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SUPERCONGRUENCES INVOLVING LUCAS SEQUENCES

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Abstract. For $A, B \in \mathbb{Z}$, the Lucas sequence $u_n(A,B)$ $(n = 0, 1, 2, \ldots)$ are defined by $u_0(A,B) = 0$, $u_1(A,B) = 1$, and $u_{n+1}(A,B) = Au_n(A,B) - Bu_{n-1}(A,B)$ $(n = 1, 2, 3, \ldots)$. For any odd prime p and positive integer n, we establish the new result

$$\frac{u_{pn}(A,B) - (\frac{A^2-4B}{p})u_n(A,B)}{pn} \in \mathbb{Z}_p,$$

where $(\frac{\cdot}{p})$ is the Legendre symbol and $\mathbb{Z}_p$ is the ring of p-adic integers.

Let p be an odd prime and let n be a positive integer. For any integer $m \not\equiv 0 \pmod p$, we prove that

$$\frac{1}{pn}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k} - \left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}\right) \in \mathbb{Z}_p$$

and furthermore

$$\frac{1}{n}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k} - \left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}\right) \equiv \frac{\binom{2n-1}{n-1}}{m^{n-1}}u_{p-(\frac{\Delta}{p})}(m-2,1) \pmod{p^2},$$

where $\Delta = m(m-4)$. We also pose some conjectures for further research.

1. Introduction

Let $p > 3$ be a prime. In 2006 H. Pan and the author [11] deduced from a sophisticated combinatorial identity the congruence

$$\sum_{k=0}^{p-1}\binom{2k}{k+d} \equiv \left(\frac{p-d}{3}\right) \pmod p \quad \text{for } d = 0, \ldots, p-1,$$

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where $(-)$ denotes the Legendre symbol. In 2011 the author and R. Tauraso [26] obtained further that

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\pmod{p^2}\quad\text{for } d=0,1. \tag{1.1}$$

Recently, J.-C. Liu [7] proved the following extension with $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$ conjectured by M. Apagodu and D. Zeilberger [1]:

$$\sum_{k=0}^{pn-1}\binom{2k}{k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{n-1}\binom{2k}{k}\pmod{p^2} \tag{1.2}$$

and

$$\sum_{k=0}^{pn-1}C_k\equiv\begin{cases}\sum_{r=0}^{n-1}C_r\pmod{p^2} & \text{if } p\equiv 1\pmod 3,\\ -\sum_{r=0}^{n-1}(3r+2)C_r\pmod{p^2} & \text{if } p\equiv 2\pmod 3,\end{cases} \tag{1.3}$$

where C_k denotes the Catalan number $\binom{2k}{k}-\binom{2k}{k+1}=\binom{2k}{k}/(k+1)$. Note that this result in the case $n=1$ yields the supercongruence (1.1).

For given integers A and B, the Lucas sequence $u_n=u_n(A,B)$ $(n\in\mathbb{N}=\{0,1,2,\ldots\})$ is given by

$$u_0=0,\ u_1=1,\quad \text{and } u_{n+1}=Au_n-Bu_{n-1} \text{ for } n=1,2,3,\ldots.$$

It is well known that $p\mid u_{p-(\frac{A^2-4B}{p})}$ for any odd prime p not dividing B (see, e.g., [17, Lemma 2.3]). In 2010 the author [17] proved that for any nonzero integer m and odd prime p not dividing m we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}\equiv\left(\frac{m(m-4)}{p}\right)+u_{p-(\frac{m(m-4)}{p})}(m-2,1)\pmod{p^2}. \tag{1.4}$$

In this paper we obtain the following general result which is a common extension of (1.1)–(1.4).

Theorem 1.1. *Let $m\in\mathbb{Z}\setminus\{0\}$, $n\in\mathbb{Z}^+$ and $\Delta=m(m-4)$. For any odd prime p not dividing m, we have*

$$\frac{1}{n}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k}-\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}\right)\equiv\frac{\binom{2n-1}{n-1}}{m^{n-1}}u_{p-(\frac{\Delta}{p})}(m-2,1)\pmod{p^2} \tag{1.5}$$

and

$$\begin{aligned}&\frac{1}{n}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k+1}}{m^k}-\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r+1}}{m^r}+\left(\frac{m}{2}-\frac{\binom{2n-1}{n-1}}{m^{n-1}}\right)\left(1-\left(\frac{\Delta}{p}\right)\right)\right)\\ \equiv&\frac{\binom{2n-1}{n-1}}{m^{n-1}}\left(1-m^{p-1}+\frac{m-2}{2}u_{p-(\frac{\Delta}{p})}(m-2,1)\right)\pmod{p^2},\end{aligned} \tag{1.6}$$

hence

$$\begin{aligned}&\frac{1}{n}\bigg(\sum_{k=0}^{pn-1}\frac{C_k}{m^k}-\Big(\frac{\Delta}{p}\Big)\sum_{r=0}^{n-1}\frac{C_r}{m^r}+\bigg(\frac{\binom{2n-1}{n-1}}{m^{n-1}}-\frac{m}{2}\bigg)\Big(1-\Big(\frac{\Delta}{p}\Big)\Big)\bigg)\\ \equiv&\frac{\binom{2n-1}{n-1}}{m^{n-1}}\bigg(m^{p-1}-1+\frac{4-m}{2}u_{p-(\frac{\Delta}{p})}(m-2,1)\bigg)\pmod{p^2}.\end{aligned}\tag{1.7}$$

Corollary 1.1. *Let p be an odd prime and let $n\in\mathbb{Z}^+$. Then*

$$\frac{1}{n}\bigg(\sum_{k=0}^{pn-1}\binom{2k}{k}-\Big(\frac{p}{3}\Big)\sum_{r=0}^{n-1}\binom{2r}{r}\bigg)\equiv 0\pmod{p^2},\tag{1.8}$$

$$\frac{1}{n}\bigg(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{2^k}-\Big(\frac{-1}{p}\Big)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{2^r}\bigg)\equiv 0\pmod{p^2},\tag{1.9}$$

and

$$\frac{1}{n}\bigg(\sum_{k=0}^{pn-1}C_k-\Big(\frac{p}{3}\Big)\sum_{r=0}^{n-1}C_r+\frac{1-(\frac{p}{3})}{2}\Big(\binom{2n}{n}-1\Big)\bigg)\equiv 0\pmod{p^2}.\tag{1.10}$$

When $p>3$, we also have

$$\frac{1}{n}\bigg(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{3^k}-\Big(\frac{p}{3}\Big)\sum_{k=0}^{n-1}\frac{\binom{2r}{r}}{3^r}\bigg)\equiv 0\pmod{p^2}.\tag{1.11}$$

Proof. By induction, $u_{2k}(0,1)=0$ and $u_{3k}(\pm1,1)=0$ for all $k\in\mathbb{N}$. Applying (1.5) with $m=1,2,3$ we obtain (1.8), (1.9) and (1.11). Note also that (1.7) with $m=1$ yields (1.10). □

Remark 1.1. Our (1.10) implies (1.3) since $\sum_{r=0}^{n-1}(3r+1)C_r=\binom{2n}{n}-1$ for all $n\in\mathbb{Z}^+$. (1.9) and (1.10) in the case $n=1$ were first proved by the author [19].

For given integers A and B, the sequence $v_n=v_n(A,B)$ $(n=0,1,2,\ldots)$ defined by

$$v_0=2,\ v_1=A,\ \text{and}\ v_{n+1}=Av_n-Bv_{n-1}\ (n=1,2,3,\ldots)$$

is called the the companion sequence of the Lucas sequence $u_n=u_n(A,B)$ $(n\in\mathbb{N})$. By induction,

$$v_n(A,B)=2u_{n+1}(A,B)-Au_n(A,B)\quad\text{for all } n\in\mathbb{N}.$$

To prove Theorem 1.1, we need the following auxiliary result on general Lucas sequences which has its own interest.

Theorem 1.2. *Let $A, B \in \mathbb{Z}$ and $\Delta = A^2 - 4B$. Let p be an odd prime and let $n \in \mathbb{Z}^+$. Then*

$$\frac{u_{pn}(A,B) - (\frac{\Delta}{p})u_n(A,B)}{pn} \in \mathbb{Z}_p \quad \text{and} \quad \frac{v_{pn}(A,B) - v_n(A,B)}{pn} \in \mathbb{Z}_p, \tag{1.12}$$

where $\mathbb{Z}_p$ denotes the ring of p-adic integers. Moreover, if $p \nmid B\Delta$ then

$$\begin{aligned}\frac{u_{pn}(A,B) - (\frac{\Delta}{p})u_n(A,B)}{pn} \equiv & \frac{u_n(A,B)}{2}\left(\frac{\Delta}{p}\right)\frac{B^{p-1}-1}{p} \\ & + \frac{v_n(A,B)}{2B^{(1-(\frac{\Delta}{p}))/2}} \cdot \frac{u_{p-(\frac{\Delta}{p})}(A,B)}{p} \pmod p\end{aligned} \tag{1.13}$$

and

$$\begin{aligned}\frac{v_{pn}(A,B) - v_n(A,B)}{pn} \equiv & \frac{v_n(A,B)}{2} \cdot \frac{B^{p-1}-1}{p} \\ & + \frac{\Delta u_n(A,B)}{2B^{(1-(\frac{\Delta}{p}))/2}}\left(\frac{\Delta}{p}\right)\frac{u_{p-(\frac{\Delta}{p})}(A,B)}{p} \pmod p.\end{aligned} \tag{1.14}$$

Remark 1.2. (1.12) in the case $n = 1$ is well known, see, e.g., [17, Lemma 2.3]. For the prime $p = 2$, (1.12) also holds if we adopt the Kronecker symbol

$$\left(\frac{\Delta}{2}\right) = \begin{cases} 1 & \text{if } \Delta \equiv 1 \pmod 8, \\ -1 & \text{if } \Delta \equiv 5 \pmod 8, \\ 0 & \text{if } \Delta \equiv 0 \pmod 2. \end{cases}$$

Motivated by Theorems 1.1 and 1.2, we give another theorem on supercongruences.

