$

and hence

$$\sum_{k=0}^{\infty}(200k^3-210k^2-137k-33)w_k=0. \tag{5.3}$$

By (1.7),

$$\sum_{k=0}^{\infty}33(4k+1)w_k=\frac{33\times 18}{\sqrt{15}\,\pi}. \tag{5.4}$$

Adding (5.3) and (5.4), we immediately obtain (1.13). □

Our method for proving parts (i)-(ii) of Theorem 1.4 also works for parts (iii)-(iv) of Theorem 1.4. We omit the proof details.

6. Conjectures for series only involving binomial coefficients

Motivated by the author's conjectural identity

$$\sum_{k=1}^{\infty}\frac{48^k}{k(2k-1)\binom{2k}{k}\binom{4k}{2k}}=\frac{15}{2}K$$

(cf. [22, 24]), we make the following conjecture.

Conjecture 6.1. *We have*

$$\sum_{k=0}^{\infty}\frac{(4k^2-30k+17)48^k}{(2k-1)\binom{2k}{k}\binom{4k}{2k}}=\frac{45K-44}{2}.$$

Remark 6.1. By induction, for each $n=1,2,3,\ldots$ we have

$$\sum_{k=1}^{n}\frac{(4k^2-28k+3)48^k}{k\binom{2k}{k}\binom{4k}{2k}}=12-\frac{12(n+1)48^n}{\binom{2n}{n}\binom{4n}{2n}}$$

and

$$\sum_{k=1}^{n}\frac{(4k^2-40k-9)48^k}{\binom{2k}{k}\binom{4k}{2k}}=-12+\frac{12(n+1)^2 48^n}{\binom{2n}{n}\binom{4n}{2n}}.$$

In view of Theorem 1.3 and the author's conjectural identity (1.5), we pose the following conjecture obtained via the PSLQ algorithm.

Conjecture 6.2. (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\binom{2k}{k+1}^5}P(k)=\frac{8103654862170335619+4368545100830839178\zeta(3)}{5}$$

where

$$\begin{aligned}P(k)=&54430524632163842275k^4-132483674356197881281k^3\\&+121816306858962351437k^2-47590274284953796032k\\&+6700636215039814272.\end{aligned}$$

(ii) *We have*

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-64)^k}{\binom{2k}{k}^4\binom{3k}{k}}(676704k^4-1205388k^3+1140374k^2-152237k+78797)\\&=35733-1344\zeta(3),\end{aligned}$$

$$\sum_{k=1}^{\infty}\frac{(-64)^{k-1}}{\binom{2k}{k+1}^4\binom{3k}{k}}Q(k)=7(14488697756+4718909979\zeta(3))$$

and

$$\sum_{k=1}^{\infty}\frac{(-64)^kR(k)}{\binom{2k}{k+1}^4\binom{3k}{k+1}}=-86166921288937568-74477398755902744\zeta(3),$$

where

$$\begin{aligned}Q(k)=&9152858507744k^4-18103487906940k^3+16104889340010k^2\\&-5519172201903k+668801335410\end{aligned}$$

and

$$\begin{aligned}R(k)=&152571345867547488k^4-325445013351260332k^3\\&+295511129648313866k^2-106449469340961699k\\&+13378286508841890\end{aligned}$$

Remark 6.2. In the spirit of our method to prove Theorems 1.1-1.3 as well as the general algorithm to deduce series of type S, parts (i) and (ii) of Conjecture 6.2 look equivalent to (1.4) and (1.5) respectively but we have not done this in details.

Motivated by B. Gourevich's conjectural identity (1.2) and Guillera's conjectural identity

$$\sum_{k=1}^{\infty}\frac{256^k}{k^7\binom{2k}{k}^7}(21k^3-22k^2+8k-1)=\frac{\pi^4}{8}$$

(cf. [13, Section 4]), we pose the following conjecture via the PSLQ algorithm.

Conjecture 6.3. (i) *We have*

$$\sum_{k=1}^{\infty}\frac{S(k)}{2^{20k}}k^7\binom{2k}{k}^7=-\frac{1}{24\pi^3}$$

and

$$\begin{aligned}\frac{1}{32}\sum_{k=1}^{\infty}\frac{T(k)}{2^{20k}}\binom{2k}{k+1}^7=&7557023139446739654574720 0\\&-\frac{23431455851330568058457047 03}{\pi^3}\end{aligned}$$

where

$$\begin{aligned}S(k):=&56448k^6-347200k^5+854280k^4\\&-1145956k^3+851214k^2-339967k+56160\end{aligned}$$

and

$$\begin{aligned}T(k) =&3623349017250475433406178 56k^6 + 1557477795272579082461315904k^5\\&+ 2387296377854823511932510816k^4 + 1269650426797215833274563064k^3\\&- 3726123596657358354698025 16k^2 - 6203959696228088793093677 22k\\&- 1705818636835338216445717 35\end{aligned}$$

