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ON CONGRUENCES INVOLVING APÉRY NUMBERS

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ABSTRACT. In this paper, we mainly establish a congruence for a sum involving Apéry numbers, which was conjectured by Z.-W. Sun. Namely, for any prime $p > 3$ and positive odd integer m, we prove that there is a p-adic integer c_m only depending on m such that

$$\sum_{k=0}^{p-1}(2k+1)^m(-1)^kA_k \equiv c_m p\left(\frac{p}{3}\right) \pmod{p^3},$$

where $A_k = \sum_{j=0}^{k}\binom{k}{j}^2\binom{k+j}{j}^2$ is the Apéry number and $(\frac{\cdot}{p})$ is the Legendre symbol.

1. INTRODUCTION

In 1979, Apéry [2] proved that $\zeta(3)$ is irrational. During his proof, he introduced the numbers

$$A_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2 \quad (n \in \mathbb{N}),$$

which are known as Apéry numbers. These numbers have many interesting congruence properties, and attracted the attention of many researchers. In 1987, F. Beukers [3] conjectured that

$$A_{(p-1)/2} \equiv a_p \pmod{p^2}$$

for any odd prime p, where a_n ($n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$) are given by the power series expansion

$$q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 = \sum_{n=1}^{\infty}a_nq^n \quad (|q| < 1).$$

Beukers' conjecture was finally confirmed by S. Ahlgren and K. Ono [1] in 2000. Recently, J.-C. Liu and C. Wang [7] determined A_{p-1} modulo p^5 for any prime $p > 5$, namely,

$$A_{p-1} \equiv 1 + p^3\left(\frac{4}{3}B_{p-3} - \frac{1}{2}B_{2p-4}\right) + \frac{1}{9}p^4B_{p-3} \pmod{p^5}, \tag{1.1}$$

where $B_0, B_1, B_2, \ldots$ denote the well-known Bernoulli numbers.

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In 2012, Z.-W. Sun [12] discovered some new divisibility results for certain sums involving Apéry numbers. In particular, he proved that

$$\sum_{k=0}^{n-1}(2k+1)A_k \equiv 0 \pmod{n}$$

for all $n \in \mathbb{Z}^+$. There are some further studies along this line (see, e.g., [9, 10, 15, 17]). For instance, V.J.W. Guo and J. Zeng [6] proved that

$$\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k \equiv 0 \pmod{n}$$

for all $n \in \mathbb{Z}^+$ and

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k A_k \equiv p\left(\frac{p}{3}\right) \pmod{p^3} \tag{1.2}$$

for any prime $p > 3$, which were both conjectured by Sun [12]. By [13, Corollary 2.2], we also have

$$\sum_{k=0}^{p-1}(2k+1)^3(-1)^k A_k \equiv -\frac{p}{3}\left(\frac{p}{3}\right) \pmod{p^3} \tag{1.3}$$

for any prime $p > 3$. Motivated by the above work, we obtain the following result conjectured by Sun [13, Remark 2.4].

Theorem 1.1. *For any prime $p > 3$ and positive odd integer m, there is a p-adic integer c_m only depending on m such that*

$$\sum_{k=0}^{p-1}(2k+1)^m(-1)^k A_k \equiv c_m p\left(\frac{p}{3}\right) \pmod{p^3}. \tag{1.4}$$

Sun [13, Remark 2.4] mentioned that he was able to prove (1.4) for any prime $p > 3$ in the cases $m = 5, 7$ with $c_5 = -13/27$ and $c_7 = 5/9$. It is worth noting that there exists a parameter m in (1.4) and this is the difficulty of this conjecture. It is unrealistic to check every value of m, so our approach is to treat it uniformly. Firstly, we look for a series of polynomials such that the left-hand side of (1.4) has a closed form if we replace $(2k+1)^m$ by such polynomials. Then we transform $(2k+1)^m$ into a linear combination of such polynomials. We shall prove Theorem 1.1 in the next section.

Our second theorem is motivated by Sun's recent work on π-series [14].