Theorem 1.3. *For any prime $p > 5$ and $n \in \mathbb{Z}^+$, we have*

$$\frac{g_{pn}(-1) - g_n(-1)}{n^2} \equiv 0 \pmod{p^3}, \tag{1.15}$$

where

$$g_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k. \tag{1.16}$$

Remark 1.3. The polynomials $g_n(x)$ $(n = 0, 1, 2, \ldots)$ were introduced by the author [23] in which the author proved for any prime $p > 5$ that

$$\sum_{k=1}^{p-1} \frac{g_k(-1)}{k} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{g_k(-1)}{k^2} \equiv 0 \pmod p;$$

see also V.J.W. Guo, G.-S. Mao and H. Pan [3] for some congruences involving the polynomials $g_n(x)$ $(n = 0, 1, 2, \ldots)$.

We will prove Theorems 1.2, 1.1 and 1.3 in Sections 2, 3 and 4 respectively. In Section 5 we pose some conjectures for further research.

2. Proof of Theorem 1.2

Let $A, B \in \mathbb{Z}$, and let

$$\alpha = \frac{A+\sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A-\sqrt{\Delta}}{2} \tag{2.1}$$

be the two roots of the quadratic equation $x^2 - Ax + B = 0$, where $\Delta = A^2 - 4B$. It is well known that

$$(\alpha-\beta)u_n(A,B) = \alpha^n - \beta^n \quad \text{and} \quad v_n(A,B) = \alpha^n + \beta^n \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

When $\Delta = 0$, by induction we have $u_n(A,B) = n(A/2)^{n-1}$ for all $n \in \mathbb{Z}^+$.

Lemma 2.1. *Let $A, B \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Then*

$$u_n(A,B) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-k}{k} A^{n-1-2k}(-B)^k \tag{2.3}$$

and

$$v_n(A,B) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k} A^{n-2k}(-B)^k. \tag{2.4}$$

Remark 2.1. Lemma 2.1 is a well known result (see, e.g., [2, (1.60)]) and the two identities (2.3) and (2.4) can be easily proved by induction on n.

Lemma 2.2 [18, Lemma 2.2]**.** *Let $A, B \in \mathbb{Z}$ and let $d \in \mathbb{Z}^+$ be an odd divisor of $\Delta = A^2 - 4B$. Then, for any $n \in \mathbb{Z}^+$, we have*

$$\frac{u_n(A,B)}{n} \equiv \left(\frac{A}{2}\right)^{n-1} + \begin{cases} (A/2)^{n-3}\Delta/3 \pmod{d} & \text{if } 3 \mid d \text{ and } 3 \mid n, \\ 0 \pmod{d} & \text{otherwise.} \end{cases} \tag{2.5}$$

Lemma 2.3 [21, Lemma 2.2]**.** *Let $A, B \in \mathbb{Z}$ and $\Delta = A^2 - 4B$. Suppose that p is an odd prime with $p \nmid B\Delta$. Then we have the congruence*

$$\left(\frac{A \pm \sqrt{\Delta}}{2}\right)^{p-(\frac{\Delta}{p})} \equiv B^{(1-(\frac{\Delta}{p}))/2} \pmod{p} \tag{2.6}$$

in the ring of algebraic integers.

Lemma 2.4. *Let $A, B \in \mathbb{Z}$ and $\Delta = A^2 - 4B$. For any odd prime p not dividing $B\Delta$, we have*

$$v_{p-(\frac{\Delta}{p})}(A,B) \equiv B^{(1-(\frac{\Delta}{p}))/2}(B^{p-1}+1) \pmod{p^2}. \tag{2.7}$$

Proof. By the proof of [22, Lemma 3.2],

$$v_{p-(\frac{\Delta}{p})}(A,B) \equiv 2\left(\frac{B}{p}\right) B^{(p-(\frac{\Delta}{p}))/2} \pmod{p^2}.$$

Thus

$$\begin{aligned} & v_{p-(\frac{\Delta}{p})}(A,B) - B^{(1-(\frac{\Delta}{p}))/2}(B^{p-1}+1) \\ \equiv & 2\left(\frac{B}{p}\right) B^{(p-(\frac{\Delta}{p}))/2} - B^{(1-(\frac{\Delta}{p}))/2}(B^{p-1}+1) \\ = & B^{(1-(\frac{\Delta}{p}))/2}\left(2\left(\left(\frac{B}{p}\right)B^{(p-1)/2}-1\right) - (B^{p-1}-1)\right) \\ \equiv & 0 \pmod{p^2} \end{aligned}$$

since

$$\begin{aligned} B^{p-1}-1 = & \left(\left(\frac{B}{p}\right)B^{(p-1)/2}+1\right)\left(\left(\frac{B}{p}\right)B^{(p-1)/2}-1\right) \\ \equiv & 2\left(\left(\frac{B}{p}\right)B^{(p-1)/2}-1\right) \pmod{p^2}. \end{aligned}$$

This concludes the proof. □

Lemma 2.5. *Let p be a prime and let $n \in \mathbb{Z}^+$. For any p-adic integer $a \not\equiv 0 \pmod p$ and positive integer n, we have*

$$\frac{a^{(p-1)n}-1}{pn} \in \mathbb{Z}_p \tag{2.8}$$

and moreover

$$\frac{a^{(p-1)n}-1}{pn} \equiv n^{\delta_{p,2}} \frac{a^{p-1}-1}{p} \pmod{p}, \tag{2.9}$$

where $\delta_{i,j}$ denotes the Kronecker delta which takes 1 or 0 according as $i = j$ or not.

Proof. Let r be the unique integer in $\{1, \ldots, p-1\}$ with $a \equiv r \pmod p$. Then $a^{p-1} \equiv r^{p-1} \equiv 1 \pmod p$ by Fermat's little theorem. Write $a^{p-1} = 1 + pt$ with $t \in \mathbb{Z}_p$. Observe that

$$\begin{aligned} \frac{a^{(p-1)n}-1}{pn} = & \frac{(1+pt)^n-1}{pn} = \frac{1}{pn}\sum_{k=1}^{n}\binom{n}{k}(pt)^k = \sum_{k=1}^{n}\binom{n-1}{k-1}\frac{p^{k-1}}{k}t^k \\ \equiv & t + (n-1)\frac{p}{2}t^2 \equiv t + (n-1)\frac{p}{2}t \equiv n^{\delta_{p,2}}t \pmod p \end{aligned}$$

since $p^{k-2}/k \in \mathbb{Z}_p$ for all $k = 3, 4, \ldots$. This concludes the proof. $\square$

Proof of Theorem 1.2. For the sake of brevity, we just write $u_k = u_k(A, B)$ and $v_k = v_k(A, B)$ for all $k \in \mathbb{N}$. Let α and β be the algebraic integers defined by (2.1).

If $p \mid \Delta$, then by Lemma 2.2 we have

$$\frac{u_{pn} - (\frac{\Delta}{p})u_n}{pn} = \frac{u_{pn}}{pn} \equiv \left(\frac{A}{2}\right)^{pn-1} + \delta_{p,3}\left(\frac{A}{2}\right)^{3n-3}\frac{\Delta}{3} \pmod p.$$

By (2.2),

$$v_n = \left(\frac{A+\sqrt{\Delta}}{2}\right)^n + \left(\frac{A-\sqrt{\Delta}}{2}\right)^n = \frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}A^{n-2k}\Delta^k.$$

When $p \mid \Delta$, we have

$$\frac{v_n - A^n/2^{n-1}}{pn} = \frac{\Delta}{p}\sum_{0<k\leqslant\lfloor n/2\rfloor}\binom{n-1}{2k-1}A^{n-2k}\frac{\Delta^{k-1}}{k2^n} \in \mathbb{Z}_p$$

and similarly

$$\frac{v_{pn} - A^{pn}/2^{pn-1}}{pn} \in \mathbb{Z}_p,$$

hence $(v_{pn} - v_n)/(pn) \in \mathbb{Z}_p$ since

$$\frac{A^{pn}/2^{pn-1} - A^n/2^{n-1}}{pn} = \frac{A^n}{2^{n-1}pn}\left(\left(\frac{A}{2}\right)^{(p-1)n} - 1\right) \in \mathbb{Z}_p$$

by Lemma 2.5.