(ii) *We have*

$$\sum_{k=0}^{\infty}\frac{U(k)256^k}{\binom{2k}{k}^7}=\frac{9984-\pi^4}{48}$$

and

$$\sum_{k=1}^{\infty}\frac{V(k)256^k}{\binom{2k}{k+1}^7}=\frac{103034584550200896962943 36-471182272146551046973 25\pi^4}{48},$$

where

$$U(k):=14112k^6-44464k^5+49490k^4-41069k^3+8155k^2-4749k-210.$$

and

$$\begin{aligned}V(k):=&10519563155172140696432 4k^6-3755223903276709721743 76k^5\\&+5080100300247690472701 38k^4-3769895731861497363467 23k^3\\&+1561393277754815825035 24k^2-3394256997774780970672 2k\\&+302710028806150203399 3.\end{aligned}$$

7. New type series involving generalized central trinomial coefficients

For $b,c\in\mathbb{Z}$ and $n\in\mathbb{N}$, the generalized central trinomial coefficient $T_n(b,c)$ denotes the coefficient of x^n in the expansion of $(x^2+bx+c)^n$. The author [21, 24, 25, 26] posed totally 9 types of conjectural series for $1/\pi$ involving generalized central trinomial coefficients. Here we consider their variants of type S.

Conjecture 7.1. *We have*

$$\sum_{k=1}^{\infty}\frac{3054600k^2-16826114k+11236485}{(-256)^k}k^3\binom{2k}{k}^2T_k(1,16)=-\frac{1952307}{5\pi}, \tag{I1S}$$

$$\sum_{k=1}^{\infty}\frac{357600k^2-239434k+401075}{(-1024)^k}k^3\binom{2k}{k}^2T_k(34,1)=\frac{25983}{10\pi}, \tag{I2S}$$

$$\begin{aligned}\sum_{k=1}^{\infty}&\frac{28823880k^2-740215234k+3516311133}{4096^k}k^3\binom{2k}{k}^2T_k(194,1)\\&=-\frac{152854918}{3\pi},\end{aligned} \tag{I3S}$$

$$\sum_{k=1}^{\infty}\frac{1336776k^2-5896258k+4457117}{4096^k}k^3\binom{2k}{k}^2T_k(62,1)=-\frac{188698\sqrt{3}}{7\pi}. \tag{I4S}$$

Also,

$$\sum_{k=1}^{\infty}\frac{12k^2-28k-9}{256^k}k\binom{2k}{k}^2T_k(8,-2)=-\frac{4(\sqrt{8+6\sqrt{2}}+3\sqrt[4]{2})}{3\pi},$$

$$\sum_{k=1}^{\infty}\frac{162k^2-969k+872}{256^k}k^2\binom{2k}{k}^2T_k(8,-2)=\frac{708\sqrt[4]{2}-1951\sqrt{8+6\sqrt{2}}}{12\pi},$$

and

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{31392k^2-277274k+594637}{256^k}k^3\binom{2k}{k}^2T_k(8,-2)\\&\quad=\frac{90563\sqrt{8+6\sqrt{2}}-221844\sqrt[4]{2}}{12\pi}.\end{aligned} \tag{I5S}$$

Conjecture 7.2. *We have the following identities:*

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{k^3(3401775k^2-37884933k+65097406)}{972^k}\binom{2k}{k}\binom{3k}{k}T_k(18,6)\\&\quad=\frac{158875\sqrt{3}}{\pi},\end{aligned} \tag{II1S}$$

$$\sum_{k=1}^{\infty}\frac{k^3a_k}{1000^k}\binom{2k}{k}\binom{3k}{k}T_k(10,1)=-\frac{3441159450\sqrt{3}}{7\pi} \tag{II2S}$$

where

$$a_k=34073404820k^2-166944861551k+136909066683,$$

$$\sum_{k=1}^{\infty}\frac{k^3a'_k}{18^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(198,1)=-\frac{1285805325750\sqrt{3}}{\pi} \tag{II3S}$$

where

$$a'_k=138317121900k^2-11341624063599k+169057291391203,$$

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{515565k^2-1452888k+614707}{24^{3k}}k^3\binom{2k}{k}\binom{3k}{k}T_k(26,729)\\&\quad=-\frac{64}{98415\pi}(195423\sqrt{3}+532925\sqrt{15}),\end{aligned} \tag{II10S}$$

