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A PARAMETRIC CONGRUENCE MOTIVATED BY ORR'S IDENTITY

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Abstract. For any $m,n\in\mathbb{N}=\{0,1,2\ldots\}$, the truncated hypergeometric series ${}_{m+1}F_m$ is defined by

$$ {}_{m+1}F_m\left[\begin{matrix}x_0 & x_1 & \ldots & x_m\\ & y_1 & \ldots & y_m\end{matrix}\bigg| z\right]_n=\sum_{k=0}^n\frac{(x_0)_k(x_1)_k\cdots(x_m)_k}{(y_1)_k\cdots(y_m)_k}\cdot\frac{z^k}{k!}, $$

where $(x)_k=x(x+1)\cdots(x+k-1)$ is the Pochhammer symbol. Let p be an odd prime. For $\alpha,z\in\mathbb{Z}_p$ with $\langle-\alpha\rangle_p\equiv0\pmod 2$, where $\langle x\rangle_p$ denotes the least nonnegative residue of x modulo p for any $x\in\mathbb{Z}_p$, we mainly prove the following congruence motivated by Orr's identity:

$$ {}_2F_1\left[\begin{matrix}\frac12\alpha & \frac32-\frac12\alpha\\ & 1\end{matrix}\bigg| z\right]_{p-1}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12-\frac12\alpha\\ & 1\end{matrix}\bigg| z\right]_{p-1}\equiv{}_3F_2\left[\begin{matrix}\alpha & 2-\alpha & \frac12\\ & 1 & 1\end{matrix}\bigg| z\right]_{p-1}\pmod{p^2}. $$

As a corollary, for any positive integer b with $p\equiv\pm1\pmod b$ and $\langle-1/b\rangle_p\equiv0\pmod 2$, we deduce that

$$ \sum_{k=0}^{p-1}(b^2k+b-1)\frac{\binom{2k}{k}}{4^k}\binom{-1/b}{k}\binom{1/b-1}{k}\equiv0\pmod{p^2}. $$

This confirms a conjectural congruence of the second author.

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1. Introduction

For any $m,n\in\mathbb{N}=\{0,1,2\ldots\}$, the truncated hypergeometric series ${}_{m+1}F_m$ is defined by

$$ {}_{m+1}F_m\left[\begin{matrix}x_0 & x_1 & \ldots & x_m\\ & y_1 & \ldots & y_m\end{matrix}\bigg| z\right]_n=\sum_{k=0}^n\frac{(x_0)_k(x_1)_k\cdots(x_m)_k}{(y_1)_k\cdots(y_m)_k}\cdot\frac{z^k}{k!}, $$

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where $(x)_k = x(x+1)\cdots(x+k-1)$ is the Pochhammer symbol. Clearly, the above truncated hypergeometric series is the truncation of the original hypergeometric series after the z^n term.

Summation and transformation formulas for hypergeometric series play an important role in the study of the congruence properties of truncated hypergeometric series (see, e.g., [2–4,6, 8–10,12,14,16,17]). Recall the well-known Clausen's identity (cf. [1, p. 116])

$$\left({}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12\beta\\ & \frac12+\frac12(\alpha+\beta)\end{matrix}\middle| z\right]\right)^2 = {}_3F_2\left[\begin{matrix}\alpha & \beta & \frac12(\alpha+\beta)\\ & \alpha+\beta & \frac12+\frac12(\alpha+\beta)\end{matrix}\middle| z\right]. \tag{1.1}$$

Letting $\beta = 1-\alpha$ in (1.1) we obtain

$$\left({}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12-\frac12\alpha\\ & 1\end{matrix}\middle| z\right]\right)^2 = {}_3F_2\left[\begin{matrix}\alpha & 1-\alpha & \frac12\\ & 1 & 1\end{matrix}\middle| z\right]. \tag{1.2}$$

Let p be an odd prime and let $\mathbb{Z}_p$ denote the ring of all p-adic integers. For any $x \in \mathbb{Z}_p$ let $\langle x\rangle_p$ denote the least nonnegative residue of x modulo p. Mao and Pan [4] proved the following parametric congruence with respect to the identity (1.2):

$$\left({}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12-\frac12\alpha\\ & 1\end{matrix}\middle| z\right]_{p-1}\right)^2 \equiv {}_3F_2\left[\begin{matrix}\alpha & 1-\alpha & \frac12\\ & 1 & 1\end{matrix}\middle| z\right]_{p-1} \pmod{p^2}, \tag{1.3}$$

where $\alpha, z \in \mathbb{Z}_p$ and $\langle -\alpha\rangle_p$ is even.