Below we assume that $p \nmid \Delta$. Note that

$$\begin{aligned} u_{pn} =& \frac{\alpha^{pn} - \beta^{pn}}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\alpha - \beta}\cdot\frac{(\alpha^n)^p - (\beta^n)^p}{\alpha^n - \beta^n} \\ =& u_n u_p(\alpha^n + \beta^n, \alpha^n\beta^n) = u_n u_p(v_n, B^n) \end{aligned}$$

and

$$v_{pn} = (\alpha^n)^p + (\beta^n)^p = v_p(\alpha^n + \beta^n, \alpha^n\beta^n) = v_p(v_n, B^n).$$

By Lemma 2.1,

$$u_p(v_n, B^n) = \sum_{k=0}^{(p-1)/2}\binom{p-1-k}{k}v_n^{p-1-2k}(-B^n)^k$$

and

$$v_p(v_n, B^n) = \sum_{k=0}^{(p-1)/2}\frac{p}{p-k}\binom{p-k}{k}v_n^{p-2k}(-B^n)^k.$$

Now suppose that $p \mid B$. Then $(-B^n)^k/(pn) \in \mathbb{Z}_p$ for all $k \in \mathbb{Z}^+$ since $p^{n-1}/n \in \mathbb{Z}_p$. Note also that $(\frac{\Delta}{p}) = (\frac{A^2}{p}) = 1$. Thus

$$\frac{u_{pn} - (\frac{\Delta}{p})u_n}{pn} - \frac{u_n}{pn}(v_n^{p-1} - 1) \in \mathbb{Z}_p \quad \text{and} \quad \frac{v_{pn} - v_n^p}{pn} \in \mathbb{Z}_p.$$

In view of (2.4),

$$v_n - A^n = -Bn \sum_{0<k\leqslant \lfloor n/2 \rfloor} \binom{n-k-1}{k-1} \frac{(-B)^{k-1}}{k} A^{n-2k}.$$

Since $p \mid B$ and $p^{k-1}/k \in \mathbb{Z}_p$ for all $k \in \mathbb{Z}^+$, we see that $v_n = A^n + pnt$ for some $t \in \mathbb{Z}_p$. By Lemma 2.5, $(A^{(p-1)n} - 1)/(pn) \in \mathbb{Z}_p$. Therefore $(v_n^{p-1} - 1)/(pn) \in \mathbb{Z}_p$ and hence $(u_{pn} - (\frac{\Delta}{p})u_n)/(pn) \in \mathbb{Z}_p$. Note also that

$$\frac{v_{pn} - v_n}{pn} = \frac{v_{pn} - v_n^p}{pn} + v_n \frac{v_n^{p-1} - 1}{pn} \in \mathbb{Z}_p.$$

Below we suppose that $p \nmid B\Delta$.

Case 1. $(\frac{\Delta}{p}) = 1$.

In this case, by Lemma 2.3 we have $\alpha^{p-1} \equiv \beta^{p-1} \equiv 1 \pmod p$ in the ring of algebraic integers. Similar to Lemma 2.5, we have

$$\frac{\alpha^{(p-1)n} - 1}{pn} \equiv \frac{\alpha^{p-1} - 1}{p} \pmod p \quad \text{and} \quad \frac{\beta^{(p-1)n} - 1}{pn} \equiv \frac{\beta^{p-1} - 1}{p} \pmod p.$$

Therefore

$$\begin{aligned}
\frac{u_{pn} - (\frac{\Delta}{p})u_n}{pn} =& \frac{\alpha^{pn} - \beta^{pn} - (\alpha^n - \beta^n)}{pn(\alpha - \beta)} \\
=& \frac{\alpha - \beta}{\Delta} \left(\alpha^n \frac{\alpha^{(p-1)n} - 1}{pn} - \beta^n \frac{\beta^{(p-1)n} - 1}{pn} \right) \\
\equiv& \frac{1}{\alpha - \beta} \cdot \frac{\alpha^n(\alpha^{p-1} - 1) - \beta^n(\beta^{p-1} - 1)}{p} \\
=& \frac{(\alpha^n - \beta^n)(\alpha^{p-1} + \beta^{p-1} - 2) + (\alpha^n + \beta^n)(\alpha^{p-1} - \beta^{p-1})}{2p(\alpha - \beta)} \\
=& \frac{u_n}{2} \cdot \frac{v_{p-1} - 2}{p} + \frac{v_n}{2} \cdot \frac{u_{p-1}}{p} \pmod p
\end{aligned}$$

and

$$\begin{aligned}\frac{v_{pn}-v_n}{pn} &= \frac{\alpha^{pn}+\beta^{pn}-(\alpha^n+\beta^n)}{pn}\\ &= \alpha^n\frac{\alpha^{(p-1)n}-1}{pn}+\beta^n\frac{\beta^{(p-1)n}-1}{pn}\\ &\equiv \frac{\alpha^n(\alpha^{p-1}-1)+\beta^n(\beta^{p-1}-1)}{p}\\ &= \frac{(\alpha^n+\beta^n)(\alpha^{p-1}+\beta^{p-1}-2)+(\alpha^n-\beta^n)(\alpha^{p-1}-\beta^{p-1})}{2p}\\ &= \frac{v_n}{2}\cdot\frac{v_{p-1}-2}{p}+\frac{\Delta u_n}{2}\cdot\frac{u_{p-1}}{p} \pmod p.\end{aligned}$$

Case 2. $(\frac{\Delta}{p})=-1$.

In this case, by Lemma 2.3 we have $\alpha^{p+1}\equiv\beta^{p+1}\equiv B \pmod p$ in the ring of algebraic integers. Thus

$$\begin{aligned}\frac{\alpha^{(p+1)n}-B^n}{pn} &= \frac{1}{pn}\sum_{k=1}^{n}\binom{n}{k}(\alpha^{p+1}-B)^kB^{n-k}\\ &= \frac{\alpha^{p+1}-B}{p}\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{(\alpha^{p+1}-B)^{k-1}}{k}B^{n-k}\\ &\equiv \frac{\alpha^{p+1}-B}{p}B^{n-1} \pmod p.\end{aligned}$$

Similarly,

$$\frac{\beta^{(p+1)n}-B^n}{pn}\equiv\frac{\beta^{p+1}-B}{p}B^{n-1} \pmod p.$$

Therefore

$$\begin{aligned}\frac{u_{pn}-(\frac{\Delta}{p})u_n}{pn} &= \frac{\alpha^{pn}-\beta^{pn}+\alpha^n-\beta^n}{pn(\alpha-\beta)}\\ &= \frac{\alpha-\beta}{pnB^n\Delta}\left((\alpha\beta)^n\alpha^{pn}-(\alpha\beta)^n\beta^{pn}+B^n(\alpha^n-\beta^n)\right)\\ &= \frac{\alpha-\beta}{B^n\Delta}\left(\beta^n\frac{\alpha^{(p+1)n}-B^n}{pn}+\alpha^n\frac{B^n-\beta^{(p+1)n}}{pn}\right)\\ &\equiv \frac{1}{(\alpha-\beta)B}\left(\beta^n\frac{\alpha^{p+1}-B}{p}+\alpha^n\frac{B-\beta^{p+1}}{p}\right)\\ &= \frac{v_n}{2B}\cdot\frac{u_{p+1}}{p}-\frac{u_n}{2B}\cdot\frac{v_{p+1}-2B}{p} \pmod p\end{aligned}$$

and

$$\begin{aligned}\frac{v_{pn}-v_n}{pn}=&\frac{(\alpha\beta)^n}{pnB^n}(\alpha^{pn}+\beta^{pn})-\frac{\alpha^n+\beta^n}{pn}\\=&\frac{1}{B^n}\left(\beta^n\frac{\alpha^{(p+1)n}-B^n}{pn}+\alpha^n\frac{\beta^{(p+1)n}-B^n}{pn}\right)\\\equiv&\frac{1}{B}\left(\beta^n\frac{\alpha^{p+1}-B}{p}+\alpha^n\frac{\beta^{p+1}-B}{p}\right)\\=&\frac{(\alpha^n+\beta^n)(\alpha^{p+1}+\beta^{p+1}-2B)-(\alpha^n-\beta^n)(\alpha^{p+1}-\beta^{p+1})}{2Bp}\\=&\frac{v_n}{2B}\cdot\frac{v_{p+1}-2B}{p}-\frac{\Delta u_n}{2B}\cdot\frac{u_{p+1}}{p}\pmod p.\end{aligned}$$

No matter $(\frac{\Delta}{p})$ is 1 or -1, we always have

$$\begin{aligned}\frac{u_{pn}-(\frac{\Delta}{p})u_n}{pn}\equiv&\left(\frac{\Delta}{p}\right)\frac{u_n}{2B^{(1-(\frac{\Delta}{p}))/2}}\cdot\frac{v_{p-(\frac{\Delta}{p})}-2B^{(1-(\frac{\Delta}{p}))/2}}{p}\\&+\frac{v_n}{2B^{(1-(\frac{\Delta}{p}))/2}}\cdot\frac{u_{p-(\frac{\Delta}{p})}}{p}\pmod p\end{aligned}$$

and

$$\begin{aligned}\frac{v_{pn}-v_n}{pn}\equiv&\frac{v_n}{2B^{(1-(\frac{\Delta}{p}))/2}}\cdot\frac{v_{p-(\frac{\Delta}{p})}-2B^{(1-(\frac{\Delta}{p}))/2}}{p}\\&+\left(\frac{\Delta}{p}\right)\frac{\Delta u_n}{2B^{(1-(\frac{\Delta}{p}))/2}}\cdot\frac{u_{p-(\frac{\Delta}{p})}}{p}\pmod p.\end{aligned}$$

By Lemma 2.4,

$$\frac{v_{p-(\frac{\Delta}{p})}-2B^{(1-(\frac{\Delta}{p}))/2}}{p}\equiv B^{(1-(\frac{\Delta}{p}))/2}\frac{B^{p-1}-1}{p}\pmod p.$$

So we have the desired (1.13) and (1.14).