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{1326294k^2-4598217k+2285731}{(-5400)^k}k^3\binom{2k}{k}\binom{3k}{k}T_k(70,3645)\\&\quad=-\frac{2}{2187\pi}(78460655\sqrt{3}+13402757\sqrt{15}),\end{aligned} \tag{II11S}$$

$$\sum_{k=1}^\infty\frac{4076131815k^2-7828831071k+7124292568}{(-13500)^k}k^3\binom{2k}{k}\binom{3k}{k}T_k(40,1548)$$
$$=\frac{25}{91854\pi}(32473732248\sqrt3-33657641611\sqrt6) \tag{II12S}$$

$$\sum_{k=1}^\infty\frac{17276571k^2-9174528k+19362029}{(-675)^k}k^3\binom{2k}{k}\binom{3k}{k}T_k(15,-5)$$
$$=\frac{159680563\sqrt{15}}{3456\pi}, \tag{II13S}$$

$$\sum_{k=1}^\infty\frac{5837598k^2-6981399k+6325061}{(-1944)^k}k^3\binom{2k}{k}\binom{3k}{k}T_k(18,-3)$$
$$=\frac{11336\sqrt3}{\pi}, \tag{II14S}$$

Conjecture 7.3. *We have the following identities:*

$$\sum_{k=1}^\infty\frac{k(133120k^2-28704k+2669)}{(-168^2)^k}\binom{2k}{k}\binom{4k}{2k}T_k(7,4096)$$
$$=\frac{21(14481\sqrt{42}+5650\sqrt{210})}{256\pi},$$

$$\sum_{k=1}^\infty\frac{k^2(2733056k^2-1413552k+1735253)}{(-168^2)^k}\binom{2k}{k}\binom{4k}{2k}T_k(7,4096)$$
$$=\frac{63(26395881\sqrt{42}+13904950\sqrt{210})}{40960\pi},$$

and

$$\sum_{k=1}^\infty\frac{k^3b_k}{(-168^2)^k}\binom{2k}{k}\binom{4k}{2k}T_k(7,4096)$$
$$=\frac{189}{1310720\pi}(181371717913\sqrt{42}-200347079650\sqrt{210}), \tag{III5S}$$

where

$$b_k=17768990720k^2-81509977344k+73509891901.$$

Remark 7.1. We omit many other conjectural identities of type S arising from the author's series for $1/\pi$ of type III (cf. [21]).

Conjecture 7.4. *We have the following identities:*

$$\sum_{k=1}^\infty\frac{k^3c_k}{(-48^2)^k}\binom{2k}{k}^2T_{2k}(7,1)=-\frac{7110645399}{5\pi} \tag{IV1S}$$

and

$$\sum_{k=1}^{\infty}\frac{k^3d_k}{4608^k}\binom{2k}{k}^2T_{2k}(10,-2)=-\frac{20525967\sqrt{6}}{4\pi},\tag{IV19S}$$

where

$$c_k=253607171350k^2-231835223289k+295912225514$$

and

$$d_k=220693694k^2-1203431385k+1673732470.$$

Remark 7.2. We omit many other conjectural identities of type S arising from the author's series for $1/\pi$ of type IV (cf. [21]).

Conjecture 7.5. *We have*

$$\sum_{k=1}^{\infty}\frac{ke_k^{(1)}}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=-\frac{318713456\sqrt{105}}{49\pi}$$

where

$$e_k^{(1)}:=24501204k^2+511623432k+17450401,$$

$$\sum_{k=1}^{\infty}\frac{k^2e_k^{(2)}}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=-\frac{11867637235544\sqrt{105}}{343\pi}$$

where

$$e_k^{(2)}:=1543522869252k^2+354135873240k+3269250841643,$$

and

$$\sum_{k=1}^{\infty}\frac{k^3e_k^{(3)}}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=-\frac{1274477364899408 92\sqrt{105}}{2401\pi}\tag{V1S}$$

where

$$e_k^{(3)}:=41310253635000948k^2-37929668348178936k+56105265685716343.$$

Also,

$$\sum_{k=1}^{\infty}\frac{(k^2+1)e_k}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=-\frac{429939571410644\sqrt{105}}{343\pi},$$

where

$$e_k:=263893057256556k^2+46715038856064k+6234077522125.$$

Remark 7.3. This is motivated by the author's conjectural identity

$$\sum_{k=0}^{\infty}\frac{1638k+277}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=\frac{44\sqrt{105}}{\pi}\tag{V1}$$

