Czechoslovak Math. J. 12 (2022), 817-823.

ON SUMS AND PRODUCTS IN A FIELD

GUANG-LIANG ZHOU AND ZHI-WEI SUN

Abstract. In this paper we study sums and products in a field. Let F be a field with $\mathrm{ch}(F) \neq 2$, where $\mathrm{ch}(F)$ is the characteristic of F. For any integer $k \geqslant 4$, we show that each $x \in F$ can be written as $a_1 + \ldots + a_k$ with $a_1, \ldots, a_k \in F$ and $a_1 \ldots a_k = 1$, and that for any $\alpha \in F \setminus \{0\}$ we can write each $x \in F$ as $a_1 \ldots a_k$ with $a_1, \ldots, a_k \in F$ and $a_1 + \ldots + a_k = \alpha$. We also prove that for any $x \in F$ and $k \in \{2, 3, \ldots\}$ there are $a_1, \ldots, a_{2k} \in F$ such that $a_1 + \ldots + a_{2k} = x = a_1 \ldots a_{2k}$.

1. Introduction

Let $\mathbb{Q}$ be the field of rational numbers. In 1749 Euler showed that any $q \in \mathbb{Q}$ can be written as $abc(a+b+c)$ with $a, b, c \in \mathbb{Q}$; equivalently, we can always write $x = -q \in \mathbb{Q}$ as $abcd$ with $a, b, c, d \in \mathbb{Q}$ and $a+b+c+d = 0$. Actually, Euler noted that the equation $abc(a+b+c) = q$ has the following rational parameter solutions:

$$\begin{aligned} a =& \frac{6qst^3(qt^4-2s^4)^2}{(4qt^4+s^4)(2q^2t^8+10qs^4t^4-s^8)}, \\ b =& \frac{3s^5(4qt^4+s^4)^2}{2t(qt^4-2s^4)(2q^2t^8+10qs^4t^4-s^8)}, \\ c =& \frac{2(2q^2t^8+10qs^4t^4-s^8)}{3s^3t(4qt^4+s^4)}. \end{aligned}$$

The reader may consult N. D. Elkies's talk [E] for a nice exposition of this curious discovery of Euler and its connection to modern topics like $K3$ surfaces. Elkies [E] found that $abcd = x$ with $a+b+c+d = 0$,

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Key words and phrases. Fields, rational functions, restricted sums, restricted products.

2020 *Mathematics Subject Classification.* Primary 11D85; Secondary 11P99, 11T99.

This research was supported by the Natural Science Foundation of China (grant 11971222).

where

$$a=\frac{(s^4+4x)^2}{2s^3(s^4-12x)},\ b=\frac{2x(3s^4-4x)^2}{s^3(s^4+4x)(s^4-12x)},$$
$$c=\frac{s(s^4-12x)}{2(3s^4-4x)},\ d=-\frac{2s^5(s^4-12x)}{(s^4+4x)(3s^4-4x)}.$$

Let F be a field. If $x=a_1\cdots a_k$ with $a_1,\ldots,a_k\in F$ and $a_1+\ldots+a_k=0$, then $a_1\ldots a_k$ is called a *balanced decomposition* of x by A. A. Klyachko and A. N. Vassilyev [KV]. Unaware of Euler's above work in 1749, Klyachko and Vassilyev [KV] showed that if $\mathrm{ch}(F)$ (the characteristic of F) is not two then for each $k=5,6,\ldots$ every $x\in F$ has a balanced decomposition $a_1\ldots a_k$ with $a_1,\ldots,a_k\in F$ and $a_1+\cdots+a_k=0$. When F is a finite field and $k>1$ is an integer, they determined completely when each $x\in F$ can be written as $a_1\ldots a_k$ with $a_1,\ldots,a_k\in F$ and $a_1+\cdots+a_k=0$. In 2016, Klyachko, A. M. Mazhuga and A. N. Ponfilenko [KMP] proved that if $\mathrm{ch}(F)\neq 2,3$ and $|F|\neq 5$ then each $x\in F$ has a balanced decomposition $a_1a_2a_3a_4$ with $a_1,a_2,a_3,a_4\in F$ and $a_1+a_2+a_3+a_4=0$; in fact, for $x\in F\setminus\{1/4,-1/8\}$ they found that $a(x)b(x)c(x)d(x)=x$ and $a(x)+b(x)+c(x)+d(x)=0$, where

$$a(x)=\frac{2(1-4x)^2}{3(1+8x)},\ b(x)=-\frac{1+8x}{6},$$
$$c(x)=-\frac{1+8x}{2(1-4x)},\ d(x)=\frac{18x}{(1-4x)(1+8x)}.$$

This is much simpler than Euler's and Elkies' rational parameter solutions to the equation $abcd=x$ with the restriction $a+b+c+d=0$.

