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q-ANALOGUES OF SOME SERIES FOR POWERS OF π

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ABSTRACT. We obtain q-analogues of several series for powers of π. For example, the identity

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}=\frac{\pi^3}{32}$$

has the following q-analogue:

$$\sum_{k=0}^{\infty}(-1)^k\frac{q^{2k}(1+q^{2k+1})}{(1-q^{2k+1})^3}=\frac{(q^2;q^4)_\infty^2(q^4;q^4)_\infty^6}{(q;q^2)_\infty^4},$$

where q is any complex number with $|q|<1$. We also give q-analogues of four new series for powers of π found by the second author.

1. INTRODUCTION

The Riemann zeta function is given by

$$\zeta(s):=\sum_{n=1}^{\infty}\frac{1}{n^s}\quad \text{for } \mathrm{Re}(s)>1.$$

Obviously,

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^s}=\left(1-\frac{1}{2^s}\right)\zeta(s)\ \ \text{for } \mathrm{Re}(s)>1.$$

As Euler proved (cf. [12, pp. 231–232]), for each $m=1,2,3,\ldots$ we have

$$\zeta(2m)=(-1)^{m-1}\frac{2^{2m-1}\pi^{2m}}{(2m)!}B_{2m},$$

where the Bernoulli numbers $B_0,B_1,\ldots$ are given by $B_0=1$ and

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k=0\quad (n=1,2,3,\ldots).$$

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As usual, for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the q-analogue of n is given by

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{0\leqslant k<n} q^k.$$

For complex numbers a and q with $|q| < 1$, we adopt the standard notation

$$(a;q)_\infty = \prod_{k=0}^\infty (1-aq^k).$$

Recently, Z.-W. Sun [14] obtained the following q-analogues of Euler's formulae $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$:

$$\sum_{k=0}^\infty \frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2} = \frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4}$$

and

$$\sum_{k=0}^\infty \frac{q^{2k}(1+4q^{2k+1}+q^{4k+2})}{(1-q^{2k+1})^4} = \frac{(q^2;q^2)_\infty^8}{(q;q^2)_\infty^8},$$

where q is any complex number with $|q| < 1$. Note that $\lim_{q\to 1} \frac{1-q}{1-q^{2k+1}} = \frac{1}{2k+1}$ and also

$$\lim_{\substack{q\to 1\\ |q|<1}} (1-q)\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} = \lim_{\substack{q\to 1\\ |q|<1}} \prod_{n=1}^\infty \frac{[2n]_q^2}{[2n-1]_q[2n+1]_q} = \frac{\pi}{2} \tag{1.1}$$

with the help of Wallis' formula

$$\prod_{n=1}^\infty \frac{4n^2}{4n^2-1} = \frac{\pi}{2}.$$

Motivated by Sun's work [14], A. Goswami [7] got q-analogues of Euler's general formula for $\zeta(2m)$ with m a positive integer, and M. L. Dawsey and K. Ono [5] gave further applications.

Let χ be a Dirichlet character modulo a positive integer m. The Dirichlet L-function associated with the character χ is given by

$$L(s,\chi) := \sum_{n=1}^\infty \frac{\chi(n)}{n^s} \quad \text{for } \mathrm{Re}(s) > 1.$$

The Dirichlet beta function is defined by

$$\beta(s) = L\left(s, \left(\frac{-4}{\cdot}\right)\right) = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^s} \quad \text{for } \mathrm{Re}(s) > 0,$$

where $(-)$ denotes the Kronecker symbol. As Euler observed,

$$\beta(2n+1) = \frac{(-1)^n E_{2n}}{4^{n+1}(2n)!}\pi^{2n+1} \tag{1.2}$$

for all $n = 0, 1, 2, \ldots$ (cf. (3.63) of [6, p. 112]), where $E_0, E_1, E_2, \ldots$ are Euler numbers defined by

$$E_0 = 1, \text{ and } \sum_{\substack{k=0\\ 2\mid k}}^{n} \binom{n}{k} E_{n-k} = 0 \text{ for } n = 1, 2, 3, \ldots.$$

In particular,

$$\beta(1) = \frac{\pi}{4},\ \beta(3) = \frac{\pi^3}{32},\ \beta(5) = \frac{5\pi^5}{1536}.$$