Theorem 1.2. *For any $n \in \mathbb{Z}^+$, we have*

$$\frac{1}{n(n+1)}\sum_{k=1}^{n}(-1)^{n-k}(9k^2+10k+3)k^2 A_k \in \{1, 3, 5, \ldots\}.$$

Moreover, for any odd prime p, we have

$$\sum_{k=1}^{p}(-1)^k(9k^2+10k+3)k^2A_k \equiv \frac{p}{3}\left(\frac{p}{3}\right)-15p^2 \pmod{p^3}. \tag{1.5}$$

2. Proof of Theorem 1.1

For the sake of convenience, we define

$$P_m(x)=\begin{cases}2x+1 & \text{if } m=1,\\ (x+1)^3x^{m-3}+(34x^3+51x^2+27x+5)(x-1)^{m-3}+x^3(x-2)^{m-3} & \text{if } m\geq 3.\end{cases}$$

Lemma 2.1. *Let $p>3$ be a prime and $m\geq 3$ an integer. Then we have*

$$\sum_{k=0}^{p-1}P_m(k)(-1)^kA_k\equiv 0 \pmod{p^3}.$$

Proof. Define

$$f(k):=(-1)^kA_k \quad (k\in\mathbb{N}).$$

By Zeilberger's algorithm [11], we obtain the recurrence:

$$(k+1)^3f(k)+(2k+3)(17k^2+51k+39)f(k+1)+(k+2)^3f(k+2)=0 \quad (k\in\mathbb{N}).$$

Multiplying both sides of the above formula by k^{m-3}, we get

$$(k+1)^3k^{m-3}f(k)+(2k+3)(17k^2+51k+39)k^{m-3}f(k+1)+(k+2)^3k^{m-3}f(k+2)=0.$$

Then summing both sides from $k=0$ to $n-2$ ($n\geq 2$), and then rearranging the summation term, we arrive at

$$\begin{aligned}\sum_{k=0}^{n}P_m(k)(-1)^kA_k =&(-1)^{n-1}(n-1)^{m-3}n^3A_{n-1}+(-1)^nn^{m-3}(n+1)^3A_n\\ &+(-1)^n(34n^3+51n^2+27n+5)(n-1)^{m-3}A_n.\end{aligned} \tag{2.1}$$

Thus (2.1) with $n=p-1$ yields that

$$\begin{aligned}\sum_{k=0}^{p-1}P_m(k)(-1)^kA_k =&-(p-1)^3(p-2)^{m-3}A_{p-2}+p^3(p-1)^{m-3}A_{p-1}\\ &+(34p^3-51p^2+27p-5)(p-2)^{m-3}A_{p-1}.\end{aligned} \tag{2.2}$$

Recall a classical result of Wolstenholme [16]:

$$H_{p-1}=\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0 \pmod{p^2} \quad\text{and}\quad H_{p-1}^{(2)}=\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0 \pmod{p}.$$

Thus

$$\begin{aligned}
A_{p-2} &= \sum_{k=0}^{p-2}\binom{p-2}{k}^2\binom{p-2+k}{k}^2\\
&= 1+(p-2)^2(p-1)^2+\sum_{k=2}^{p-2}\binom{p-2}{k}^2\binom{p-2+k}{k}^2\\
&\equiv 5-12p+13p^2+\sum_{k=2}^{p-2}\frac{(k+1)^2p^2}{(k-1)^2k^2} \pmod{p^3}\\
&\equiv 5-12p+13p^2+p^2\sum_{k=2}^{p-2}\left(\frac{4}{(k-1)^2}-\frac{4}{k-1}+\frac{1}{k^2}+\frac{4}{k}\right) \pmod{p^3}\\
&\equiv 5-12p+13p^2+p^2\left(5H_{p-1}^{(2)}-\frac{4}{(p-2)^2}-\frac{5}{(p-1)^2}+\frac{4}{p-2}-5\right) \pmod{p^3}\\
&\equiv 5-12p \pmod{p^3}.
\end{aligned}$$

Combining this with (1.1) and (2.2), we get that

$$\begin{aligned}
\sum_{k=0}^{p-1}P_m(k)(-1)^kA_k &\equiv -(p-1)^3(5-12p)(p-2)^{m-3}+(-51p^2+27p-5)(p-2)^{m-3} \pmod{p^3}\\
&\equiv 0 \pmod{p^3}.
\end{aligned}$$

Thus we obtain the desired result. □

Inspired by L. Ghidelli's nice answer [5] in the MathOverflow, we deduce the following result.