When Orr discussed the differential equation satisfied by the product of two hypergeometric series, he discovered the following formula (cf. [1, p. 180]) similar to (1.1):

$$ {}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12\beta\\ & \frac12(\alpha+\beta)-\frac12\end{matrix}\middle| z\right] {}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12\beta-1\\ & \frac12(\alpha+\beta)-\frac12\end{matrix}\middle| z\right] = {}_3F_2\left[\begin{matrix}\alpha & \beta-1 & \frac12(\alpha+\beta)-1\\ & \alpha+\beta-2 & \frac12(\alpha+\beta)-\frac12\end{matrix}\middle| z\right]. \tag{1.4}$$

Putting $\beta = 3-\alpha$ in (1.4) we have

$$ {}_2F_1\left[\begin{matrix}\frac12\alpha & \frac32-\frac12\alpha\\ & 1\end{matrix}\middle| z\right] {}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12-\frac12\alpha\\ & 1\end{matrix}\middle| z\right] = {}_3F_2\left[\begin{matrix}\alpha & 2-\alpha & \frac12\\ & 1 & 1\end{matrix}\middle| z\right]. \tag{1.5}$$

The main purpose of this paper is to establish a parametric congruence corresponding to (1.5).

Theorem 1.1. *Let p be an odd prime. Then, for $\alpha, z \in \mathbb{Z}_p$ with $\langle -\alpha\rangle_p \equiv 0 \pmod 2$ we have*

$$ {}_2F_1\left[\begin{matrix}\frac12\alpha & \frac32-\frac12\alpha\\ & 1\end{matrix}\middle| z\right]_{p-1} {}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12-\frac12\alpha\\ & 1\end{matrix}\middle| z\right]_{p-1} \equiv {}_3F_2\left[\begin{matrix}\alpha & 2-\alpha & \frac12\\ & 1 & 1\end{matrix}\middle| z\right]_{p-1} \pmod{p^2}. \tag{1.6}$$

Recently, the second author [11, Conjecture 19] posed the following conjecture.

Conjecture 1.1. *Let $b, n \in \mathbb{Z}^+$ and let p be a prime with $p \equiv \pm 1 \pmod b$ and $\langle -1/b\rangle_p \equiv 0 \pmod 2$. Then*

$$\frac{1}{n^2\binom{-1/b}{n}\binom{1/b-1}{n}} \sum_{k=0}^{pn-1} (b^2k+b-1)\frac{\binom{2k}{k}}{4^k}\binom{-1/b}{k}\binom{1/b-1}{k} \equiv 0 \pmod{p^2}. \tag{1.7}$$

Note that Conjecture 1.1 with $n = 1$ and $b \in \{2, 3, 4, 6\}$ was first stated by the second author in [8, Conjecture 5.9]. Our following result confirms Conjecture 1.1 for $n = 1$.

Corollary 1.1. *Let $b \in \mathbb{Z}^+$, and let p be a prime with $p \equiv \pm 1 \pmod b$ and $\langle -1/b\rangle_p \equiv 0 \pmod 2$. Then*

$$\sum_{k=0}^{p-1}(b^2k+b-1)\frac{\binom{2k}{k}}{4^k}\binom{-1/b}{k}\binom{1/b-1}{k} \equiv 0 \pmod{p^2}. \tag{1.8}$$

The relation between Theorem 1.1 and Corollary 1.1 becomes more evident when we write (1.7) as a difference of truncated hypergeometric series. Note that

$$(x)_k/(1)_k = \binom{-x}{k}(-1)^k,\ (1/2)_k/(1)_k = \binom{2k}{k}/4^k \text{ and } \frac{(1+\frac{1}{b})_k}{(\frac{1}{b})_k} = bk+1.$$

Thus we have

$$\begin{aligned}&\sum_{k=0}^{p-1}(b^2k+b-1)\frac{\binom{2k}{k}}{4^k}\binom{-1/b}{k}\binom{1/b-1}{k}\\ &= b\,{}_3F_2\left[\begin{matrix}1+\frac{1}{b} & 1-\frac{1}{b} & \frac{1}{2}\\ & 1 & 1\end{matrix}\middle|\,1\right]_{p-1} - {}_3F_2\left[\begin{matrix}\frac{1}{b} & 1-\frac{1}{b} & \frac{1}{2}\\ & 1 & 1\end{matrix}\middle|\,1\right]_{p-1}.\end{aligned} \tag{1.9}$$

In view of (1.3) and Theorem 1.1, in order to show Corollary 1.1, it suffices to evaluate some truncated ${}_2F_1$ series modulo p^2.