(cf. [21, 24]) confirmed in [28]. We also conjecture that

$$\sum_{k=0}^{\infty}(47675628k^3+995541624k^2+437210809k+79678364)\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)}{(-240)^{3k}}=0.$$

Conjecture 7.6. (i) *We have*

$$\sum_{k=1}^\infty\frac{5939142726k^2-50217677843k+47679239989}{(2^{11}3^3)^k}k^3T_k(10,11^2)^3 =\frac{125366398162515\sqrt2}{644204\pi}, \tag{VI1S}$$

(ii) *We have*

$$\sum_{k=1}^\infty\frac{k^3f(k)}{(-80)^{3k}}T_k(22,21^2)^3=-\frac{317993969514116005\sqrt5}{64827\pi}, \tag{VI2S}$$

where

$$f(k):=182075646906594k^2-658193121766971k+498776294291290.$$

(iii) *We have*

$$\sum_{k=1}^\infty\frac{k^3g(k)}{(-288)^{3k}}T_k(62,95^2)^3=-\frac{9}{30008125\pi}(a\sqrt2+b\sqrt{14}), \tag{VI3S}$$

where

$$a=27785710057239522248342134 58\ ,\ b=233563959215647713372779062 5,$$

and

$$g(k):=29814661986490996566930k^2-13827052413593207967842 5k + 9175668541887014008043 9.$$

Conjecture 7.7. (i) *We have*

$$\sum_{k=1}^\infty\frac{k^3h_k}{450^k}\binom{2k}{k}T_k(6,2)^2=-\frac{15255169187996007504 00}{7\pi}, \tag{VII1S}$$

where

$$h_k:=17175624536354715956 98k^2-20741022469043431508721k + 2993065177502089651689 5.$$

Also,

$$\sum_{k=1}^\infty\frac{761948702208k^2-4717770389584k+2473620838841}{28^{2k}}k^3\binom{2k}{k}T_k(4,9)^2 =-\frac{49}{46656\pi}(1744223791168\sqrt3+307854304805\sqrt6), \tag{VII2S}$$

and

$$\sum_{k=1}^\infty\frac{k^3\ell_k}{22^{2k}}\binom{2k}{k}T_k(5,1)^2=-\frac{861143519145937597955\sqrt7}{1536\pi}, \tag{VII3S}$$

where

$$\ell_k := 258187375547938944 00k^2 - 14815685500933220 8624k + 13669954671855350 2681.$$

(ii) *We have*

$$\sum_{k=1}^{\infty} \frac{k^3 m_k}{46^{2k}} \binom{2k}{k} T_k(7,1)^2 = -\frac{675829671880947233751995193\sqrt{7}}{2458624\pi}, \tag{VII4S}$$

where

$$m_k := 60405940502779625588736k^2 - 19422679003406151098 2928k + 11344530206880618404 8447.$$

Also,

$$\sum_{k=1}^{\infty} \frac{23575340288k^2 - 30724817208k + 70092391411}{(-108)^k} k^3 \binom{2k}{k} T_k(3,-3)^2 = \frac{83673484461\sqrt{7}}{512\pi}, \tag{VII5S}$$

and

$$\sum_{k=1}^{\infty} \frac{k^3 n_k}{(-5177196)^k} \binom{2k}{k} T_k(171,-171)^2 = \frac{350688948772953536866681889843315704 8375\sqrt{7}}{1024\pi}, \tag{VII6S}$$

where

$$n_k := 10084570997509032944421348793965204251648k^2 - 19349061584246199246547660635419016030648k + 10272620865354618637287798800250376143247.$$

Moreover,

$$\sum_{k=1}^{\infty} \frac{k^3 p(k)}{434^{2k}} \binom{2k}{k} T_k(73,576)^2 = -\frac{1941824970252133568742562638158781654482433600879227\sqrt{6}}{28311552\pi}, \tag{VII7S}$$

where

$$p(k) := 848697060734207218041016898212857790098672 8448k^2 - 336802994867722964408074512804039621216263 43552k + 173291774654923578683359642422675140811716 15873.$$

Conjecture 7.8. *We have*

$$\sum_{k=1}^{\infty}\frac{4158282880k^2-22813966696k+40778481375}{(-50)^k}k^3T_k(4,1)T_k(1,-1)^2$$
$$=-\frac{29231055627965\sqrt{15}}{124416\pi}, \tag{VIII1S}$$

$$\sum_{k=1}^{\infty}\frac{q(k)}{3240^k}k^3T_k(7,1)T_k(10,10)^2=-\frac{260284882234938220643237793 6\sqrt5}{7\pi}, \tag{VIII2S}$$

where

$$q(k):=498872838526220266559278065k^2-759417631728571773272510548 1k$$
$$+151117694789961557184766131894.$$