Lemma 2.2. *For any positive odd number m, we can write*

$$(2x+1)^m=\sum_{3\le k\le m}a_kP_k(x)+a_1P_1(x),$$

where $a_1,a_3,\ldots,a_m\in\mathbb{Q}$.

Proof. Clearly, the case $m=1$ holds. So we now assume $m\ge 3$. Note that $P_n(x)$ $(n\ge 3)$ is a polynomial of degree n with leading coefficient 36. According to the Euclidean division, there is a unique coefficient $a_m\in\mathbb{Q}$ and a unique polynomial $r_{m-1}(x)\in\mathbb{Q}[x]$ of degree not exceeding $m-1$ such that $(2x+1)^m=a_mP_m(x)+r_{m-1}(x)$. Continue to use the Euclidean division, we get a unique coefficient $a_{m-1}\in\mathbb{Q}$ and a unique polynomial $r_{m-2}(x)\in\mathbb{Q}[x]$ of degree not exceeding $m-2$ such that $r_{m-1}(x)=a_{m-1}P_{m-1}(x)+r_{m-2}(x)$. Repeating the process, we finally get $a_m,a_{m-1},\ldots,a_3\in\mathbb{Q}$ and $r_{m-1}(x),r_{m-2}(x),\ldots,r_2(x)\in\mathbb{Q}(x)$ such that

$$(2x+1)^m=\sum_{k=3}^{m}a_kP_k(x)+r_2(x).$$

It remains to prove that $r_2(x)$ is a multiple of $2x+1$ by a coefficient $a_1 \in \mathbb{Q}$. Changing the variable $z = 2x+1$, we obtain

$$z^m = \sum_{k=3}^{m} a_k P_k\left(\frac{z-1}{2}\right) + r_2\left(\frac{z-1}{2}\right).$$

Noting that $r_2(x)$ is a polynomial of degree not exceeding 2, so it suffices to show that $r_2(\frac{z-1}{2})$ is an odd function with regard to the variable z.

By the definition of $P_k(x)$, we have

$$\begin{aligned}\sum_{k=3}^{m} a_k P_k\left(\frac{z-1}{2}\right) =& \sum_{k=3}^{m} a_k \frac{(z+1)^3}{8}\left(\frac{z-1}{2}\right)^{k-3} \\ &+ \sum_{k=3}^{m} a_k \left(\frac{17z^3}{4}+\frac{3z}{4}\right)\left(\frac{z-3}{2}\right)^{k-3} + \sum_{k=3}^{m} a_k \frac{(z-1)^3}{8}\left(\frac{z-5}{2}\right)^{k-3}.\end{aligned} \tag{2.3}$$

Define

$$g(z) := \sum_{k=3}^{m} a_k \left(\frac{z-3}{2}\right)^{k-3}. \tag{2.4}$$

Since $g(z) \in \mathbb{Q}(z)$ and $\deg g(z) \le m-3$, we have $g(z) = \sum_{k=3}^{m} c_k z^{k-3}$ for some $c_3, c_4, \ldots, c_m \in \mathbb{Q}$. Substituting (2.4) into (2.3), we have

$$\begin{aligned}\sum_{k=3}^{m} a_k P_k\left(\frac{z-1}{2}\right) =& \frac{(z+1)^3}{8} g(z+2) + \left(\frac{17z^3}{4}+\frac{3z}{4}\right) g(z) + \frac{(z-1)^3}{8} g(z-2) \\ =& \sum_{k=3}^{m} c_k \left(\frac{(z+1)^3}{8}(z+2)^{k-3} + \left(\frac{17z^3}{4}+\frac{3z}{4}\right) z^{k-3} + \frac{(z-1)^3}{8}(z-2)^{k-3}\right).\end{aligned}$$