In view of (1.1), we may view Ramanujan's formula

$$\sum_{k=0}^{\infty} \frac{(-q)^k}{1-q^{2k+1}} = \frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2} \quad (|q|<1)$$

(equivalent to Example (iv) in [2, p. 139]) as a q-analogue of Leibniz's identity $\beta(1) = \pi/4$. Recently, Q.-H. Hou, C. Krattenthaler and Z.-W. Sun [11] obtained the following new q-analogue of Leibniz's identity:

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+3)/2}}{1-q^{2k+1}} = \frac{(q^2;q^2)_\infty (q^8;q^8)_\infty}{(q;q^2)_\infty (q^4;q^8)_\infty} \quad \text{for } |q|<1.$$

Motivated by the above work, we seek a q-analogue of the identity

$$\beta(3) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}. \tag{1.3}$$

This leads to our following result.

Theorem 1.1. *For $|q|<1$ we have*

$$\sum_{k=0}^{\infty} (-1)^k \frac{q^{2k}(1+q^{2k+1})}{(1-q^{2k+1})^3} = \frac{(q^2;q^4)_\infty^2 (q^4;q^4)_\infty^6}{(q;q^2)_\infty^4}. \tag{1.4}$$

Remark 1.1. (1.4) is a q-analogue of (1.3) because

$$\begin{aligned}
&\lim_{\substack{q\to 1\\ |q|<1}} (1-q)^3 \frac{(q^2;q^4)_\infty^2 (q^4;q^4)_\infty^6}{(q;q^2)^4} \\
&= \lim_{\substack{q\to 1\\ |q|<1}} (1-q)^2 \frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4} \times \lim_{\substack{q\to 1\\ |q|<1}} \frac{1-q^2}{1+q} \cdot \frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2} \\
&= \frac{\pi^2}{4} \times \frac{\pi}{4} = \frac{\pi^3}{16}
\end{aligned}$$

in view of (1.1).

How to give q-analogues of (1.2) for $n = 2, 3, 4, \ldots$? This problem looks sophisticated.

We will prove Theorem 1.1 in the next section and present more similar results in Section 3.

Recently, Z.-W. Sun [15] established the following new identities:

$$\sum_{k=0}^{\infty}\frac{k(4k-1)\binom{2k}{k}^3}{(2k-1)^2(-64)^k}=-\frac{1}{\pi}, \tag{1.5}$$

$$\sum_{k=0}^{\infty}\frac{(4k-1)\binom{2k}{k}^3}{(2k-1)^3(-64)^k}=\frac{2}{\pi}, \tag{1.6}$$

$$\sum_{k=0}^{\infty}\frac{(12k^2-1)\binom{2k}{k}^3}{(2k-1)^2 256^k}=-\frac{2}{\pi}, \tag{1.7}$$

$$\sum_{k=1}^{\infty}\frac{(3k+1)16^k}{(2k+1)^2k^3\binom{2k}{k}^3}=\frac{\pi^2-8}{2}. \tag{1.8}$$

They are (1.1), (1.2), (1.3) and (1.77) of Sun [15] respectively. In our second theorem we give q-analogues of these four identities.

Theorem 1.2. *For $|q|<1$ we have*

$$\sum_{k=0}^{\infty}(-1)^k q^{k^2}\frac{[2k]_q([4k]_q-1)}{([2k]_q-1)^2}\cdot\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3}=-\frac{(q;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty^2}, \tag{1.9}$$

$$\sum_{k=0}^{\infty}(-1)^k q^{k^2+2k}\frac{[4k]_q-1}{([2k]_q-1)_q^3}\cdot\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3}=\frac{(q;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty^2}, \tag{1.10}$$

$$\sum_{k=0}^{\infty}\frac{P_k(q)q^{k^2}}{(1-q)^3([2k]_q-1)^2}\cdot\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3}=2q(1+q)\frac{(q^2;q^4)_\infty(q^6;q^4)_\infty}{(q^4;q^4)_\infty^2}, \tag{1.11}$$

where $P_k(q)$ denotes

$$q^{12k+1}-3q^{10k+2}+3(2q^2-1)q^{8k+1}-(3q^4-1)q^{6k}+3q^{4k+1}-3q^{2k+2}+2q^3-q,$$

and also

$$q\sum_{k=0}^{\infty}\frac{[3k+4]_q}{[2k+3]_q^2}\cdot\frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3}q^{k(k+5)/2}=(1-q)^2\frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4}-1-q. \tag{1.12}$$

We will prove Theorem 1.2 in Section 4 via the difference operator and some known q-identities.