Taking $b = 2, 3, 4, 6$ in Corollary 1.1 and noting that

$$\binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k},\ \binom{-1/3}{k}\binom{-2/3}{k} = \frac{\binom{2k}{k}\binom{3k}{k}}{27^k},$$
$$\binom{-1/4}{k}\binom{-3/4}{k} = \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k},\ \binom{-1/6}{k}\binom{-5/6}{k} = \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k},$$

we obtain the following congruences which confirm [8, Conjecture 5.9] with $a = 1$.

Corollary 1.2. *Let p be an odd prime. If $p \equiv 1 \pmod 3$, then*

$$\sum_{k=0}^{p-1}\frac{9k+2}{108^k}\binom{2k}{k}^2\binom{3k}{k} \equiv 0 \pmod{p^2}. \tag{1.10}$$

If $p \equiv 1, 3 \pmod 8$, then

$$\sum_{k=0}^{p-1}\frac{16k+3}{256^k}\binom{2k}{k}^2\binom{4k}{2k} \equiv 0 \pmod{p^{2+\delta_{p,3}}}, \tag{1.11}$$

where $\delta_{p,q}$ denotes the Kronecker delta symbol which takes 1 or 0 according as $p=q$ or not. If $p\equiv 1 \pmod 4$, then

$$\sum_{k=0}^{p-1}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv 0 \pmod{p^2} \tag{1.12}$$

and

$$\sum_{k=0}^{p-1}\frac{36k+5}{12^{3k}}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}\equiv 0 \pmod{p^{2+\delta_{p,5}}}. \tag{1.13}$$

In 2019, the authors [15] established the modulus p^3 congruence for

$${}_4F_3\left[\begin{matrix}\alpha & 1+\frac12\alpha & \alpha & \alpha\\ & \frac12\alpha & 1 & 1\end{matrix}\middle| 1\right]_{p-1}$$

which is a parametric extension of (1.12), where p is an odd prime and α is a p-adic unit.

We shall prove Theorem 1.1 and Corollary 1.1 in the next section.

2. Proofs of Theorem 1.1 and Corollary 1.1

To show Theorem 1.1 we need the following result due to Tauraso [13, Theorem 2].

Lemma 2.1. *For any prime $p>3$ and p-adic integer x we have*

$$\left(\sum_{k=1}^{p-1}\binom{2k}{k}x^k\right)\left(\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k}\right)\equiv 2\sum_{k=1}^{p-1}\binom{2k}{k}(H_{2k-1}-H_k)x^k \pmod p, \tag{2.1}$$

where $H_k=\sum_{j=1}^k 1/j$ is the kth harmonic number.

Proof of Theorem 1.1. Throughout the proof, we always set $a=\langle-\alpha\rangle_p$.

Case 1. $a\le p-3$.

Let

$$\Phi(x)={}_2F_1\left[\begin{matrix}\frac12(-a+x) & \frac12(a+3-x)\\ & 1\end{matrix}\middle| z\right]_{p-1}{}_2F_1\left[\begin{matrix}\frac12(-a+x) & \frac12(a+1-x)\\ & 1\end{matrix}\middle| z\right]_{p-1}$$
$$-{}_3F_2\left[\begin{matrix}-a+x & a+2-x & \frac12\\ & 1 & 1\end{matrix}\middle| z\right]_{p-1},$$

where $x\in\mathbb{Z}_p$. Expanding $\Phi(x)$ we have

$$\Phi(x)=\sum_{k=0}^{p-1}\sum_{l=0}^{p-1}\frac{(\frac12(-a+x))_k(\frac12(a+3-x))_k(\frac12(-a+x))_l(\frac12(a+1-x))_l}{k!^2l!^2}\cdot z^{k+l}$$
$$-\sum_{j=0}^{p-1}\frac{(-a+x)_j(a+2-x)_j(\frac12)_j}{j!^3}\cdot z^j.$$