For any integer $3 \le k \le m$, define

$$G_k(z) := \frac{(z+1)^3}{8}(z+2)^{k-3} + \left(\frac{17z^3}{4}+\frac{3z}{4}\right) z^{k-3} + \frac{(z-1)^3}{8}(z-2)^{k-3}.$$

Thus

$$\sum_{k=3}^{m} a_k P_k\left(\frac{z-1}{2}\right) = \sum_{k=3}^{m} c_k G_k(z),$$

and hence

$$z^m = \sum_{k=3}^{m} c_k G_k(z) + r_2\left(\frac{z-1}{2}\right). \tag{2.5}$$

Note that $G_k(z)$ is a polynomial of degree k and it satisfies the equation

$$G_k(-z) = (-1)^k G_k(z).$$

Thus $G_k(z)$ is either an odd function (when k is odd) or an even function (when k is even). We claim that $c_k = 0$ for all even $3 \le k \le m$. If not, we take the biggest even number $3 \le t \le m$ such that $c_t \ne 0$. Consider the coefficient of z^t on both sides of (2.5). Since m is odd, the coefficient of z^t on the left-hand side is 0. Obviously, $r_2\left(\frac{z-1}{2}\right)$ has no z^t term since its degree is not exceeding 2. As we know, a polynomial function is an odd function if and only if it has no even terms. Thus $c_kG_k(z)$ has no z^t term when k is odd. In addition, $c_kG_k(z)$ has no z^t term when $k < t$ is even. To sum up, the coefficient of z^t on the right-hand size is $c_t \ne 0$. That's a contradiction. Therefore $c_k = 0$ for all even $3 \le k \le m$. That is to say, $\sum_{k=3}^m c_kG_k(z)$ is an odd function. Since the function z^m is odd, we obtain that $r_2\left(\frac{z-1}{2}\right) = z^m - \sum_{k=3}^m c_kG_k(z)$ is also an odd function. The proof of Lemma 2.2 is now complete. □

Proof of Theorem 1.1. In light of (1.2), (1.4) holds for $m = 1$ with $c_1 = 1$. We now assume that $m \ge 3$ is an odd integer and $p > 3$ is a prime. By means of Lemma 2.2, we can write

$$(2x+1)^m = \sum_{k=3}^m a_kP_k(x) + a_1P_1(x), \tag{2.6}$$

where a_1 depends only on m. For the sake of clarity, we denote a_1 by c_m. Changing the variable $y = 2x$, we arrive at

$$(y+1)^m = \sum_{k=3}^m a_kP_k\left(\frac{y}{2}\right) + c_m(y+1).$$

Define $Q_k(y) := 2^{k-2}P_k\left(\frac{y}{2}\right)$ for $3 \le k \le m$. By some simple calculations, we find that the coefficient of x^{k-1} in $P_k(x)$ is even and the leading coefficient is 36. Therefore $Q_k(y) \in \mathbb{Z}[y]$ is of degree k with leading coefficient 9. Now we have

$$(y+1)^m = \sum_{k=3}^m b_kQ_k(y) + c_m(y+1), \tag{2.7}$$

where $b_k = a_k/2^{k-2}$ for $3 \le k \le m$. Considering the coefficient of y^m on both sides of (2.7), we get that $b_m = 1/9$. So b_m is a p-adic integer and 3 is the only prime factor of its denominator. Similarly, we can get by induction that $b_{m-1}, b_{m-2}, \ldots, b_3, a_1 = c_m$ are all p-adic integers and their denominators are powers of three. Thus $a_k = 2^{k-2}b_k$ are all p-adic integers for $3 \le k \le m$. This, together with (1.2), (2.6) and Lemma 2.1, gives that

$$\begin{aligned}\sum_{k=0}^{p-1}(2k+1)^m(-1)^kA_k &= \sum_{k=0}^{p-1}\left(\sum_{j=3}^m a_jP_j(k) + c_mP_1(k)\right)(-1)^kA_k \\ &= \sum_{j=3}^m a_j\sum_{k=0}^{p-1}P_j(k)(-1)^kA_k + c_m\sum_{k=0}^{p-1}(2k+1)(-1)^kA_k \\ &\equiv c_mp\left(\frac{p}{3}\right) \pmod{p^3}.\end{aligned}$$