Also,

$$\sum_{k=1}^{\infty}\frac{r_k}{(-2430)^k}k^3T_k(8,1)T_k(5,-5)^2=\frac{411463803058011921348 9\sqrt{15}}{7168\pi}, \tag{VIII3S}$$

and

$$\sum_{k=1}^{\infty}\frac{s_k}{(-29700)^k}k^3T_k(14,1)T_k(11,-11)^2$$
$$=\frac{2378258646892144658342016823285 65\sqrt5}{3584\pi}, \tag{VIII4S}$$

where

$$r_k:=192149117669867382528 0k^2-287765492662177090853 6k$$
$$+2447787507626240957299$$

and

$$s_k:=468213330859000529926217259379 20k^2$$
$$-715802437316982308600351967669 52k$$
$$+470984732519855837343784826127 35.$$

Conjecture 7.9. *We have*

$$\sum_{k=1}^{\infty}\frac{P_1(k)}{3136^k}k^3\binom{2k}{k}T_k(14,1)T_k(17,16)$$
$$=-\frac{70662375766828100018060759}{24\pi} \tag{IX1S}$$

and

$$\sum_{k=1}^\infty\frac{P_2(k)}{3136^k}k^3\binom{2k}{k}T_k(2,81)T_k(14,81)$$
$$=-\frac{49}{2916\pi}(16031198128567258016+6632521068118111355\sqrt5),\quad \text{(IX2S)}$$

where

$$P_1(k):=856641661945015641791811 0k^2-7321197350639766560601200 3k+10928986868870691256458210 6$$

and

$$P_2(k):=6503739411338308880k^2-11795256486234065124k+5657835264086004473.$$

Those $T_n=T_n(1,1)$ $(n\in\mathbb{N})$ are central trinomial coefficients. Motivated by the author's conjectural identity

$$\sum_{k=1}^\infty\frac{(105k-44)T_{k-1}}{k^2\binom{2k}{k}^2 3^{k-1}}=6\log3+\frac{5\pi}{\sqrt3}$$

(cf. [25, (10.1)]), we pose the following conjecture.

Conjecture 7.10. *We have*

$$\sum_{k=1}^\infty\frac{(33k^2+32k+8)T_{k-1}(8,-2)}{k(2k+1)^2\binom{2k}{k}^2 18^{k-1}}=3\log2$$

and

$$\sum_{k=1}^\infty\frac{(15k^3+8k^2-4k-2)T_{k-1}(2,-1)}{k^2(2k+1)^2\binom{2k}{k}^2}=\frac{\pi}{4\sqrt2}.$$

Also,

$$\sum_{k=0}^\infty\frac{(32340k^2-28975k-63)T_k}{\binom{2k}{k}^2 3^k}=1024+216\log3-\frac{388}{9}\sqrt3\,\pi,$$

$$\sum_{k=0}^\infty\frac{(625k^3-2125k^2+1735k-249)T_k(2,-1)}{\binom{2k}{k}^2}=\frac{9\pi}{\sqrt2}-384,$$

$$\sum_{k=0}^\infty\frac{(11979k^3-50303k^2+43769k-9725)T_k(8,-2)}{\binom{2k}{k}^2 18^k}=36(\log2-288).$$

8. Other series of type S

The author [25, (8.1)] observed the identity

$$\sum_{k=0}^\infty\frac{145k+9}{900^k}\beta_kT_k(52,1)=\frac{285}{\pi}.$$

Motivated by this we formulate the following conjecture.

Conjecture 8.1. (i) *We have*

$$\sum_{k=0}^{\infty}\frac{kt_k^{(1)}}{900^k}\beta_kT_k(52,1)=-\frac{498507105}{\pi}$$

$$\sum_{k=0}^{\infty}\frac{k^2t_k^{(2)}}{900^k}\beta_kT_k(52,1)=-\frac{64232846618769}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{k^3t_k^{(3)}}{900^k}\beta_kT_k(52,1)=-\frac{1002142163415074991}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{(k^2+1)t_k^{+}}{900^k}\beta_kT_k(52,1)=-\frac{2512943751517782}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{(k^2-1)t_k^{-}}{900^k}\beta_kT_k(52,1)=-\frac{213755454707472}{\pi},$$

where

$$t_k^{(1)}=31642625k^2-170395425k-234814102,$$

$$t_k^{(2)}=4936966494550k^2-65077136905125k+134224328497289,$$

$$\begin{aligned}t_k^{(3)}=&2822066506655501225k^2-4544582337949545 6435k\\&+1349161938863226 18984,\end{aligned}$$

$$\begin{aligned}t_k^{+}=&71236503338400k^2-613367241395375k+111921391484847,\\t_k^{-}=&40085552483775k^2-295218039299125k-125724302470098.\end{aligned}$$