Clearly, $\Phi(x)$ is a rational function in x and $\Phi(0)\in\mathbb{Z}_p$. Therefore, by [6, Lemma 4.1] we have

$$\Phi(tp)\equiv\Phi(0)+tp\Phi'(0)\pmod{p^2}\tag{2.2}$$

for any $t\in\mathbb{Z}_p$. In particular, we have

$$\Phi(p)\equiv\Phi(0)+p\Phi'(0)\pmod{p^2}.\tag{2.3}$$

Note that $-a/2,\ -a\in\{1-p,\ldots,-1,0\}$. In view of (1.5), we have

$$\Phi(0)={}_2F_1\left[\begin{matrix}-\frac12 a & \frac12(a+3)\\ & 1\end{matrix}\middle| z\right]{}_2F_1\left[\begin{matrix}-\frac12 a & \frac12(a+1)\\ & 1\end{matrix}\middle| z\right]-{}_3F_2\left[\begin{matrix}-a & a+2 & \frac12\\ & 1 & 1\end{matrix}\middle| z\right]=0.$$

Since $a\le p-3$ we have $(a+3-p)/2,\ (a+1-p)/2,\ a+2-p\in\{1-p,\ldots,-1,0\}$. By (1.5) we also have

$$\begin{aligned}\Phi(p)&={}_2F_1\left[\begin{matrix}\frac12(-a+p) & \frac12(a+3-p)\\ & 1\end{matrix}\middle| z\right]{}_2F_1\left[\begin{matrix}\frac12(-a+p) & \frac12(a+1-p)\\ & 1\end{matrix}\middle| z\right]\\&\quad-{}_3F_2\left[\begin{matrix}-a+p & a+2-p & \frac12\\ & 1 & 1\end{matrix}\middle| z\right]\\&=0.\end{aligned}$$

Substituting these into (2.2) and (2.3), we get

$$\Phi(tp)\equiv\Phi(0)=0\pmod{p^2}.$$

Putting $t=(a+\alpha)/p$ we immediately obtain the desired result.

Case 2. $a=p-1$.

We can directly verify this case for $p=3$. Below we assume $p>3$. In this case, we may write $\alpha=1+pt$ for some $t\in\mathbb{Z}_p$ since $\alpha\equiv1\pmod p$. Then we have

$$\begin{aligned}{}_2F_1\left[\begin{matrix}\frac12\alpha & \frac32-\frac12\alpha\\ & 1\end{matrix}\middle| z\right]_{p-1}&=\sum_{k=0}^{p-1}\frac{(\frac12+\frac12pt)_k(1-\frac12pt)_k}{k!^2}\cdot z^k\\&\equiv\sum_{k=0}^{p-1}\frac{(\frac12)_k}{k!}\left(1+pt\sum_{j=0}^{k-1}\frac{1}{2j+1}-\frac12ptH_k\right)z^k\\&\equiv\sum_{k=0}^{p-1}\binom{2k}{k}(1+ptH_{2k}-ptH_k)\left(\frac z4\right)^k\pmod{p^2}.\end{aligned}\tag{2.4}$$

Similarly, we also have

$${}_2F_1\left[\begin{matrix}\frac12\alpha & \frac12-\frac12\alpha\\ & 1\end{matrix}\middle| z\right]_{p-1}\equiv1-\frac12pt\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\left(\frac z4\right)^k\pmod{p^2}\tag{2.5}$$

and

$${}_3F_2\left[\begin{matrix}\alpha & 2-\alpha & \frac12\\ & 1 & 1\end{matrix}\middle| z\right]_{p-1}\equiv\sum_{k=0}^{p-1}\binom{2k}{k}\left(\frac z4\right)^k\pmod{p^2}.\tag{2.6}$$