Note that $Q_k(0) = 2^{k-2}P_k(0)$ is even for $3 \le k \le m$. Set $y = 0$ in (2.7), and then we obtain $c_m \equiv 1 \pmod 2$. Therefore, we get that c_m is a p-adic integer with denominator a power of three and numerator an odd integer. As a result, we complete the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

Lemma 3.1. *For any $n \in \mathbb{Z}^+$, we have*

$$\sum_{k=1}^{n}(-1)^{n-k}(9k^2+10k+3)k^2A_k > 0.$$

Proof. Let a_n denote the left-hand side of the above inequality. Then $a_1 = 110$, $a_2 = 17118$, and

$$a_{n+1} - a_{n-1} = (9n^2+28n+22)(n+1)^2A_{n+1} - (9n^2-8n+2)(n-1)^2A_{n-1} > 0$$

for all $n \ge 2$. So $a_n > 0$ for all $n \in \mathbb{Z}^+$. □

Lemma 3.2. *Let $n \in \mathbb{Z}^+$. Then*

$$\sum_{m=0}^{n-1}\binom{2m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{m+k}{k}\binom{n-1}{m+k}\binom{n+m+k}{m+k} \equiv (-1)^{n-1}n \pmod 3.$$

Proof. Let S_n denote the left-hand side of the above congruence. It suffices to prove $S_{3n} \equiv 0 \pmod 3$ and $S_{3n+1} \equiv S_{3n+2} \equiv (-1)^n \pmod 3$.

For any $m \in \mathbb{N}$ and $k \in \mathbb{Z}^+$, if $m/k \equiv 0 \pmod 3$, then we get

$$\binom{m}{k} = \frac{m}{k}\binom{m-1}{k-1} \equiv 0 \pmod 3.$$

By this little trick and direct calculations, we arrive at

$$\begin{aligned}
S_{3n} &= \sum_{m=0}^{3n-1}\binom{2m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{m+k}{k}\binom{3n-1}{m+k}\binom{3n+m+k}{m+k} \\
&\equiv \sum_{m=0}^{n-1}\binom{6m}{3m}\sum_{k=0}^{3m}\binom{3m}{k}\binom{3m+k}{k}\binom{3n-1}{3m+k}\binom{3n+3m+k}{3m+k} + \sum_{m=0}^{n-1}\binom{6m+2}{3m+1}\sigma_m \pmod 3,
\end{aligned}$$

where

$$\begin{aligned}
\sigma_m &= \sum_{k=0}^{3m+1}\binom{3m+1}{k}\binom{3m+k+1}{k}\binom{3n-1}{3m+k+1}\binom{3n+3m+k+1}{3m+k+1} \\
&\equiv \sum_{k=0}^{m}\binom{3m+1}{3k}\binom{3m+3k+1}{3k}\binom{3n-1}{3m+3k+1}\binom{3n+3m+3k+1}{3m+3k+1} \\
&\quad + \sum_{k=0}^{m}\binom{3m+1}{3k+1}\binom{3m+3k+2}{3k+1}\binom{3n-1}{3m+3k+2}\binom{3n+3m+3k+2}{3m+3k+2} \pmod 3.
\end{aligned}$$