In view of the above, we have completed the proof of Theorem 1.2. □

3. Proof of Theorem 1.1

We first give an easy lemma on Lucas sequences which is essentially known.

Lemma 3.1. *Let $A,B\in\mathbb{Z}$ and $\Delta=A^2-4B$. For any $k,l\in\mathbb{N}$ with $k\geqslant l$, we have*

$$u_k(A,B)v_l(A,B)-u_l(A,B)v_k(A,B)=2B^lu_{k-l}(A,B)\tag{3.1}$$

and

$$v_k(A,B)v_l(A,B)-\Delta u_k(A,B)u_l(A,B)=2B^lv_{k-l}(A,B).\tag{3.2}$$

Proof. (i) Clearly (3.1) holds for $l = 0$. If $k \in \mathbb{Z}^+$, then

$$\begin{aligned}&u_k(A,B)v_1(A,B) - u_1(A,B)v_k(A,B)\\=&Au_k(A,B) - v_k(A,B) = Au_k(A,B) - (2u_{k+1}(A,B) - Au_k(A,B))\\=&2(Au_k(A,B) - u_{k+1}(A,B)) = 2Bu_{k-1}(A,B).\end{aligned}$$

Now let $k \geqslant l \geqslant 2$ and assume that the identity (3.1) with l replaced by $l-j$ holds for each $j = 1, 2$. Then

$$\begin{aligned}&u_k(A,B)v_l(A,B) - u_l(A,B)v_k(A,B)\\=&u_k(A,B)(Av_{l-1}(A,B) - Bv_{l-2}(A,B)) - (Au_{l-1}(A,B) - Bu_{l-2}(A,B))v_k(A,B)\\=&A(u_k(A,B)v_{l-1}(A,B) - u_{l-1}(A,B)v_k(A,B))\\&- B(u_k(A,B)v_{l-2}(A,B) - u_{l-2}(A,B)v_k(A,B))\\=&A \times 2B^{l-1}u_{k-(l-1)}(A,B) - B \times 2B^{l-2}u_{k-(l-2)}(A,B)\\=&2B^{l-1}(Au_{k-l+1}(A,B) - u_{k-l+2}(A,B)) = 2B^l u_{k-l}(A,B).\end{aligned}$$

This proves (3.1) by induction on l.

(ii) We prove (3.2) in another way. Let α and β be the two roots of the equation $x^2 - Ax + B = 0$. Then $\alpha\beta = B$ and $\Delta = (\alpha - \beta)^2$. Hence

$$\begin{aligned}&v_k(A,B)v_l(A,B) - \Delta u_k(A,B)u_l(A,B)\\=&(\alpha^k + \beta^k)(\alpha^l + \beta^l) - (\alpha^k - \beta^k)(\alpha^l - \beta^l)\\=&2(\alpha^k\beta^l + \alpha^l\beta^k) = 2(\alpha\beta)^l(\alpha^{k-l} + \beta^{k-l}) = 2B^l v_{k-l}(A,B).\end{aligned}$$

This concludes the proof. □

Lemma 3.2. *For any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, we have*

$$\sum_{r=0}^{n-1}\binom{2n}{r}u_{n-r}(m-2,1) = \sum_{k=0}^{n-1}\binom{2k}{k}m^{n-1-k}, \tag{3.3}$$

$$\sum_{r=0}^{n-1}\binom{2n}{r}v_{n-r}(m-2,1) = m^n - \binom{2n}{n}, \tag{3.4}$$

$$\sum_{r=0}^{n-1}\binom{2n-1}{r}u_{n-r}(m-2,1) = \frac{1}{2}\sum_{k=0}^{n-1}\binom{2k}{k}m^{n-1-k} + \frac{m^{n-1}}{2}, \tag{3.5}$$

$$\sum_{r=0}^{n-1}\binom{2n-1}{r}v_{n-r}(m-2,1) = \frac{m-4}{2}\sum_{k=0}^{n-1}\binom{2k}{k}m^{n-1-k} + \frac{m^n}{2}. \tag{3.6}$$

Proof. [25, (2.1)] with $d = 0$ yields (3.3). Also, [25, (2.1)] with $d = 1$ gives

$$\sum_{r=0}^{n}\binom{2n}{r}u_{n+1-r}(m-2,1) = \sum_{k=0}^{n-1}\binom{2k}{k+1}m^{n-1-k} + m^n. \tag{3.7}$$

Hence

$$\begin{aligned}\sum_{r=0}^{n-1}\binom{2n}{r}v_{n-r}(m-2,1)=&\sum_{r=0}^{n-1}\binom{2n}{r}(2u_{n+1-r}(m-2,1)-(m-2)u_{n-r}(m-2,1))\\ =&2\bigg(\sum_{k=0}^{n-1}\binom{2k}{k+1}m^{n-1-k}+m^n-\binom{2n}{n}\bigg)\\ &-(m-2)\sum_{k=0}^{n-1}\binom{2k}{k}m^{n-1-k}\\ =&2m^n-2\binom{2n}{n}+\sum_{k=0}^{n-1}\left(2\binom{2k}{k+1}+(2-m)\binom{2k}{k}\right)m^{n-1-k}\end{aligned}$$

and thus (3.4) follows since

$$\begin{aligned}&\sum_{k=0}^{n-1}\left(2\binom{2k+1}{k+1}-m\binom{2k}{k}\right)m^{n-1-k}\\ =&\sum_{k=0}^{n-1}\binom{2(k+1)}{k+1}m^{n-(k+1)}-\sum_{k=0}^{n-1}\binom{2k}{k}m^{n-k}=\binom{2n}{n}-m^n.\end{aligned}$$

If $n>1$, then by substituting $n-1$ for n in (3.3) and (3.7) we get

$$\sum_{r=0}^{n-1}\binom{2(n-1)}{r}u_{n-1-r}(m-2,1)=\sum_{0\leqslant k<n-1}\binom{2k}{k}m^{n-2-k}$$

and

$$\sum_{s=0}^{n-1}\binom{2(n-1)}{s}u_{(n-1)+1-s}(m-2,1)=\sum_{0\leqslant k<n-1}\binom{2k}{k+1}m^{n-2-k}+m^{n-1};$$

it is easy to see that these two equalities also hold when $n=1$. Adding the last two identities and noting that

$$\binom{2k}{k}+\binom{2k}{k+1}=\binom{2k+1}{k}=\frac{1}{2}\binom{2(k+1)}{k+1}\quad\text{for all } k\in\mathbb{N},$$

we obtain

$$\begin{aligned}&\sum_{0\leqslant k<n-1}\frac{1}{2}\binom{2(k+1)}{k+1}m^{n-2-k}+m^{n-1}\\ =&\sum_{r=0}^{n-1}\binom{2(n-1)}{r}u_{n-1-r}(m-2,1)\\ &+\sum_{0\leqslant r<n-1}\binom{2(n-1)}{r+1}u_{n-1-r}(m-2,1)+u_n(m-2,1)\\ =&\sum_{0\leqslant r<n-1}\binom{2n-1}{r+1}u_{n-1-r}(m-2,1)+u_n(m-2,1)\end{aligned}$$

and hence (3.5) holds. (3.3) minus (3.5) gives

$$\sum_{0<r<n}\binom{2n-1}{r-1}u_{n-r}(m-2,1)=\frac{1}{2}\sum_{k=0}^{n-1}\binom{2k}{k}m^{n-1-k}-\frac{m^{n-1}}{2}. \tag{3.8}$$

By induction,

$$v_k(m-2,1)=(m-2)u_k(m-2,1)-2u_{k-1}(m-2,1)\quad\text{for all } k\in\mathbb{Z}^+.$$

Therefore

$$\begin{aligned}&\sum_{r=0}^{n-1}\binom{2n-1}{r}v_{n-r}(m-2,1)\\ =&(m-2)\sum_{r=0}^{n-1}\binom{2n-1}{r}u_{n-r}(m-2,1)-2\sum_{r=0}^{n-1}\binom{2n-1}{(r+1)-1}u_{n-(r+1)}(m-2,1)\\ =&\frac{m-2}{2}\sum_{k=0}^{n-1}\binom{2k}{k}m^{n-1-k}+\frac{m-2}{2}m^{n-1}-\sum_{k=0}^{n-1}\binom{2k}{k}m^{n-1-k}+m^{n-1}\\ =&\frac{m-4}{2}\sum_{k=0}^{n-1}\binom{2k}{k}m^{n-1-k}+\frac{m^n}{2}\end{aligned}$$

with the aid of (3.5) and (3.8). This proves (3.6). □

Lemma 3.3. *Let $m\in\mathbb{Z}\setminus\{0\}$ and $\Delta=m(m-4)$. And let p be an odd prime with $p\nmid\Delta$. Then*

$$\sum_{k=1}^{p-1}\binom{p}{k}u_k(m-2,1)\equiv\left(\frac{\Delta}{p}\right)\frac{m^{p-1}-1}{2}+\frac{4-m}{4}u_{p-(\frac{\Delta}{p})}(m-2,1)\pmod{p^2} \tag{3.9}$$

and

$$\sum_{k=1}^{p-1}\binom{p}{k}v_k(m-2,1)\equiv\frac{m}{2}(m^{p-1}-1)-\frac{\Delta}{4}\left(\frac{\Delta}{p}\right)u_{p-(\frac{\Delta}{p})}(m-2,1)\pmod{p^2}. \tag{3.10}$$