Combining (2.4)–(2.6) with Lemma 2.1, we arrive at

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix}\frac{1}{2}\alpha & \frac{3}{2}-\frac{1}{2}\alpha\\ & 1\end{matrix}\middle| z\right]_{p-1} {}_2F_1\left[\begin{matrix}\frac{1}{2}\alpha & \frac{1}{2}-\frac{1}{2}\alpha\\ & 1\end{matrix}\middle| z\right]_{p-1} - {}_3F_2\left[\begin{matrix}\alpha & 2-\alpha & \frac{1}{2}\\ & 1 & 1\end{matrix}\middle| z\right]_{p-1}\\
&\equiv -\frac{1}{2}pt\left(\sum_{k=0}^{p-1}\binom{2k}{k}\left(\frac{z}{4}\right)^k\right)\left(\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\left(\frac{z}{4}\right)^k\right) + pt\sum_{k=0}^{p-1}\binom{2k}{k}(H_{2k}-H_k)\left(\frac{z}{4}\right)^k\\
&\equiv -\frac{1}{2}pt\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\left(\frac{z}{4}\right)^k - pt\sum_{k=1}^{p-1}\binom{2k}{k}(H_{2k-1}-H_k)\left(\frac{z}{4}\right)^k + pt\sum_{k=0}^{p-1}\binom{2k}{k}(H_{2k}-H_k)\left(\frac{z}{4}\right)^k\\
&= 0 \pmod{p^2}.
\end{aligned}
$$

The proof of Theorem 1.1 is now complete. □

Let us recall the definition and the main properties of the p-adic Gamma function introduced by Morita [5] as a p-adic analogue of the classical Gamma function. Let p be an odd prime. For any $n \in \mathbb{N}$ define

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1\le k<n\\ p\nmid k}} k.$$

In particular, set $\Gamma_p(0) = 1$. Clearly, the values of $\Gamma_p(n)$ belong to the group $\mathbb{Z}_p^\times$ of p-adic units. It is known that the definition of $\Gamma_p(n)$ can be extended to $\mathbb{Z}_p$ since $\mathbb{N}$ is a dense subset of $\mathbb{Z}_p$ in the sense of p-adic norm $|\cdot|_p$. That is, for all $x \in \mathbb{Z}_p$ we can define

$$\Gamma_p(x) = \lim_{\substack{n\in\mathbb{N}\\ |x-n|_p\to 0}} \Gamma_p(n).$$

Similar to the classical Gamma function, the p-adic Gamma function has some interesting properties. For example, for any $x \in \mathbb{Z}_p$ we have

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x & \text{if } p\nmid x,\\ -1 & \text{if } p\mid x,\end{cases} \tag{2.7}$$

and

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{p-\langle -x\rangle_p}. \tag{2.8}$$

The reader may consult [7] for more properties of the p-adic Gamma function.

Lemma 2.2 (Mao and Pan [4, Theorem 1.1])**.** *Let p be an odd prime and $\alpha, \beta \in \mathbb{Z}_p$. If $\langle -\alpha\rangle_p + \langle -\beta\rangle_p < p$, then*

$${}_2F_1\left[\begin{matrix}\alpha & \beta\\ & 1\end{matrix}\middle| 1\right]_{p-1} \equiv -\frac{\Gamma_p(1-\alpha-\beta)}{\Gamma_p(1-\alpha)\Gamma_p(1-\beta)} \pmod{p^2}. \tag{2.9}$$

If $\langle -\alpha\rangle_p + \langle -\beta\rangle_p \ge p$, then

$${}_2F_1\left[\begin{matrix}\alpha & \beta\\ & 1\end{matrix}\middle| 1\right]_{p-1} \equiv (\alpha+\beta+\langle -\alpha\rangle_p+\langle -\beta\rangle_p-p)\frac{\Gamma_p(1-\alpha-\beta)}{\Gamma_p(1-\alpha)\Gamma_p(1-\beta)} \pmod{p^2}. \tag{2.10}$$

Lemma 2.3. *Let $b\in\{2,3,4,\ldots\}$, and let p be a prime with $p\equiv\pm1 \pmod b$ and $\langle -1/b\rangle_p\equiv 0 \pmod 2$. Then*

$$b\,{}_3F_2\left[\begin{matrix}1+\frac1b & 1-\frac1b & \frac12\\ & 1 & 1\end{matrix}\middle|1\right]_{p-1}\equiv {}_3F_2\left[\begin{matrix}\frac1b & 1-\frac1b & \frac12\\ & 1 & 1\end{matrix}\middle|1\right]_{p-1} \pmod{p^2}. \tag{2.11}$$

Proof. It is clear that p is odd and

$$\left\langle\frac1b-1\right\rangle_p=p-1-\left\langle-\frac1b\right\rangle_p\equiv 0 \pmod 2.$$