2. Proof of Theorem 1.1

Lemma 2.1. *Let χ be any Dirichlet character. For $|q|<1$ we have*

$$\sum_{n=1}^{\infty}\chi(n)\frac{q^{n-1}(1+q^n)}{(1-q^n)^3}=\sum_{m=1}^{\infty}\left(\sum_{d\mid m}\chi\left(\frac{m}{d}\right)d^2\right)q^{m-1}. \tag{2.1}$$

Proof. For any positive integer n, we have

$$\frac{1}{(1-q^n)^3}=\sum_{k=0}^{\infty}\binom{-3}{k}(-q^n)^k=\sum_{k=0}^{\infty}\binom{k+2}{2}q^{kn}.$$

Thus

$$\begin{aligned}
&\sum_{n=1}^{\infty}\chi(n)\frac{q^{n-1}(1+q^n)}{(1-q^n)^3}\\
=&\sum_{n=1}^{\infty}\chi(n)q^{n-1}\left(\sum_{k=0}^{\infty}\binom{k+2}{2}q^{kn}+\sum_{k=0}^{\infty}\binom{k+2}{2}q^{(k+1)n}\right)\\
=&\sum_{n=1}^{\infty}\chi(n)q^{n-1}\left(1+\sum_{k=1}^{\infty}\left(\binom{k+2}{2}+\binom{k+1}{2}\right)q^{kn}\right)\\
=&\sum_{n=1}^{\infty}\chi(n)q^{n-1}\sum_{k=0}^{\infty}(k+1)^2q^{kn}=\sum_{n=1}^{\infty}\chi(n)\sum_{d=1}^{\infty}d^2q^{dn-1}\\
=&\sum_{m=1}^{\infty}\left(\sum_{d\mid m}\chi\left(\frac{m}{d}\right)d^2\right)q^{m-1}.
\end{aligned}$$

This concludes the proof of (2.1). ∎

Proof of Theorem 1.1. For $n=1,2,3,\ldots$ let $\chi(n)$ be the Kronecker symbol $(\frac{-4}{n})$. With the help of Lemma 2.1,

$$\sum_{k=0}^{\infty}(-1)^k\frac{q^{2k}(1+q^{2k+1})}{(1-q^{2k+1})^3}=\sum_{n=1}^{\infty}\chi(n)\frac{q^{n-1}(1+q^n)}{(1-q^n)^3}=\sum_{m=1}^{\infty}\left(\sum_{d\mid m}\chi\left(\frac{m}{d}\right)d^2\right)q^{m-1}.$$

On the other hand, we have

$$\sum_{m=1}^{\infty}\left(\sum_{d\mid m}\left(\frac{-4}{m/d}\right)d^2\right)q^{m-1}=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^6(1-q^{4n})^4}{(1-q^n)^4}$$

by Carlitz [4, 4.1] (or [1, Theorem 2.5]). Therefore

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k\frac{q^{2k}(1+q^{2k+1})}{(1-q^{2k+1})^3} &= \prod_{n=1}^{\infty}\frac{(1-q^{2n})^2(1-q^{4n})^4}{(1-q^{2n-1})^4}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{4n-2})^2(1-q^{4n})^6}{(1-q^{2n-1})^4}\\ &= \frac{(q^2;q^4)_\infty^2(q^4;q^4)_\infty^6}{(q;q^2)_\infty^4}.\end{aligned}$$

This concludes the proof of (1.4). ∎

3. Other results similar to Theorem 1.1

For any positive integers d and m with $(-1)^m d \equiv 0, 1 \pmod 4$, it is known (cf. [16]) that

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^m d}{n}\right)\frac{1}{n^m} \in \frac{\pi^m}{\sqrt{d}}\mathbb{Q},$$

where $\mathbb{Q}$ is the field of rational numbers.