By the well-known Lucas congruence [8], we have

$$\binom{3a+s}{3b+t} \equiv \binom{a}{b}\binom{s}{t} \pmod{3}$$

for any $a, b \in \mathbb{N}$ and $s, t \in \{0, 1, 2\}$. Applying this congruence, we deduce from the above that

$$\begin{aligned}S_{3n} \equiv &\sum_{m=0}^{n-1}\binom{2m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{m+k}{k}\binom{n-1}{m+k}\binom{2}{0}\binom{n+m+k}{m+k}\\ &+\sum_{m=0}^{n-1}\binom{2m}{m}\binom{2}{1}\sum_{k=0}^{m}\binom{m}{k}\binom{1}{0}\binom{m+k}{k}\binom{1}{0}\binom{n-1}{m+k}\binom{2}{1}\binom{n+m+k}{m+k}\binom{1}{1}\\ &+\sum_{m=0}^{n-1}\binom{2m}{m}\binom{2}{1}\sum_{k=0}^{m}\binom{m}{k}\binom{1}{1}\binom{m+k}{k}\binom{2}{1}\binom{n-1}{m+k}\binom{2}{2}\binom{n+m+k}{m+k}\binom{2}{2} \pmod{3}\\ \equiv &S_n + 4S_n + 4S_n = 9S_n \equiv 0 \pmod{3}.\end{aligned}$$

Similarly, dealing with S_{3n+1} modulo 3, we obtain

$$S_{3n+1} \equiv \sum_{m=0}^{n}\binom{2m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{m+k}{k}\binom{n}{m+k}\binom{n+m+k}{m+k} \pmod{3}.$$

Define

$$B_n := \sum_{m=0}^{n}\binom{2m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{m+k}{k}\binom{n}{m+k}\binom{n+m+k}{m+k}.$$

Using the same method, we obtain $B_{3n} \equiv B_n \pmod{3}$, $B_{3n+1} \equiv -B_n \pmod{3}$ and $B_{3n+2} \equiv B_n \pmod{3}$. Note that $B_1 = 5 \equiv -1 \pmod{3}$, $B_2 = 73 \equiv 1 \pmod{3}$ and $B_3 = 1445 \equiv -1 \pmod{3}$. By induction, we can get $B_n \equiv (-1)^n \pmod{3}$ easily. That is to say, $S_{3n+1} \equiv (-1)^n \pmod{3}$. In a similar way, we can prove $S_{3n+2} \equiv (-1)^n \pmod{3}$. This completes the proof of Lemma 3.2. □

Recall that Gessel [4] investigated some congruence properties of Apéry numbers. Namely, he proved that $A_n \equiv (-1)^n \pmod{3}$, $A_{2n} \equiv 1 \pmod{8}$ and $A_{2n+1} \equiv 5 \pmod{8}$ for any $n \in \mathbb{N}$. Thus we immediately obtain $A_n + A_{n-1} \equiv 0 \pmod{3}$ and $A_n - A_{n-1} \equiv 4 \pmod{8}$ for any $n \in \mathbb{Z}^+$. Our next lemma gives a further refinement of these results.

Lemma 3.3. *For any $n \in \mathbb{Z}^+$ with $3 \nmid n$, we have $A_n + A_{n-1} \equiv (-1)^n 3n \pmod{9}$. Similarly, for any $n \in \mathbb{Z}^+$ with $2 \nmid n$, we have $A_n - A_{n-1} \equiv 4(-1)^{(n-1)/2} \pmod{16}$.*

Proof. These two results can be proved in a similar way even though the second result is a bit more difficult. Here we only prove the first result in detail. The Zeilberger algorithm gives the recurrence relation of A_n as follows:

$$(n+1)^3 A_n - (3+2n)(39+51n+17n^2)A_{n+1} + (n+2)^3 A_{n+2} = 0 \quad (n \in \mathbb{N}). \tag{3.1}$$

Note that $(n+1)^3$, $-(3+2n)(39+51n+17n^2)$ and $(n+2)^3$ modulo 9 are periodic with period 3. Let $k \in \mathbb{N}$. Considering the above equation modulo 9, we find that

$$\begin{cases} 4A_{3k} + A_{3k+1} \equiv 0 \pmod 9, \\ -A_{3k+1} - 4A_{3k+2} \equiv 0 \pmod 9, \\ A_{3k} - A_{3k+2} \equiv 0 \pmod 9. \end{cases}$$

Thus using Gessel's result, we obtain $A_{3k+1} + A_{3k} \equiv -3A_{3k} \equiv -3(-1)^{3k} = (-1)^{3k+1}3(3k+1) \pmod 9$, and $A_{3k+2} + A_{3k+1} \equiv -3A_{3k+2} \equiv -3(-1)^{3k+2} = (-1)^{3k+2}3(3k+2) \pmod 9$. These conclude the proof.