(ii) *We have*

$$\sum_{k=0}^{\infty}\frac{k^j\beta_kT_k(52,1)}{900^k}P_j(k)=0$$

for all $j=0,1,2,$ *where*

$$\begin{aligned}P_0(k)=&601209875k^3-3237513075k^2+357434077k+299104263,\\P_1(k)=&3493703960875k^3-48937877104875k^2\\&+110522556574445k+21410948872923,\\P_2(k)=&450165200053206075k^3-7261629784552340695k^2\\&+21683271651287795913k-3340473878050249 97.\end{aligned}$$

Motivated by Theorem 1.4, we pose the following two conjectures.

Conjecture 8.2. *We have*

$$\sum_{k=1}^{\infty}\frac{(k-1)(9k+1)}{(-32)^k}k^2\mathrm{Domb}(k)=\frac{4}{3\pi},$$

$$\sum_{k=1}^{\infty}\frac{27k^2-12k+17}{(-32)^k}k^3\mathrm{Domb}(k)=\frac{2}{\pi},$$

$$\sum_{k=1}^{\infty}\frac{51k^2-60k+89}{(-32)^k}k^4\mathrm{Domb}(k)=-\frac{2}{3\pi},$$

$$\sum_{k=1}^{\infty}\frac{801k^2-796k+1371}{(-32)^k}k^5\mathrm{Domb}(k)=-\frac{248}{3\pi},$$

$$\sum_{k=1}^{\infty}\frac{12339k^2-19308k+46729}{(-32)^k}k^6\mathrm{Domb}(k)=\frac{4886}{3\pi},$$

$$\sum_{k=1}^{\infty}\frac{15k^2-51k+22}{64^k}k^3\mathrm{Domb}(k)=-\frac{8\sqrt{3}}{27\pi},$$

$$\sum_{k=1}^{\infty}\frac{110k^2-453k+229}{64^k}k^4\mathrm{Domb}(k)=-\frac{8\sqrt{3}}{45\pi},$$

$$\sum_{k=1}^{\infty}\frac{1145k^2-7119k+8368}{64^k}k^5\mathrm{Domb}(k)=\frac{7832\sqrt{3}}{135\pi},$$

$$\sum_{k=1}^{\infty}\frac{376560k^2-3010921k+4944543}{64^k}k^6\mathrm{Domb}(k)=\frac{3050552\sqrt{3}}{405\pi}.$$

Also,

$$\sum_{k=0}^{\infty}\frac{k\mathrm{Domb}(k)}{(-32)^k}(27k^3-6k^2+9k+2)=0,$$

$$\sum_{k=0}^{\infty}\frac{k^2\mathrm{Domb}(k)}{(-32)^k}(54k^3-51k^2+58k+3)=0,$$

$$\sum_{k=0}^{\infty}\frac{k^3\mathrm{Domb}(k)}{(-32)^k}(9k^3-9k^2+15k+1)=0,$$

$$\sum_{k=0}^{\infty}\frac{k^4\mathrm{Domb}(k)}{(-32)^k}(9k^3-80k^2+99k-124)=0,$$

$$\sum_{k=0}^{\infty}\frac{k^5\mathrm{Domb}(k)}{(-32)^k}(1116k^3-319k^2+2808k+2443)=0,$$

$$\sum_{k=0}^{\infty}\frac{k^6\mathrm{Domb}(k)}{(-32)^k}(21987k^3-214467k^2+316525k-651453)=0,$$

and

$$\begin{aligned}\sum_{k=0}^\infty\frac{k\mathrm{Domb}(k)}{64^k}(45k^3-75k^2-21k-5)=&0,\\ \sum_{k=0}^\infty\frac{k^2\mathrm{Domb}(k)}{64^k}(45k^3-153k^2+71k-3)=&0,\\ \sum_{k=0}^\infty\frac{k^3\mathrm{Domb}(k)}{64^k}(25k^3-105k^2+59k-3)=&0,\\ \sum_{k=0}^\infty\frac{k^4\mathrm{Domb}(k)}{64^k}(15k^3+377k^2-1827k+979)=&0,\\ \sum_{k=0}^\infty\frac{k^5\mathrm{Domb}(k)}{64^k}(132165k^3-1108949k^2+2059839k-381319)=&0,\\ \sum_{k=0}^\infty\frac{k^6\mathrm{Domb}(k)}{64^k}(1143957k^3-6418845k^2-10015633k+43834497)=&0.\end{aligned}$$