Motivated by the above work, we obtain the following new results.

Theorem 1.1. *Let F be any field with $\mathrm{ch}(F)\neq 2$, and let $\alpha\in F\setminus\{0\}$ and $k\in\{4,5,\ldots\}$. Then each $x\in F$ can be written as $a_1\ldots a_k$ with $a_1,\ldots,a_k\in F$ with $a_1+\ldots+a_k=\alpha$.*

Remark 1.1. This theorem with $\alpha=1$ implies that for any $x\in\mathbb{Q}$ and $k\in\{4,5,\ldots\}$ there are $a_1\ldots a_k\in\mathbb{Q}$ such that $a_1\ldots a_k(a_1+\ldots+a_k)=x$, this extension of Euler's work was asked by D. van der Zypen [Z].

Theorem 1.2. *Let F be a filed with $\mathrm{ch}(F)\neq 2$, and let $k\geqslant 4$ be an integer.*

(i) *If $\mathrm{ch}(F)\neq 3$, then any $x\in F$ can be written as $a_1+\ldots+a_k$ with $a_1,\ldots,a_k\in F$ and $a_1\ldots a_k=-1$.*

(ii) *Any $x\in F$ can be written as $a_1+\ldots+a_k$ with $a_1,\ldots,a_k\in F$ and $a_1\ldots a_k=1$.*

Remark 1.2. It seems that there are no $a, b, c \in \mathbb{Q}$ with $a+b+c = 1 = abc$.

Let F be any field and k be a positive integer. Clearly, any $x \in F$ can be written as $a_1 + \ldots + a_{2k+1}$ with $a_1, \ldots, a_{2k+1} \in F$ and $a_1 \ldots a_{2k+1} = (-1)^k x$; in fact, $x + k(1-1) = x$ and $x \times 1^k \times (-1)^k = (-1)^k x$. If $a^2 = -1$ for some $a \in F$, then any $x \in F$ can be written as $a_1 + \ldots + a_{2k+1}$ with $a_1, \ldots, a_{2k+1} \in F$ and $a_1 \ldots a_{2k+1} = (-1)^{k-1} x$; in fact, $x + (a-a) + (k-1)(1-1) = x$ and

$$x \times a \times (-a) \times 1^{k-1} \times (-1)^{k-1} = (-1)^{k-1} x.$$

Theorem 1.3. *Let F be a field with $\mathrm{ch}(F) \neq 2$, and let $k \geqslant 2$ be an integer. Then, for any $x \in F$ there are $a_1, \ldots, a_{2k} \in F$ such that $a_1 + \ldots + a_{2k} = x = a_1 \ldots a_{2k}$.*

Remark 1.3. If F is a field with $\mathrm{ch}(F) \neq 2$, and $k \geqslant 2$ is an integer, then for any $x \in F$, by Theorem 1.3 there are $a_1, \ldots, a_{2k} \in F$ with $a_1 + \ldots + a_{2k} = -x = a_1 \ldots a_{2k}$, hence

$$(-a_1) + \ldots + (-a_{2k}) = x \quad \text{and} \quad (-a_1) \ldots (-a_{2k}) = -x.$$

Motivated by Theorem 1.3 and Remark 1.3, we propose the following conjecture based on our computation.

Conjecture 1.1. *Let F be any field with $\mathrm{ch}(F) \neq 2, 3$. Then, for any $x \in F$ there are $a, b, c, d \in F$ such that $a + b + c + d - 1 = x = abcd$.*

For example, in any field F with $\mathrm{ch}(F) \neq 2, 3$, we have

$$-2 + \frac{9}{2} - \frac{2}{3} + \frac{1}{6} - 1 = 1 = (-2) \times \frac{9}{2} \times \left(-\frac{2}{3}\right) \times \frac{1}{6}.$$

Motivated by our proof of Theorem 1.2(ii), we obtain the following result.

Theorem 1.4. *Let F be a field with $\mathrm{ch}(F) \neq 2$, and let m be any nonzero integer. Then any $x \in F \setminus \{0\}$ can be written as $a + b + c + d$ with $a, b, c, d \in F$ and $abcd = x^m$.*

2. Proofs of Theorems 1.1-1.4

Proof of Theorem 1.1. We distinguish three cases.

Case 1. $k = 4$.

If each $q \in F$ can be written as $abcd$ with $a, b, c, d \in F$ and $a + b + c + d = 1$, then for any $x \in F$ we can write $x/\alpha^4 = abcd$ with $a, b, c, d \in F$ and $a + b + c + d = 1$ and hence $x = (a\alpha)(b\alpha)(c\alpha)(d\alpha)$ with $a\alpha + b\alpha + c\alpha + d\alpha = \alpha$. So, it suffices to work with $\alpha = 1$.