□

Proof of Theorem 1.2. Let $n \in \mathbb{Z}^+$. Denote $\sum_{k=1}^n (-1)^{n-k}(9k^2+10k+3)k^2 A_k$ by T_n. First of all, we show that T_n/n is an integer, which is odd if $2 \mid n$. It is routine to verify that

$$(9k^2+10k+3)k^2 = \frac{1}{4}P_4(k) + \frac{11}{72}P_3(k) + \frac{1}{3}P_1(k),$$

where $P_m(x)$ is defined at the beginning of Section 2. Hence (2.1) yields

$$T_n = \frac{(-1)^n}{3}\sum_{k=0}^{n-1}(-1)^k(2k+1)A_k + \frac{1}{72}n^2(630n^2+745n+216)A_n - \frac{1}{72}n^3(18n-7)A_{n-1}. \tag{3.2}$$

Recall that V.J.W. Guo and J. Zeng [6] obtained the following amazing combinatorial identity:

$$\frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2k+1)A_k = (-1)^{n-1}\sum_{m=0}^{n-1}\binom{2m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{m+k}{k}\binom{n-1}{m+k}\binom{n+m+k}{m+k}. \tag{3.3}$$

With the help of the above identity, we obtain

$$\begin{aligned} \frac{T_n}{n} = &-\frac{1}{3}\sum_{m=0}^{n-1}\binom{2m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{m+k}{k}\binom{n-1}{m+k}\binom{n+m+k}{m+k} \\ &+ \frac{1}{72}n(630n^2+745n+216)A_n - \frac{1}{72}n^2(18n-7)A_{n-1}. \end{aligned}$$

In order to prove $n \mid T_n$, we need to show that the right-hand side of the above equation is a 3-adic integer and also a 2-adic integer. Firstly, we show that T_n/n is a 3-adic integer. According to Lemma 3.2, it suffices to show that

$$\frac{1}{24}n^2(630n^2+745n+216)A_n - \frac{1}{24}n^3(18n-7)A_{n-1} \equiv (-1)^{n-1}n \pmod 3.$$

By simple calculations, we get

$$\frac{1}{24}n^2(630n^2+745n+216)A_n - \frac{1}{24}n^3(18n-7)A_{n-1} \equiv \frac{2}{3}n^2(A_n + A_{n-1}) \pmod 3.$$

Clearly, $2n^2(A_n+A_{n-1})/3 \equiv (-1)^{n-1}n \equiv 0 \pmod 3$ when $n \equiv 0 \pmod 3$. When $3 \nmid n$, we obtain by Lemma 3.3 that

$$\frac{2}{3}n^2(A_n+A_{n-1}) \equiv \frac{2}{3}n^2(-1)^n 3n \equiv (-1)^{n-1}n^3 \equiv (-1)^{n-1}n \pmod 3,$$

where the last congruence follows from Fermat's little theorem. Secondly, we show that T_n/n is a 2-adic integer. In fact, we just need to prove

$$\frac{1}{36}n(630n^2+745n+216)A_n - \frac{1}{36}n^2(18n-7)A_{n-1} \equiv 0 \pmod 2.$$