Remark 8.1. Though we have proved (1.11), we are unable to prove the first identity in Conjecture 8.2. This is because we could not prove the auxilliary identity

$$\sum_{k=0}^\infty(27k^3-6k^2+9k+2)v_k'=0 \tag{8.5}$$

with $v_k'=k\mathrm{Domb}(k)/(-32)^k$. By the Zeilberger algorithm, we find the recurrence

$$(n+1)^4v_n'+n(2n+3)(5n^2+15n+12)v_{n+1}'+16n(n+1)(n+2)^2v_{n+2}'=0$$

from which we have

$$\sum_{k=0}^\infty(27k^4-39k^3+32k^2-3k-1)v_k'=0.$$

Note that this is different from the desired (8.5).

Conjecture 8.3. *We have the following identities:*

$$\sum_{k=1}^\infty\frac{1048k^2-5526k+2651}{36^k}k^3f_k^{(4)}=-\frac{3618\sqrt{15}}{625\pi}, \tag{Y1S}$$

$$\sum_{k=1}^\infty\frac{304k^2-576k+521}{(-64)^k}k^3f_k^{(4)}=-\frac{25696\sqrt{15}}{151875\pi}, \tag{Y2S}$$

$$\sum_{k=1}^\infty\frac{157260600k^2-373237478k+150818747}{196^k}k^3f_k^{(4)}=-\frac{952574\sqrt{7}}{5\pi}, \tag{Y3S}$$

$$\sum_{k=1}^{\infty}\frac{1691296k^2-3241287k+1756217}{(-324)^k}k^3f_k^{(4)}=\frac{20434599\sqrt{5}}{20000\pi},\tag{Y4S}$$

$$\sum_{k=1}^{\infty}\frac{833112800k^2-1793574801k+878937679}{1296^k}k^3f_k^{(4)}=-\frac{1499005521\sqrt{2}}{10240\pi},\tag{Y5S}$$

$$\sum_{k=1}^{\infty}\frac{47808294003072k^2-102482715691400k+52422407372915}{5776^k}k^3f_k^{(4)}$$
$$=-\frac{122626206796\sqrt{95}}{625\pi}.\tag{Y6S}$$

Conjecture 8.4. (i) *We have*

$$\sum_{k=1}^{\infty}\frac{6700k^2-25077k+6239}{96^k}k^3\binom{2k}{k}f_k=-\frac{5787\sqrt{2}}{40\pi},$$
$$\sum_{k=1}^{\infty}\frac{100563606k^2-179847747k+97593215}{(-400)^k}k^3\binom{2k}{k}f_k=\frac{642350}{3\pi},$$

where f_k denotes the Franel number $\sum_{j=0}^{k}\binom{k}{j}^3$.

(ii) *We have*

$$\sum_{k=1}^{\infty}\frac{21k^2-109k-26}{32^k}kT_kZ_k=\frac{8(250+259\sqrt{5})}{375\pi},$$
$$\sum_{k=1}^{\infty}\frac{1170k^2-14621k+19673}{32^k}k^2T_kZ_k=-\frac{88(51250+24079\sqrt{5})}{375\pi},$$
$$\sum_{k=1}^{\infty}\frac{885285k^2-14121391k+29366404}{32^k}k^3T_kZ_k=\frac{8(21456450+3244519\sqrt{5})}{125\pi},$$

where

$$Z_k=\sum_{j=0}^{k}\binom{k}{j}\binom{2j}{j}\binom{2(k-j)}{k-j}$$

as introduced by D. Zagier [29].

(iii) *We have*

$$\sum_{k=1}^{\infty}\frac{148778208k^2-813461721k+717359335}{(-100)^k}k^3S_k(1,25)=-\frac{19433301932825}{995328\pi},$$

where

$$S_k(b,c)=\sum_{j=0}^{k}\binom{k}{j}^2T_j(b,c)T_{k-j}(b,c).$$

Remark 8.2. For known $\frac{1}{\pi}$-series involving Franel numbers, see [5, 6]. Part (ii) of Conjecture 8.4 is motivated by the author's discovery (cf. [25, (5.1)])

$$\sum_{k=0}^{\infty}\frac{5k+1}{32^k}T_kZ_k=\frac{8}{3\pi}(2+\sqrt{5}),$$

while part (iii) is inspired by the identity

$$\sum_{k=0}^{\infty}(3k+1)\frac{S_k(1,25)}{(-100)^k}=\frac{25}{8\pi}$$

obtained by the author (cf. [25, (1.91)]).