Proof. Let α and β be the two roots of the equation $x^2-(m-2)x+1=0$. Then

$$\begin{aligned}&2+v_p(m-2,1)+\sum_{k=1}^{p-1}\binom{p}{k}v_k(m-2,1)\\ =&\sum_{k=0}^{p}\binom{p}{k}(\alpha^k+\beta^k)=(1+\alpha)^p+(1+\beta)^p=v_p(m,m),\end{aligned} \tag{3.11}$$

since

$$(1+\alpha)+(1+\beta)=2+m-2=m$$

and

$$(1+\alpha)(1+\beta)=1+(\alpha+\beta)+\alpha\beta=1+(m-2)+1=m.$$

Similarly,

$$\begin{aligned}&u_p(m-2,1)+\sum_{k=1}^{p-1}\binom{p}{k}u_k(m-2,1)\\&=\sum_{k=0}^{p}\binom{p}{k}\frac{\alpha^k-\beta^k}{\alpha-\beta}=\frac{(1+\alpha)^p-(1+\beta)^p}{(1+\alpha)-(1+\beta)}=u_p(m,m).\end{aligned}$$

In view of [17, Lemma 2.4],

$$2u_p(m,m)-\left(\frac{\Delta}{p}\right)m^{p-1}\equiv u_p(m-2,1)+u_{p-(\frac{\Delta}{p})}(m-2,1)\pmod{p^2}.$$

By [20, (3.6)] we have

$$u_p(m-2,1)-\left(\frac{\Delta}{p}\right)\equiv\frac{m-2}{2}u_{p-(\frac{\Delta}{p})}(m-2,1)\pmod{p^2}. \tag{3.12}$$

Therefore,

$$\begin{aligned}2\sum_{k=1}^{p-1}\binom{p}{k}u_k(m-2,1)=&2(u_p(m,m)-u_p(m-2,1))\\\equiv&\left(\frac{\Delta}{p}\right)m^{p-1}-u_p(m-2,1)+u_{p-(\frac{\Delta}{p})}(m-2,1)\\\equiv&\left(\frac{\Delta}{p}\right)(m^{p-1}-1)+\left(\frac{2-m}{2}+1\right)u_{p-(\frac{\Delta}{p})}(m-2,1)\pmod{p^2}.\end{aligned}$$

This proves (3.9).

By the paragraph following [17, (2.10)],

$$\left(\frac{\Delta}{p}\right)\frac{v_p(m,m)}{m}\equiv\left(\frac{\Delta}{p}\right)m^{p-1}-(u_p(m,m)-u_p(m-2,1))\pmod{p^2}.$$

Combining this with the last paragraph, we obtain

$$\begin{aligned}\left(\frac{\Delta}{p}\right)\frac{v_p(m,m)}{m}&\equiv\left(\frac{\Delta}{p}\right)\left(m^{p-1}-\frac{m^{p-1}-1}{2}\right)-\frac{4-m}{4}u_{p-(\frac{\Delta}{p})}(m-2,1)\\&=\left(\frac{\Delta}{p}\right)\frac{m^{p-1}+1}{2}+\frac{m-4}{4}u_{p-(\frac{\Delta}{p})}(m-2,1)\pmod{p^2}.\end{aligned} \tag{3.13}$$

As

$$\begin{aligned}v_p(m-2,1) =&2u_{p+1}(m-2,1)-(m-2)u_p(m-2,1)\\ =&(m-2)u_p(m-2,1)-2u_{p-1}(m-2,1),\end{aligned}$$

we have

$$\begin{aligned}\left(\frac{\Delta}{p}\right)v_p(m-2,1) =&(m-2)u_p(m-2,1)-2u_{p-(\frac{\Delta}{p})}(m-2,1)\\ \equiv&(m-2)\left(\left(\frac{\Delta}{p}\right)+\frac{m-2}{2}u_{p-(\frac{\Delta}{p})}(m-2,1)\right)-2u_{p-(\frac{\Delta}{p})}(m-2,1)\\ =&(m-2)\left(\frac{\Delta}{p}\right)+\frac{\Delta}{2}u_{p-(\frac{\Delta}{p})}(m-2,1) \pmod{p^2}\end{aligned}$$

with the aid of (3.12). Combining this with (3.11) and (3.13), we finally get

$$\begin{aligned}\sum_{k=1}^{p-1}\binom{p}{k}v_k(m-2,1) =&v_p(m,m)-v_p(m-2,1)-2\\ \equiv&\frac{m}{2}(m^{p-1}+1)+\left(\frac{\Delta}{p}\right)\frac{\Delta}{4}u_{p-(\frac{\Delta}{p})}(m-2,1)\\ &-(m-2)-\left(\frac{\Delta}{p}\right)\frac{\Delta}{2}u_{p-(\frac{\Delta}{p})}(m-2,1)-2\\ =&\frac{m}{2}(m^{p-1}-1)-\frac{\Delta}{4}\left(\frac{\Delta}{p}\right)u_{p-(\frac{\Delta}{p})}(m-2,1) \pmod{p^2}.\end{aligned}$$

This proves (3.10).

In view of the above, we have completed the proof. □

Lemma 3.4. *Let $m\in\mathbb{Z}$, and let p be an odd prime p not dividing $\Delta=m(m-4)$. Then, for any $n\in\mathbb{Z}^+$ we have*

$$\begin{aligned}&\sum_{r=0}^{n-1}\binom{2n}{r}u_{p(n-r)}(m-2,1)\\ \equiv&\left(\left(\frac{\Delta}{p}\right)+\frac{m-4}{2}nu_{p-(\frac{\Delta}{p})}(m-2,1)\right)\sum_{k=0}^{n-1}\binom{2k}{k}m^{n-1-k}\\ &+n\binom{2n-1}{n-1}u_{p-(\frac{\Delta}{p})}(m-2,1) \pmod{p^{2+\mathrm{ord}_p(n)}},\end{aligned}\tag{3.14}$$

where $\mathrm{ord}_p(n)=\max\{a\in\mathbb{N}:\ p^a\mid n\}$ *is the p-adic order of n.*

Proof. For simplicity, we write $u_k=u_k(m-2,1)$ and $v_k=v_k(m-2,1)$ for all $k\in\mathbb{N}$. For each $r=1,\ldots,n$, by Theorem 1.2 we have

$$u_{pr}\equiv\left(\frac{\Delta}{p}\right)u_r+\frac{pr}{2}v_r\frac{u_{p-(\frac{\Delta}{p})}}{p} \pmod{p^{2+\mathrm{ord}_p(r)}}\tag{3.15}$$

and

$$v_{pr} \equiv v_r + \frac{pr}{2}\Delta u_r \left(\frac{\Delta}{p}\right) \frac{u_{p-(\frac{\Delta}{p})}}{p} \pmod{p^{2+\mathrm{ord}_p(r)}}. \tag{3.16}$$

Since $r\binom{2n}{r} = 2n\binom{2n-1}{r-1}$ for all $r \in \mathbb{Z}^+$, by (3.15) and (3.16) we get

$$\begin{aligned}
&\sum_{r=0}^{n-1}\binom{2n}{r}u_{pr} = \sum_{0<r<n} 2n\binom{2n-1}{r-1}\frac{u_{pr}}{r}\\
\equiv&\sum_{0<r<n}\binom{2n}{r}\left(\left(\frac{\Delta}{p}\right)u_r + \frac{r}{2}v_r u_{p-(\frac{\Delta}{p})}\right)\\
=&\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\binom{2n}{r}u_r + nu_{p-(\frac{\Delta}{p})}\sum_{0<r<n}\binom{2n-1}{r-1}v_r \pmod{p^{2+\mathrm{ord}_p(n)}}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{r=0}^{n-1}\binom{2n}{r}v_{pr} = v_0 + \sum_{0<r<n} 2n\binom{2n-1}{r-1}\frac{v_{pr}}{r}\\
\equiv&\sum_{r=0}^{n-1}\binom{2n}{r}\left(v_r + \frac{r}{2}\Delta u_r\left(\frac{\Delta}{p}\right)u_{p-(\frac{\Delta}{p})}\right)\\
=&\sum_{r=0}^{n-1}\binom{2n}{r}v_r + n\Delta\left(\frac{\Delta}{p}\right)u_{p-(\frac{\Delta}{p})}\sum_{0<r<n}\binom{2n-1}{r-1}u_r \pmod{p^{2+\mathrm{ord}_p(n)}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{r=0}^{n-1}\binom{2n}{r}(u_{pn}v_{pr} - v_{pn}u_{pr})\\
\equiv&u_{pn}\sum_{r=0}^{n-1}\binom{2n}{r}v_r - v_{pn}\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\binom{2n}{r}u_r\\
&+ u_{pn}\times n\Delta\left(\frac{\Delta}{p}\right)u_{p-(\frac{\Delta}{p})}\sum_{0<r<n}\binom{2n-1}{r-1}u_r\\
&- v_{pn}\times nu_{p-(\frac{\Delta}{p})}\sum_{0<r<n}\binom{2n-1}{r-1}v_r \pmod{p^{2+\mathrm{ord}_p(n)}}.
\end{aligned}$$