Let $x \in F$ with $x \neq \pm 1$. Define

$$a(x) = -\frac{(1-x)^2}{2(1+x)},\ b(x) = \frac{1+x}{2},\ c(x) = \frac{1+x}{1-x},\ d(x) = \frac{4x}{x^2-1}.$$

It is easy to verify that

$$a(x)b(x)c(x)d(x) = x \quad \text{and} \quad a(x)+b(x)+c(x)+d(x) = 1.$$

For $x = -1$, we note that

$$-1 = \frac{1}{2} \times \frac{1}{2} \times 2 \times (-2) \quad \text{with } \frac{1}{2} + \frac{1}{2} + 2 - 2 = 1.$$

For $x = 1$, if $\mathrm{ch}(F) = 3$ then

$$1 = 1 \times 1 \times 1 \times 1 \quad \text{with } 1+1+1+1 = 1,$$

if $\mathrm{ch}(F) \neq 3$ then

$$1 = \frac{3}{2} \times \left(-\frac{3}{2}\right) \times \left(-\frac{1}{3}\right) \times \frac{4}{3} \quad \text{with } \frac{3}{2} - \frac{3}{2} - \frac{1}{3} + \frac{4}{3} = 1.$$

This proved Theorem 1.1 for $k = 4$.

Case 2. $k = 5$.

As $\mathrm{ch}(F) \neq 2$, we have $\alpha - \varepsilon \neq 0$ for some $\varepsilon \in \{\pm 1\}$. Let $x \in F$. By Theorem 1.1 for $k = 4$, we can write εx as $abcd$ with $a, b, c, d \in F$ and $a+b+c+d = \alpha - \varepsilon$. Hence $x = abcd\varepsilon$ with $a+b+c+d+\varepsilon = \alpha$. So Theorem 1.1 also holds for $k = 5$.

Case 3. $k \geqslant 6$.

Let $x \in F$. If k is even, then by Theorem 1.1 for $k = 4$ there are $a, b, c, d \in F$ with $a+b+c+d = \alpha$ such that $abcd = (-1)^{(k-4)/2}x$, hence

$$x = abcd \times 1^{(k-4)/2} \times (-1)^{(k-4)/2}$$

with

$$a+b+c+d+\frac{k-4}{2}(1-1) = \alpha.$$

When k is odd, by Theorem 1.1 for $k = 5$ there are $a, b, c, d, e \in F$ with $a+b+c+d+e = \alpha$ such that $abcde = (-1)^{(k-5)/2}x$, hence

$$x = abcde \times 1^{(k-5)/2} \times (-1)^{(k-5)/2}$$

with

$$a+b+c+d+e+\frac{k-5}{2}(1-1) = \alpha.$$

Combining the above, we have completed the proof of Theorem 1.1. □

Proof of Theorem 1.2. For any $m \in \mathbb{Z}$, if each $x \in F$ can be written as $a+b+c+d$ with $a, b, c, d \in F$ and $abcd = m$, then for any $x \in F$ and $k \in \{4, 5, \ldots\}$ there are $a_1, a_2, a_3, a_4 \in F$ such that $a_1 + a_2 + a_3 + a_4 =$

$x-(k-4)$ and $a_1a_2a_3a_4 = m$, hence $a_1+\ldots+a_k = x$ and $a_1 \ldots a_k = m$, where $a_j = 1$ for $4 < j \leqslant k$. Thus it suffices to show parts (i) and (ii) in the case $k = 4$.

(i) For $x \in F \setminus \{-1, -3\}$, we define

$$a(x) = \frac{(x+1)^2}{2(x+3)},\ b(x) = \frac{x+3}{2},\ c(x) = -\frac{x+3}{x+1},\ d(x) = \frac{4}{(x+1)(x+3)},$$

and it is easy to verify that

$$a(x)b(x)c(x)d(x) = -1 \quad \text{and} \quad a(x)+b(x)+c(x)+d(x) = x.$$

Observe that

$$-1 = 2 - 2 - \frac{1}{2} - \frac{1}{2} \quad \text{with } 2 \times (-2) \times \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) = -1.$$

If $\mathrm{ch}(F) \neq 3$, then

$$-3 = \frac{2}{3} - \frac{2}{3} - \frac{3}{2} - \frac{3}{2} \quad \text{with } \frac{2}{3} \times \left(-\frac{2}{3}\right) \times \left(-\frac{3}{2}\right) \times \left(-\frac{3}{2}\right) = -1.$$

This concludes the proof of Theorem 1.2(i).