Conjecture 8.5. (i) *For $n\in\mathbb{N}$ let*

$$a_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}.$$

Then

$$\sum_{k=0}^{\infty}\frac{k\binom{2k}{k}a_k}{2160^k}(3750642k^2-27417879-200413)=\frac{6028830}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{k^2\binom{2k}{k}a_k}{2160^k}(7297838982k^2+44777877957k-249182264449)=\frac{56371645440}{\pi},$$

and

$$\sum_{k=0}^{\infty}\frac{k^3\binom{2k}{k}a_k}{2160^k}(1296246139663698k^2-604571242044753k+1636493480946725)$$
$$=-\frac{56023236343530}{7\pi}.$$

Also,

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}a_k}{2160^k}(36414k^3-266193k^2-234124k-66987)=0,$$

$$\sum_{k=0}^{\infty}\frac{k\binom{2k}{k}a_k}{2160^k}(30114378k^3+40060179k^2+29646217k+7732736)=0,$$

and

$$\sum_{k=0}^{\infty}\frac{k^2\binom{2k}{k}a_k}{2160^k}P(k)=0$$

where

$P(k)=281579848704k^3-131104215018k^2+356871786301k-7684943257.$

(ii) *For $n\in\mathbb{N}$ let*

$$b_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}\left(-\frac{2}{3}\right)^{3k}.$$

Then

$$\sum_{k=0}^{\infty}\frac{k\binom{2k}{k}b_k}{(-20)^k}(1440k^2-10980k-559)=\frac{3}{8\pi}(4255\sqrt{6}+5904\sqrt{15}),$$

$$\sum_{k=0}^{\infty}\frac{k^2\binom{2k}{k}b_k}{(-20)^k}(6192k^2-99894k+37303)=-\frac{3}{64\pi}(168935\sqrt{6}+171648\sqrt{15}),$$

and

$$\sum_{k=0}^{\infty}\frac{k^3\binom{2k}{k}b_k}{(-20)^k}(93108288k^2-1995628400k+241469595)$$
$$=\frac{3}{128\pi}(3562267392\sqrt{15}-403810135\sqrt{6}).$$

Remark 8.3. Conjecture 8.5 is motivated by the author's conjectural series

$$\sum_{k=0}^{\infty}\frac{357k+103}{2160^k}\binom{2k}{k}a_k=\frac{90}{\pi}\ \text{ and }\ \sum_{k=0}^{\infty}\frac{24n+5}{(-20)^k}\binom{2k}{k}b_k=\frac{3}{2\pi}(5\sqrt{6}+4\sqrt{15})$$

(cf. [22, (4.34)] and [26, (5.7)]).

Conjecture 8.6. *For $n\in\mathbb{N}$ let*

$$c_n=\sum_{k=0}^{n}5^k\frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}.$$

Then

$$\sum_{k=0}^{\infty}\frac{k\binom{2k}{k}c_k}{576^k}(68992k^2-129336k-9691)=\frac{27}{2\pi}(2092+1577\sqrt{2}),$$

$$\sum_{k=0}^{\infty}\frac{k^2\binom{2k}{k}c_k}{576^k}(60781952k^2-396958056k+220310929)$$
$$=-\frac{81}{4\pi}(3688232+2390989\sqrt{2}),$$

$$\sum_{k=0}^{\infty}\frac{(k^2+1)\binom{2k}{k}c_k}{576^k}(25180969856k^2-118523107336k-20012768875)$$
$$=-\frac{9}{4\pi}(37415939872+18625995101\sqrt{2}),$$

$$\sum_{k=0}^{\infty}\frac{k^3\binom{2k}{k}c_k}{576^k}(197398592384k^2-1384797901272k+855551417353)$$
$$=\frac{243}{56\pi}(4385109711\sqrt{2}-1128081200),$$

and

$$\sum_{k=0}^{\infty}\frac{k^4\binom{2k}{k}c_k}{576^k}Q(k)=\frac{243}{112\pi}(204062865832352+524948955975809\sqrt{2}),$$

where

$$Q(k) = 5366018489638016k^2 - 5037737067775448k + 5402824588325370 7.$$

Remark 8.4. Conjecture 8.6 is motivated by the author's conjectural series

$$\sum_{k=0}^\infty \frac{28k+5}{576^k}\binom{2k}{k}c_k = \frac{9}{\pi}(2+\sqrt{2})$$

(cf. [21, (8)]).

As there are many Ramanujan-type series, we cannot list all of their variants of type S here. We now conclude our paper and hope that our conjectures will stimulate further research.

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