Therefore, by (1.3) and Theorem 1.1 we have

$${}_3F_2\left[\begin{matrix}\frac1b & 1-\frac1b & \frac12\\ & 1 & 1\end{matrix}\middle|1\right]_{p-1}\equiv {}_2F_1\left[\begin{matrix}\frac1{2b} & \frac12-\frac1{2b}\\ & 1\end{matrix}\middle|1\right]_{p-1}^2 \pmod{p^2}$$

and

$${}_3F_2\left[\begin{matrix}1+\frac1b & 1-\frac1b & \frac12\\ & 1 & 1\end{matrix}\middle|1\right]_{p-1}\equiv {}_2F_1\left[\begin{matrix}1+\frac1{2b} & \frac12-\frac1{2b}\\ & 1\end{matrix}\middle|1\right]_{p-1}{}_2F_1\left[\begin{matrix}\frac1{2b} & \frac12-\frac1{2b}\\ & 1\end{matrix}\middle|1\right]_{p-1} \pmod{p^2}.$$

Now we assume $p\equiv 1 \pmod b$. Note that

$$\left\langle-\frac1{2b}\right\rangle_p+\left\langle\frac1{2b}-\frac12\right\rangle_p=\frac{p-1}{2b}+\frac{p-1}{2}-\frac{p-1}{2b}<p$$

and

$$\left\langle-1-\frac1{2b}\right\rangle_p+\left\langle\frac1{2b}-\frac12\right\rangle_p=\frac{p-1}{2b}-1+\frac{p-1}{2}-\frac{p-1}{2b}<p.$$

Thus, by Lemma 2.2 we deduce that

$$\begin{aligned}&b\,{}_3F_2\left[\begin{matrix}1+\frac1b & 1-\frac1b & \frac12\\ & 1 & 1\end{matrix}\middle|1\right]_{p-1}-{}_3F_2\left[\begin{matrix}\frac1b & 1-\frac1b & \frac12\\ & 1 & 1\end{matrix}\middle|1\right]_{p-1}\\ \equiv& b\cdot\frac{\Gamma_p(-\frac12)\Gamma_p(\frac12)}{\Gamma_p(\frac12+\frac1{2b})^2\Gamma_p(-\frac1{2b})\Gamma_p(1-\frac1{2b})}-\frac{\Gamma_p(\frac12)^2}{\Gamma_p(1-\frac1{2b})^2\Gamma_p(\frac12+\frac1{2b})^2}\\ =&0 \pmod{p^2},\end{aligned}$$

where we have used the facts

$$\Gamma_p\left(\frac12\right)=\frac12\Gamma_p\left(-\frac12\right)\text{ and }\Gamma_p\left(1-\frac1{2b}\right)=\frac1{2b}\Gamma_p\left(-\frac1{2b}\right).$$

Below we suppose that $p\equiv -1 \pmod b$. Note that

$$\left\langle-\frac1{2b}\right\rangle_p+\left\langle\frac1{2b}-\frac12\right\rangle_p=\frac12\left(p-\frac{p+1}{b}\right)+\frac{p-1}{2}-\frac12\left(p-\frac{p+1}{b}\right)<p$$

and

$$\left\langle-1-\frac1{2b}\right\rangle_p+\left\langle\frac1{2b}-\frac12\right\rangle_p=\frac12\left(p-\frac{p+1}{b}\right)-1+\frac{p-1}{2}-\frac12\left(p-\frac{p+1}{b}\right)<p.$$

Similarly, as in the case $p \equiv 1 \pmod{b}$, we also have

$$b\,{}_3F_2\left[\begin{matrix}1+\frac{1}{b} & 1-\frac{1}{b} & \frac{1}{2}\\ & 1 & 1\end{matrix}\middle|\, 1\right]_{p-1} - {}_3F_2\left[\begin{matrix}\frac{1}{b} & 1-\frac{1}{b} & \frac{1}{2}\\ & 1 & 1\end{matrix}\middle|\, 1\right]_{p-1} \equiv 0 \pmod{p^2}.$$

This completes the proof. □

Proof of Corollary 1.1. Clearly, (1.8) holds for $b = 1$. If $b \geq 2$, then the desired result easily follows from Lemma 2.3. □

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