Let q be any complex number with $|q| < 1$. By Carlitz [4, (2.1)] (or [1, Theorem 2.3]),

$$\sum_{m=1}^{\infty}\left(\sum_{d\mid m}\left(\frac{-3}{m/d}\right)d^2\right)q^{m-1} = \prod_{n=1}^{\infty}\frac{(1-q^{3n})^9}{(1-q^n)^3}.$$

Combining this with Lemma 2.1 we obtain that

$$\sum_{n=0}^{\infty}\left(\frac{n}{3}\right)\frac{q^{n-1}(1+q^n)}{(1-q^n)^3} = \frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3}, \tag{3.1}$$

which is a q-analogue of the identity

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{1}{n^3} = \frac{4\pi^3}{81\sqrt{3}}. \tag{3.2}$$

Similar to Lemma 2.1, for any Dirichlet character χ, we have

$$\sum_{n=1}^{\infty}\frac{\chi(n)q^{n-1}}{(1-q^n)^2} = \sum_{n=1}^{\infty}\chi(n)\sum_{k=0}^{\infty}(k+1)q^{(k+1)n-1} = \sum_{m=1}^{\infty}\left(\sum_{d\mid m}\chi\left(\frac{m}{d}\right)d\right)q^{m-1}. \tag{3.3}$$

Combining this with Ramanujan's identity

$$\sum_{m=1}^{\infty}\left(\sum_{d\mid m}\left(\frac{5}{m/d}\right)d\right)q^{m-1} = \prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{1-q^n},$$

we recover Ramanujan's result

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^{n-1}}{(1-q^n)^2}=\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} \tag{3.4}$$

(cf. [3, p. 107]), which can be viewed as a q-analogue of the identity

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{1}{n^2}=\frac{4\pi^2}{25\sqrt{5}}. \tag{3.5}$$

Ramanujan used (3.4) to deduce his famous congruence $p(5n+4)\equiv 0 \pmod 5$, where n is any nonnegative integer and $p(\cdot)$ is the well-known partition function.

By [1, Theorems 3.5 and 3.7],

$$\sum_{m=1}^{\infty}\left(\sum_{d\mid m}\left(\frac{8}{m/d}\right)d\right)q^{m-1}=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^3(1-q^{4n})(1-q^{8n})^2}{(1-q^n)^2}$$

and

$$\sum_{m=1}^{\infty}\left(\sum_{d\mid m}\left(\frac{12}{m/d}\right)d\right)q^{m-1}=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2(1-q^{3n})^2(1-q^{4n})(1-q^{12n})}{(1-q^n)^2}.$$

Combining this with (3.3) we obtain

$$\sum_{k=0}^{\infty}(-1)^{k(k+1)/2}\frac{q^{2k}}{(1-q^{2k+1})^2}=\frac{(q^2;q^2)_\infty^3(q^4;q^4)_\infty(q^8;q^8)_\infty^2}{(q;q)_\infty^2} \tag{3.6}$$

and

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\frac{q^{6k}}{(1-q^{6k+1})^2}-\frac{q^{6k+4}}{(1-q^{6k+5})^2}\right)\\ =&\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^2(q^4;q^4)_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty^2},\end{aligned} \tag{3.7}$$

which are q-analogues of the identities

$$\sum_{k=0}^{\infty}\frac{(-1)^{k(k+1)/2}}{(2k+1)^2}=\frac{\pi^2}{8\sqrt{2}}\ \text{and}\ \sum_{k=0}^{\infty}\left(\frac{3}{2k+1}\right)\frac{1}{(2k+1)^2}=\frac{\pi^2}{6\sqrt{3}}. \tag{3.8}$$

We are unable to find q-analogues of many identities including the following ones:

$$\sum_{k=0}^{\infty}\frac{(-1)^{k(k+3)/2}}{(2k+1)^3}=\frac{3\pi^3}{64\sqrt{2}},$$
$$\sum_{k=0}^{\infty}\frac{(-1)^{k(k+1)/2}}{(2k+1)^4}=\frac{11\pi^4}{768\sqrt{2}},$$
$$\sum_{k=0}^{\infty}\left(\frac{3}{2k+1}\right)\frac{1}{(2k+1)^4}=\frac{23\pi^4}{1296\sqrt{3}},$$
$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{1}{n^5}=\frac{4\pi^5}{729\sqrt{3}}.$$

4. Proof of Theorem 1.2

Proof of Theorem 1.2. For $k\in\mathbb{N}$ let

$$a_k(q)=(-1)^kq^{k^2}[4k+1]_q\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3}.$$