In view of (3.15) and (3.16) with $r = n$, from the above we have

$$\begin{aligned}&\sum_{r=0}^{n-1}\binom{2n}{r}(u_{pn}v_{pr}-v_{pn}u_{pr})\\ \equiv&\left(\left(\frac{\Delta}{p}\right)u_n+\frac{n}{2}v_nu_{p-(\frac{\Delta}{p})}\right)\sum_{r=0}^{n-1}\binom{2n}{r}v_r\\ &-\left(v_n+\frac{n}{2}\Delta u_n\left(\frac{\Delta}{p}\right)u_{p-(\frac{\Delta}{p})}\right)\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\binom{2n}{r}u_r\\ &+nu_n\Delta u_{p-(\frac{\Delta}{p})}\sum_{0<r<n}\binom{2n-1}{r-1}u_r\\ &-nv_nu_{p-(\frac{\Delta}{p})}\sum_{0<r<n}\binom{2n-1}{r-1}v_r \pmod{p^{2+\mathrm{ord}_p(n)}}\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{r=0}^{n-1}\binom{2n}{r}(u_{pn}v_{pr}-v_{pn}u_{pr})\\ \equiv&\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\binom{2n}{r}(u_nv_r-v_nu_r)\\ &+\frac{n}{2}u_{p-(\frac{\Delta}{p})}\sum_{r=0}^{n-1}\binom{2n}{r}(v_nv_r-\Delta u_nu_r)\\ &-nu_{p-(\frac{\Delta}{p})}\sum_{0<r<n}\binom{2n-1}{r-1}(v_nv_r-\Delta u_nu_r) \pmod{p^{2+\mathrm{ord}_p(n)}}.\end{aligned}$$

Thus, with the aid of Lemma 3.1, we obtain

$$\begin{aligned}\sum_{r=0}^{n-1}\binom{2n}{r}u_{pn-pr}\equiv&\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\binom{2n}{r}u_{n-r}+\frac{n}{2}u_{p-(\frac{\Delta}{p})}\sum_{r=0}^{n-1}\binom{2n}{r}v_{n-r}\\ &-nu_{p-(\frac{\Delta}{p})}\sum_{r=0}^{n-1}\left(\binom{2n}{r}-\binom{2n-1}{r}\right)v_{n-r}\\ \equiv&nu_{p-(\frac{\Delta}{p})}\left(\sum_{r=0}^{n-1}\binom{2n-1}{r}v_{n-r}-\frac{1}{2}\sum_{r=0}^{n-1}\binom{2n}{r}v_{n-r}\right)\\ &+\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\binom{2n}{r}u_{n-r} \pmod{p^{2+\mathrm{ord}_p(n)}}.\end{aligned}$$

Combining this with Lemma 3.2, we immediately get (3.14). □

Lemma 3.5. *For any $m\in\mathbb{Z}\setminus\{0\}$ and $n\in\mathbb{Z}^+$, we have*

$$\sum_{k=0}^{n-1}\frac{\binom{2k}{k+1}}{m^k}=\frac{\binom{2n-1}{n-1}}{m^{n-1}}-\frac{m}{2}+\frac{m-2}{2}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}}{m^k}. \tag{3.17}$$

Proof. Observe that

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{\binom{2k}{k}+\binom{2k}{k+1}}{m^k}=&\sum_{k=0}^{n-1}\frac{\binom{2k+1}{k}}{m^k}=\frac{\binom{2n-1}{n-1}}{m^{n-1}}+\frac{m}{2}\sum_{0\leqslant k<n-1}\frac{\binom{2k+2}{k+1}}{m^{k+1}}\\ =&\frac{\binom{2n-1}{n-1}}{m^{n-1}}+\frac{m}{2}\bigg(\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}-1\bigg).\end{aligned}$$

So (3.17) follows. □

Proof of Theorem 1.1. For convenience, we set $u_k=u_k(m-2,1)$ and $v_k=v_k(m-2,1)$ for all $k\in\mathbb{N}$. We first handle the case $\Delta=m(m-4)\equiv0\pmod p$. By [18, Theorem 1.1],

$$\frac{1}{pn}\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k}\equiv\frac{\binom{2pn-1}{pn-1}}{4^{pn-1}}+\delta_{p,3}\frac{m-4}{3}\binom{2n/p^{\mathrm{ord}_p(n)}-1}{n/p^{\mathrm{ord}_p(n)}-1}\pmod p.$$

By Lucas' theorem (cf. [6]),

$$\binom{2pn-1}{pn-1}=\frac12\binom{2pn}{pn}\equiv\frac12\binom{2n}{n}\equiv\frac12\binom{2n/p^{\mathrm{ord}_p(n)}}{n/p^{\mathrm{ord}_p(n)}}=\binom{2n/p^{\mathrm{ord}_p(n)}-1}{n/p^{\mathrm{ord}_p(n)}-1}\pmod p.$$

Since $m\equiv4\pmod p$, we have $m\equiv1\pmod p$ if $p=3$. Therefore

$$\begin{aligned}\frac{1}{pn}\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k}\equiv&\frac{\binom{2n-1}{n-1}}{4^{pn-1}}+\delta_{p,3}\frac{m-4}{3}\binom{2n-1}{n-1}\\ \equiv&\frac{\binom{2n-1}{n-1}}{4^{n-1}}+\delta_{p,3}\frac{m(m-4)}{3m^{n-1}}\binom{2n-1}{n-1}\\ \equiv&\frac{\binom{2n-1}{n-1}}{m^{n-1}}\left(1+\delta_{p,3}\frac{\Delta}{3}\right)\pmod p.\end{aligned}$$

By Lemma 2.2,

$$\frac{u_p}{p}\equiv\left(\frac{m-2}{2}\right)^{p-1}+\delta_{p,3}\frac{\Delta}{3}\equiv1+\delta_{p,3}\frac{\Delta}{3}\pmod p.$$

So

$$\frac1n\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k}\equiv\frac{\binom{2n-1}{n-1}}{m^{n-1}}u_p\pmod{p^2}$$

as desired.

Below we assume that $p\nmid\Delta$. By [26, Lemma 2.1], for any $r=1,\ldots,n$ we have

$$\binom{2pn}{pr}\Big/\binom{2n}{r}\equiv 1\pmod{p^{2+\mathrm{ord}_p(r)}}$$

and hence

$$\binom{2pn}{pr}\equiv\binom{2n}{r}=\frac{2n}{r}\binom{2n-1}{r-1}\pmod{p^{2+\mathrm{ord}_p(n)}}.\tag{3.18}$$

By Lucas' theorem, for any $r\in\mathbb{N}$ and $k\in\{1,\ldots,p\}$ we have

$$\binom{p(2n-1)+p-1}{pr+k-1}\equiv\binom{2n-1}{r}\binom{p-1}{k-1}\pmod{p}.$$

So, in view of Lemmas 3.1, 3.2 and 3.4, we have

$$\begin{aligned}
&\sum_{k=0}^{pn-1}\binom{2k}{k}m^{pn-1-k}\\
=&\sum_{s=0}^{pn-1}\binom{2pn}{s}u_{pn-s}=\sum_{r=0}^{n-1}\sum_{k=0}^{p-1}\binom{2pn}{pr+k}u_{pn-(pr+k)}\\
=&\sum_{r=0}^{n-1}\binom{2pn}{pr}u_{p(n-r)}+\sum_{r=0}^{n-1}\sum_{k=1}^{p-1}\frac{2pn}{pr+k}\binom{p(2n-1)+p-1}{pr+k-1}u_{p(n-r)-k}\\
\equiv&\sum_{r=0}^{n-1}\binom{2n}{r}u_{p(n-r)}+\sum_{r=0}^{n-1}\sum_{k=1}^{p-1}\frac{2pn}{k}\binom{2n-1}{r}\binom{p-1}{k-1}u_{p(n-r)-k}\\
\equiv&\left(\left(\frac{\Delta}{p}\right)+\frac{m-4}{2}nu_{p-(\frac{\Delta}{p})}\right)\sum_{k=0}^{n-1}\binom{2k}{k}m^{n-1-k}+n\binom{2n-1}{n-1}u_{p-(\frac{\Delta}{p})}\\
&+n\sum_{r=0}^{n-1}\binom{2n-1}{r}\sum_{k=1}^{p-1}\binom{p}{k}(u_{p(n-r)}v_k-u_kv_{p(n-r)})\pmod{p^{2+\mathrm{ord}_p(n)}}.
\end{aligned}$$

and hence

$$\begin{aligned}
\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k}\equiv&\left(\left(\frac{\Delta}{p}\right)m^{(1-p)n}+\frac{m-4}{2}nu_{p-(\frac{\Delta}{p})}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}+\frac{n}{m^{n-1}}\binom{2n-1}{n-1}u_{p-(\frac{\Delta}{p})}\\
&+\frac{n}{m^{n-1}}\sum_{r=0}^{n-1}\binom{2n-1}{r}\sum_{k=1}^{p-1}\binom{p}{k}(u_{p(n-r)}v_k-u_kv_{p(n-r)})\pmod{p^{2+\mathrm{ord}_p(n)}}.
\end{aligned}$$