(ii) If $x \in F \setminus \{0, \pm 1\}$, then it is easy to verify that

$$\frac{1-x^2}{x^2} + \frac{x^2-1}{x^2} + \frac{x}{1-x^2} + \frac{x^3}{x^2-1} = x$$

and

$$\frac{1-x^2}{x^2} \times \frac{x^2-1}{x^2} \times \frac{x}{1-x^2} \times \frac{x^3}{x^2-1} = 1.$$

Note that $0 = 1+1-1-1$ and $1 \times 1 \times (-1) \times (-1) = 1$. If $\mathrm{ch}(F) = 3$, then $1+1+1+1 = 1$ and $1 \times 1 \times 1 \times 1 = 1$, and also $-1-1-1-1 = -1$ and $(-1)(-1)(-1)(-1) = 1$. If $\mathrm{ch}(F) \neq 3$, then

$$\frac{3}{2} - \frac{3}{2} - \frac{1}{3} + \frac{4}{3} = 1 \ \text{ and } \ \frac{3}{2} \times \left(-\frac{3}{2}\right) \times \left(-\frac{1}{3}\right) \times \frac{4}{3} = 1,$$

and also

$$\frac{3}{2} - \frac{3}{2} + \frac{1}{3} - \frac{4}{3} = -1 \ \text{ and } \ \frac{3}{2} \times \left(-\frac{3}{2}\right) \times \frac{1}{3} \times \left(-\frac{4}{3}\right) = 1.$$

So Theorem 1.2(ii) also holds.

In view of the above, the proof of Theorem 1.2 is now complete. □

Proof of Theorem 1.3. We first handle the case $k = 2$. For $x \in F \setminus \{\pm 1\}$, we define

$$a(x) = \frac{(x+1)^2}{2(x-1)},\ b(x) = \frac{x-1}{2},\ c(x) = \frac{1-x}{1+x},\ d(x) = \frac{4x}{1-x^2},$$

and it is easy to verify that

$$a(x)b(x)c(x)d(x) = x = a(x) + b(x) + c(x) + d(x).$$

Clearly,

$$-1 = -\frac{1}{2} - \frac{1}{2} + 2 - 2 \quad \text{with } -\frac{1}{2} \times \left(-\frac{1}{2}\right) \times 2 \times (-2) = -1.$$

If $\mathrm{ch}(F) \neq 3$, then

$$1 = \frac{3}{2} - \frac{3}{2} - \frac{1}{3} + \frac{4}{3} \quad \text{with } \frac{3}{2} \times \left(-\frac{3}{2}\right) \times \left(-\frac{1}{3}\right) \times \frac{4}{3} = 1.$$

When $\mathrm{ch}(F) = 3$, we have

$$1 = 1 + 1 + 1 + 1 \quad \text{with } 1 \times 1 \times 1 \times 1 = 1.$$

This proves Theorem 1.3 for $k = 2$.

Now we consider the case $k \geqslant 3$. By Theorem 1.3 for $k = 2$, there are $a, b, c, d \in F$ such that $a + b + c + d = (-1)^k x = abcd$. Thus

$$(-1)^k a + (-1)^k b + (-1)^k c + (-1)^k d + (k-2)(1-1) = x$$

and

$$(-1)^k a \times (-1)^k b \times (-1)^k c \times (-1)^k d \times 1^{k-2} \times (-1)^{k-2} = abcd(-1)^k = x.$$

This proves Theorem 1.3 for $k \geqslant 3$.

By the above, we have completed the proof of Theorem 1.3. □

Proof of Theorem 1.4. If m is odd and $x \in F \setminus \{0, 1\}$, then, for $n = (3-m)/2$ we have

$$\frac{1-x}{x^n} + \frac{x-1}{x^n} + \frac{x}{1-x} + \frac{x^2}{x-1} = x$$

and

$$\frac{1-x}{x^n} \times \frac{x-1}{x^n} \times \frac{x}{1-x} \times \frac{x^2}{x-1} = x^{3-2n} = x^m.$$

If m is even and $x \in F \setminus \{0, \pm 1\}$, then, for $n = (4-m)/2$ we have

$$\frac{1-x^2}{x^n} + \frac{x^2-1}{x^n} + \frac{x}{1-x^2} + \frac{x^3}{x^2-1} = x$$

and

$$\frac{1-x^2}{x^n} \times \frac{x^2-1}{x^n} \times \frac{x}{1-x^2} \times \frac{x^3}{x^2-1} = x^{4-2n} = x^m.$$

As in the proof of Theorem 1.2(ii), any $x \in \{\pm 1\}$ can be written as $a + b + c + d$ with $a, b, c, d \in F$ and $abcd = 1$. So we have the desired result. □

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(Guang-Liang Zhou) Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

E-mail address: guangliangzhou@126.com

(Zhi-Wei Sun) Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

E-mail address: zwsun@nju.edu.cn