It is known that (see, e.g., [8, Eq. (1.5)])

$$\sum_{k=0}^{\infty}a_k(q)=\frac{(q;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty^2},$$

which is the q-analogue of Bauer's formula

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{\binom{2k}{k}^3}{64^k}=\frac{2}{\pi}.$$

Let

$$b_k(q)=(-1)^kq^{k^2}\frac{[2k]_q([4k]_q-1)}{([2k]_q-1)^2}\cdot\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3},$$

be the summand on the left hand side of (1.9). By the q-Gosper algorithm (implemented in `Maple` by the first author [10] or Koepf [13]), we find that

$$a_k(q)+b_k(q)=\Delta_k\left((-1)^{k+1}q^{k^2-1}\frac{(1-q^{2k})^3}{(1-q^{2k-1})^2(1-q)}\cdot\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3}\right)$$

for all $k\in\mathbb{N}$, where Δ_k is the difference operator defined by

$$\Delta_kf(k)=f(k+1)-f(k).$$

Hence

$$\sum_{k=0}^{\infty}b_k(q)=-\sum_{k=0}^{\infty}a_k(q)=-\frac{(q;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty^2}.$$

This proves (1.9).

Similarly, we set

$$c_k(q) = (-1)^k q^{k^2+2k} \frac{[4k]_q - 1}{([2k]_q - 1)^3} \cdot \frac{(q;q^2)_k^3}{(q^2;q^2)_k^3}$$

be the summand on the left hand side of (1.10) and note that

$$-a_k(q) + c_k(q) = \Delta_k \left((-1)^k q^{k^2-2} \frac{(1-q^{2k})^3(q^2-q^{2k})}{(1-q^{2k-1})^3(1-q)} \cdot \frac{(q;q^2)_k^3}{(q^2;q^2)_k^3} \right),$$

for all $k \in \mathbb{N}$. Hence

$$\sum_{k=0}^{\infty} c_k(q) = \sum_{k=0}^{\infty} a_k(q) = \frac{(q;q^2)_\infty (q^3;q^2)_\infty}{(q^2;q^2)_\infty^2}.$$

This proves (1.10).

For $k \in \mathbb{N}$ let

$$s_k(q) = \frac{P_k(q) q^{k^2+1}}{(1-q)^3([2k]_q - 1)^2} \cdot \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^4;q^4)_k^3}$$

and

$$t_k(q) = [6k+1]_q \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^4;q^4)_k^3} q^{k^2}.$$

Then

$$s_k(q) - 2q^2 t_k(q) = \Delta_k \left(\frac{(1-q^{4k})^3 q^{k^2}}{(1-q^{2k-1})^2(1-q)} \cdot \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^4;q^4)_k^3} \right).$$

for all $k \in \mathbb{N}$. Therefore,

$$\sum_{k=0}^{\infty} s_k(q) = 2q^2 \sum_{k=0}^{\infty} t_k(q) = 2q^2(1+q) \frac{(q^2;q^4)_\infty (q^6;q^4)_\infty}{(q^4;q^4)_\infty^2}$$

with the aid of [9, Theorem 1.1]. This proves (1.11).

Finally, let us consider (1.12). It is easy to verify that

$$\begin{aligned} &\frac{q^{2k+1}[3k+4]_q}{[2k+3]_q^2} \cdot \frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3} q^{k(k+1)/2} - [3k+2]_q \frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3} q^{k(k+1)/2} \\ &\qquad = \Delta_k \left(\frac{(1+q^{k+1})(1-q^{2k+1})}{1-q} \cdot \frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3} q^{k(k+1)/2} \right) \end{aligned}$$

for all $k \in \mathbb{N}$. Therefore,

$$q \sum_{k=0}^{\infty} \frac{[3k+4]_q}{[2k+3]_q^2} \cdot \frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3} q^{k(k+5)/2} - \sum_{k=0}^{\infty} [3k+2]_q \frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3} q^{k(k+1)/2}$$

coincides with $-1-q$. By [11, (1.9)],

$$\sum_{k=0}^{\infty}[3k+2]_q\frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3}q^{k(k+1)/2}=(1-q)^2\frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4}.$$

So we have (1.12).

In view of the above, we have completed the proof of Theorem 1.2. ∎

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