It is easy to check that

$$\frac{n}{36}(630n^2+745n+216)A_n - \frac{n^2}{36}(18n-7)A_{n-1}$$

is congruent to

$$h(n) := \frac{n^2}{4}(A_n-A_{n-1}) + \frac{n^3}{6}(A_n-A_{n-1}) + n^3A_{n-1} + 2nA_n$$

modulo 4. If $2 \mid n$, then $h(n) \equiv 0 \pmod 4$ by means of Gessel's result $A_n - A_{n-1} \equiv 4 \pmod 8$. If $2 \nmid n$, by Lemma 3.3 and Gessel's result $A_n \equiv 1 \pmod 4$, we have

$$h(n) \equiv \frac{n^2}{4} \times 4(-1)^{(n-1)/2} + \frac{n^3}{6} \times 4(-1)^{(n-1)/2} + n^3 + 2n \equiv 2 \pmod 4.$$

So far we have got that $n \mid T_n$. Now we consider the case $2 \mid n$. Notice that $\binom{2m}{m} = 2\binom{2m-1}{m-1} \equiv 0 \pmod 2$ for any $m \in \mathbb{Z}^+$. Thus

$$\sum_{m=0}^{n-1}\binom{2m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{m+k}{k}\binom{n-1}{m+k}\binom{n+m+k}{m+k} \equiv 1 \pmod 2,$$

and hence $T_n/n \equiv 1 \pmod 2$.

We now show that $T_n/(n+1)$ is also an integer, which is odd if $2 \nmid n$. In view of (3.1), we have

$$n^3A_{n-1} = (2n+1)(17n^2+17n+5)A_n - (n+1)^3A_{n+1}.$$

Substituting the above expression into (3.2), we deduce

$$T_n = (-1)^n\frac{1}{3}\sum_{k=0}^{n}(-1)^k(2k+1)A_k + \frac{1}{72}(n+1)^3(18n+11)A_n + \frac{1}{72}(n+1)^3(18n-7)A_{n+1}.$$

Then, using V.J.W. Guo and J. Zeng's amazing identity (3.3), we obtain

$$\begin{aligned}\frac{T_n}{n+1} =& \frac{1}{3}\sum_{m=0}^{n}\binom{2m}{m}\sum_{k=0}^{m}\binom{m}{k}\binom{m+k}{k}\binom{n-1}{m+k}\binom{n+m+k}{m+k}\\ &+ \frac{1}{72}(n+1)^2(18n+11)A_n + \frac{1}{72}(n+1)^2(18n-7)A_{n+1}.\end{aligned}$$

As a result, $(n+1) \mid T_n$ if the right-hand size of the above expression is a 3-adic integer and also a 2-adic integer, which follows in a similar way as done in the case $n \mid T_n$. Moreover, $T_n/(n+1)$ is odd if $2 \nmid n$.

Since $(n, n+1) = 1$, we immediately deduce $n(n+1) \mid T_n$. When n is odd, then $T_n/(n+1)$ is odd, and hence $T_n/(n(n+1))$ is odd. When n is even, then both T_n/n and $n+1$ are odd, and hence $T_n/(n(n+1))$ is odd. As a result, $T_n/(n(n+1))$ is odd. In view of Lemma 3.1, we conclude that $T_n/(n(n+1))$ is a positive odd integer.

Let p be an odd prime. We can verify (1.5) for $p = 3$ easily. So we assume $p > 3$ below. With the aid of Lemma 2.1 and (1.2), we obtain

$$\begin{aligned}\sum_{k=1}^{p-1}(-1)^k(9k^2+10k+3)k^2A_k =& \sum_{k=0}^{p-1}(-1)^k(9k^2+10k+3)k^2A_k \\ =& \sum_{k=0}^{p-1}\left(\frac{1}{4}P_4(k)+\frac{11}{72}P_3(k)+\frac{1}{3}P_1(k)\right)(-1)^kA_k \\ \equiv& \frac{p}{3}\left(\frac{p}{3}\right) \pmod{p^3}.\end{aligned}$$

Noting that

$$A_p = \sum_{k=0}^{p}\binom{p}{k}^2\binom{p+k}{k}^2 \equiv 1+\binom{2p}{p}^2 = 1+4\binom{2p-1}{p-1}^2 \equiv 5 \pmod{p},$$

we obtain from the above the desired result (1.5). Hence we complete the proof of Theorem 1.2. □

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