By Lemma 2.5,

$$\frac{1}{m^{(p-1)n}}\equiv\frac{1}{1+n(m^{p-1}-1)}\equiv 1-n(m^{p-1}-1)\pmod{p^{2+\mathrm{ord}_p(n)}}.\tag{3.19}$$

Therefore

$$\begin{aligned}
&\frac{1}{n}\bigg(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k}-\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}\bigg)-\frac{\binom{2n-1}{n-1}}{m^{n-1}}u_{p-(\frac{\Delta}{p})}\\
&\equiv\bigg(\left(\frac{\Delta}{p}\right)(1-m^{p-1})+\frac{m-4}{2}u_{p-(\frac{\Delta}{p})}\bigg)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}\\
&\quad+\frac{1}{m^{n-1}}\sum_{r=0}^{n-1}\binom{2n-1}{r}\sum_{k=1}^{p-1}\binom{p}{k}(u_{p(n-r)}v_k-u_kv_{p(n-r)})\pmod{p^2}.
\end{aligned}\tag{3.20}$$

Note that $p\mid\binom{p}{k}$ for all $k=1,\ldots,p-1$. In light of Theorem 1.2 and (3.5)–(3.6),

$$\begin{aligned}
&\sum_{r=0}^{n-1}\binom{2n-1}{r}\sum_{k=1}^{p-1}\binom{p}{k}(u_{p(n-r)}v_k-u_kv_{p(n-r)})\\
&\equiv\sum_{r=0}^{n-1}\binom{2n-1}{r}\sum_{k=1}^{p-1}\binom{p}{k}\left(\left(\frac{\Delta}{p}\right)u_{n-r}v_k-u_kv_{n-r}\right)\\
&=\left(\frac{\Delta}{p}\right)\sum_{k=1}^{p-1}\binom{p}{k}v_k\sum_{r=0}^{n-1}\binom{2n-1}{r}u_{n-r}-\sum_{k=1}^{p-1}\binom{p}{k}u_k\sum_{r=0}^{n-1}\binom{2n-1}{r}v_{n-r}\\
&=\left(\frac{\Delta}{p}\right)\sum_{k=1}^{p-1}\binom{p}{k}v_k\bigg(\frac{1}{2}\sum_{r=0}^{n-1}\binom{2r}{r}m^{n-1-r}+\frac{m^{n-1}}{2}\bigg)\\
&\quad-\sum_{k=1}^{p-1}\binom{p}{k}u_k\bigg(\frac{m-4}{2}\sum_{r=0}^{n-1}\binom{2r}{r}m^{n-1-r}+\frac{m^n}{2}\bigg)\pmod{p^2}.
\end{aligned}$$

Combining this with (3.20) we have reduced (1.5) to the congruence

$$\begin{aligned}
&\bigg(\left(\frac{\Delta}{p}\right)(m^{p-1}-1)+\frac{4-m}{2}u_{p-(\frac{\Delta}{p})}\bigg)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}\\
&\equiv\left(\frac{\Delta}{p}\right)\sum_{k=1}^{p-1}\binom{p}{k}\frac{v_k}{2}\bigg(\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}+1\bigg)\\
&\quad-\sum_{k=1}^{p-1}\binom{p}{k}\frac{u_k}{2}\bigg((m-4)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}+m\bigg)\pmod{p^2}.
\end{aligned}$$

This indeed holds since

$$\left(\frac{\Delta}{p}\right)\sum_{k=1}^{p-1}\binom{p}{k}v_k\equiv m\sum_{k=1}^{p-1}\binom{p}{k}u_k\pmod{p^2}$$

and

$$2\sum_{k=1}^{p-1}\binom{p}{k}u_k\equiv\left(\frac{\Delta}{p}\right)(m^{p-1}-1)+\frac{4-m}{2}u_{p-(\frac{\Delta}{p})}\pmod{p^2}$$

by Lemma 3.3. So we finally obtain (1.5).

Next we deduce (1.6) and (1.7) via (1.5). By Lemma 3.5,

$$\sum_{k=0}^{pn-1}\frac{\binom{2k}{k+1}}{m^k}=\frac{\binom{2pn-1}{pn-1}}{m^{pn-1}}-\frac{m}{2}+\frac{m-2}{2}\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k}. \tag{3.21}$$

Since

$$\binom{2pn-1}{pn-1}=\frac{1}{2}\binom{2pn}{pn}\equiv\frac{1}{2}\binom{2n}{n}=\binom{2n-1}{n-1}\pmod{p^{2+\mathrm{ord}_p(n)}}$$

by (3.18), from (3.19), (3.21) and (1.5) we get

$$\begin{aligned}&\sum_{k=0}^{pn-1}\frac{\binom{2k}{k+1}}{m^k}-\frac{\binom{2n-1}{n-1}}{m^{n-1}}(1-n(m^{p-1}-1))+\frac{m}{2}\\ \equiv&\frac{m-2}{2}\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}+\frac{m-2}{2}\cdot\frac{n}{m^{n-1}}\binom{2n-1}{n-1}u_{p-(\frac{\Delta}{p})}\pmod{p^{2+\mathrm{ord}_p(n)}}.\end{aligned}$$

Combining this with (3.17) we immediately obtain (1.6). As $C_k=\binom{2k}{k}-\binom{2k}{k+1}$ for $k\in\mathbb{N}$, (1.7) follows from (1.5) and (1.6).

The proof of Theorem 1.1 is now complete. □

4. Proof of Theorem 1.3

Lemma 4.1. *Let $p>3$ be a prime. Then, for any integers $n\geqslant k\geqslant 0$, we have*

$$\frac{\binom{pn}{pk}}{\binom{n}{k}}\in 1+p^3nk(n-k)\mathbb{Z}_p. \tag{4.1}$$

Remark 4.1. This is a useful known result, see, e.g., [13].

Lemma 4.2. *Let p be a prime, and let $j\in\mathbb{N}$ and $k\in\{0,\ldots,p-1\}$. Then*

$$\binom{2jp+2k}{jp+k}\equiv\binom{2j}{j}\binom{2k}{k}\pmod{p}. \tag{4.2}$$

Proof. If $2k<p$ then (4.2) follows from Lucas' theorem. If $2k\geqslant p$, then $p\mid\binom{2k}{k}$, and by Lucas' theorem we have

$$\begin{aligned}\binom{2jp+2k}{jp+k}=&\binom{(2j+1)p+(2k-p)}{jp+k}\\ \equiv&\binom{2j+1}{j}\binom{2k-p}{k}=0\equiv\binom{2j}{j}\binom{2k}{k}\pmod{p}.\end{aligned}$$

This concludes the proof. □

Proof of Theorem 1.3. Since

$$\binom{n}{k}^2 nk(n-k) = n^3\binom{n-1}{k-1}\binom{n-1}{n-k-1}$$

for any positive integer $k<n$, by Lemma 4.1 we have

$$\binom{pn}{pk}^2 - \binom{n}{k}^2 \in p^3n^3\mathbb{Z}_p \quad \text{for all } k=0,\ldots,n.$$

Thus

$$\sum_{k=0}^{n}\binom{pn}{pk}^2\binom{2pk}{pk}(-1)^{pk} \equiv \sum_{k=0}^{n}\binom{n}{k}^2\binom{2pk}{pk}(-1)^k \pmod{p^{3+3\mathrm{ord}_p(n)}}.$$

For each $k=1,2,3,\ldots$, clearly

$$\binom{2pk}{pk} - \binom{2k}{k} \in p^3k^3\mathbb{Z}_p$$

by Lemma 4.1, and

$$\binom{n}{k}^2 k^2 = n^2\binom{n-1}{k-1}^2 \equiv 0 \pmod{n^2}.$$

Therefore,

$$\begin{aligned}\sum_{\substack{k=0\\ p\mid k}}^{pn}\binom{pn}{k}^2\binom{2k}{k}(-1)^k =&\sum_{k=0}^{n}\binom{pn}{pk}^2\binom{2pk}{pk}(-1)^{pk}\\ \equiv&\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}(-1)^k = g_n(-1) \pmod{p^{3+2\,\mathrm{ord}_p(n)}}.\end{aligned}$$

In view of Lemma 4.2 and Lucas' theorem,

$$\begin{aligned}&\sum_{\substack{k=0\\ p\nmid k}}^{n}\binom{pn}{k}^2\binom{2k}{k}(-1)^k\\ =&\sum_{r=0}^{n-1}\sum_{k=1}^{p-1}\frac{(pn)^2}{(rp+k)^2}\binom{(n-1)p+p-1}{rp+k-1}^2\binom{2rp+2k}{rp+k}(-1)^{rp+k}\\ \equiv&\sum_{r=0}^{n-1}\sum_{k=1}^{p-1}\frac{(pn)^2}{k^2}\binom{n-1}{r}^2\binom{p-1}{k-1}^2\binom{2r}{r}\binom{2k}{k}(-1)^{r+k}\\ =&(pn)^2\sum_{r=0}^{n-1}\binom{n-1}{r}^2\binom{2r}{r}(-1)^r\sum_{k=1}^{p-1}\frac{(-1)^{(k-1)2+k}}{k^2}\binom{2k}{k} \pmod{p^{3+2\,\mathrm{ord}_p(n)}}.\end{aligned}$$

Note that

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0 \pmod p$$

by [11] or [25], and that

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}\equiv 0 \pmod p$$

by [27].

Combining the above arguments, we get

$$\begin{aligned}g_{pn}(-1)=&\sum_{k=0}^{pn}\binom{pn}{k}^2\binom{2k}{k}(-1)^k\\ \equiv&\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}(-1)^k=g_n(-1) \pmod{p^{3+2\mathrm{ord}_p(n)}}\end{aligned}$$

So (1.15) holds. This concludes our proof of Theorem 1.3. □

5. Some conjectures

In this section we pose some related conjectures. After the initial version of this paper was posted to arXiv in Oct. 2016 with the ID `arXiv:1610.03384`, some conjectures below were studied by others and we mention related partial progress in remarks.

Conjecture 5.1. *Let p be an odd prime. For any integer $m\not\equiv 0 \pmod p$ and positive integer n, we have*

$$\frac{1}{n\binom{2n-1}{n-1}}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k}-\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}\right)\equiv\frac{u_{p-(\frac{\Delta}{p})}(m-2,1)}{m^{n-1}} \pmod{p^2}, \tag{5.1}$$

where $\Delta=m(m-4)$.

Remark 5.1. (5.1) is stronger than (1.5).

Conjecture 5.2. (i) *Let p be an odd prime. For any $n\in\mathbb{Z}^+$ we have*

$$\frac{\sum_{k=0}^{pn-1}\binom{2k}{k}-(\frac{p}{3})\sum_{r=0}^{n-1}\binom{2r}{r}}{n^2\binom{2n-1}{n-1}}\equiv\sum_{k=0}^{p-1}\binom{2k}{k}-\left(\frac{p}{3}\right) \pmod{p^4}. \tag{5.2}$$

(ii) *Let $p>3$ be a prime, and let $m\in\{2,3\}$ and $\Delta=m(m-4)$. Then there is a p-adic integer $c_p^{(m)}$ only depending on p and m such that for any $n\in\mathbb{Z}^+$ we have*

$$\frac{m^{n-1}}{n^2\binom{2n-1}{n-1}}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}}{m^k}-\left(\frac{\Delta}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}}{m^r}\right)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}-\left(\frac{\Delta}{p}\right)+p^3c_p^{(m)}(n-1) \pmod{p^4}. \tag{5.3}$$

Remark 5.2. In 1992 N. Strauss, J. Shallit and D. Zagier [15] proved that

$$\frac{\sum_{k=0}^{n-1}\binom{2k}{k}}{n^2\binom{2n}{n}}\equiv -1\pmod 3\quad\text{for any } n\in\mathbb{Z}^+.$$

In 2011 the author [19] showed that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}\equiv\left(\frac{-1}{p}\right)-p^2E_{p-3}\pmod{p^3}\quad\text{for any odd prime } p,$$

where $E_0, E_1, E_2, \ldots$ are the Euler numbers defined by

$$\frac{2}{e^x+e^{-x}}=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}\quad\left(|x|<\frac{\pi}{2}\right).$$

Y. Zhang and H. Pan [28] recently proved for any prime $p>3$ and $a\in\mathbb{Z}^+$ the congruence

$$\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}}{2^k}\equiv\left(\frac{-1}{p}\right)\sum_{r=0}^{p^{a-1}-1}\frac{\binom{2r}{r}}{2^r}\pmod{p^{2a}},$$

which is weaker than part (ii) of our Conjecture 5.2.

Conjecture 5.3. (i) *Let p be an odd prime. For any integer $m\not\equiv 0\pmod p$ and positive integer n, we have*

$$\frac{1}{pn}\left(\sum_{k=0}^{pn-1}\binom{pn-1}{k}\frac{\binom{2k}{k}}{(-m)^k}-\left(\frac{m(m-4)}{p}\right)\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\binom{2r}{r}}{(-m)^r}\right)\in\mathbb{Z}_p.\qquad(5.4)$$

(ii) *For any prime $p\neq 3$ and $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{p^2n^2}\left(\sum_{k=0}^{pn-1}\binom{pn-1}{k}\frac{\binom{2k}{k}}{(-3)^k}-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\binom{2r}{r}}{(-3)^r}\right)\in\mathbb{Z}_p.\qquad(5.5)$$

Remark 5.3. The author [20] determined $\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}/(-m)^k$ modulo p^2 for any odd prime p and integer $m\not\equiv 0\pmod p$. Q.-H. Hou and Y.-S. Wang [5] made progress on Conjecture 5.3(ii) by proving that (5.5) with n^2 replace by n holds.

Conjecture 5.4. *Let $p>3$ be a prime and let $n\in\mathbb{Z}^+$. Then*

$$\frac{16^n}{n^2\binom{2n}{n}^2}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}^2}{16^k}-\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}^2}{16^r}\right)\equiv -4p^2E_{p-3}\pmod{p^3},\qquad(5.6)$$

$$\frac{27^n}{n^2\binom{2n}{n}\binom{3n}{n}}\left(\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}-\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}\binom{3r}{r}}{27^r}\right)\equiv -\frac{3}{2}p^2B_{p-2}\left(\frac13\right)\pmod{p^3},\qquad(5.7)$$

$$\frac{64^n}{n^2\binom{4n}{2n}\binom{2n}{n}}\left(\sum_{k=0}^{pn-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}-\left(\frac{-2}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{4r}{2r}\binom{2r}{r}}{64^r}\right)\equiv -p^2E_{p-3}\left(\frac14\right)\pmod{p^3},\qquad(5.8)$$

$$\frac{432^n}{n^2\binom{6n}{3n}\binom{3n}{n}}\left(\sum_{k=0}^{pn-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}-\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{6r}{3r}\binom{3r}{r}}{432^r}\right)\equiv -20p^2E_{p-3}\pmod{p^3},\qquad(5.9)$$

where $E_{p-3}(x)$ is the Euler polynomial of degree $p-3$.

Remark 5.4. Let $p>3$ be a prime. Recently, J.-C. Liu [8] proved that for any $n\in\mathbb{Z}^+$ we have

$$\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}^2}{16^r}\pmod{p^2},$$

$$\sum_{k=0}^{pn-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\equiv\left(\frac{p}{3}\right)\sum_{r=0}^{n-1}\frac{\binom{2r}{r}\binom{3r}{r}}{27^r}\pmod{p^2},$$

$$\sum_{k=0}^{pn-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{64^k}\equiv\left(\frac{-2}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{4r}{2r}\binom{2r}{r}}{64^r}\pmod{p^2},$$

$$\sum_{k=0}^{pn-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\equiv\left(\frac{-1}{p}\right)\sum_{r=0}^{n-1}\frac{\binom{6r}{3r}\binom{3r}{r}}{432^r}\pmod{p^2},$$

the case $n=1$ of which is a conjecture of F. Rodriguez-Villegas [14] first confirmed by E. Morterson [9]. In the case $n=1$, (5.6) was established by the author [19], and (5.7)-(5.9) were conjectured by the author and later confirmed by Z.-H. Sun [16]. In the special case $n=p^{a-1}$ with $a\in\mathbb{Z}^+$, Guo and W. Zudilin [4] proved the modulo p^2 version of (5.6), and H.-X. Ni [10] confirmed (5.7)–(5.9) modulo p^2.

Conjecture 5.5. *For any odd prime p and positive integer n, we have*

$$\frac{1}{n^3\binom{2n}{n}^3}\left(\frac1p\sum_{k=0}^{pn-1}(21k+8)\binom{2k}{k}^3-\sum_{r=0}^{n-1}(21r+8)\binom{2r}{r}^3\right)\equiv 0\pmod{p^3}.\quad(5.10)$$

Remark 5.5. (5.10) in the case $n=1$ was proved by the author in [19]. We guess that all those Ramanujan-type supercongruences should have extensions involving $n\in\mathbb{Z}^+$ similar to (5.10).

Conjecture 5.6. *For any prime $p>5$ and $n\in\mathbb{Z}^+$, we have*

$$\frac{g_{pn}(-1)-g_n(-1)}{(pn)^3}\in\mathbb{Z}_p.\quad(5.11)$$

Remark 5.6. This is stronger than Theorem 1.3. We note the new identity

$$g_n(-1)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}^2\binom{2k}{k}(-1)^{n-k},$$

which can be easily proved via the Zeilberger algorithm (cf. [12, pp. 101-119]).

In the same spirit, we have many other conjectures similar to the above ones (see, e.g., Conjectures 12, 22-24, 26-32, 60-63 and 82 of [24]). In our opinion, almost all previous known congruences should have such extensions involving a parameter $n\in\mathbb{Z}^+$.

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