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LEGENDRE SYMBOLS RELATED TO CERTAIN DETERMINANTS

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ABSTRACT. Let p be an odd prime. For $b,c\in\mathbb{Z}$, Sun introduced the determinant

$$D_p(b,c)=\left|(i^2+bij+cj^2)^{p-2}\right|_{1\leqslant i,j\leqslant p-1},$$

and investigated the Legendre symbol $(\frac{D_p(b,c)}{p})$. Recently Wu, She and Ni proved that $(\frac{D_p(1,1)}{p})=(\frac{-2}{p})$ if $p\equiv 2\pmod 3$, which confirms a previous conjecture of Sun. In this paper we determine $(\frac{D_p(1,1)}{p})$ in the case $p\equiv 1\pmod 3$. Sun proved that $D_p(2,2)\equiv 0\pmod p$ if $p\equiv 3\pmod 4$, in contrast we prove that $(\frac{D_p(2,2)}{p})=1$ if $p\equiv 1\pmod 8$, and $(\frac{D_p(2,2)}{p})=0$ if $p\equiv 5\pmod 8$. Our tools include generalized trinomial coefficients and Lucas sequences.

1. INTRODUCTION

For an $n\times n$ matrix $[a_{ij}]_{1\leqslant i,j\leqslant n}$ over a commutative ring, we use $|a_{ij}|_{1\leqslant i,j\leqslant n}$ to denote its determinant.

Let p be an odd prime, and let $(\frac{\cdot}{p})$ be the Legendre symbol. Carlitz [3] determined the characteristic polynomial of the matrix

$$\left[x+\left(\frac{i-j}{p}\right)\right]_{1\leqslant i,j\leqslant p-1},$$

and Chapman [4] evaluated the determinant

$$\left|x+\left(\frac{i+j-1}{p}\right)\right|_{1\leqslant i,j\leqslant (p-1)/2}.$$

Vsemirnov [12, 13] confirmed a challenging conjecture of Chapman by evaluating the determinant

$$\left|\left(\frac{j-i}{p}\right)\right|_{1\leqslant i,j\leqslant (p+1)/2}.$$

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Sun [9] studied some determinants whose entries have the form $(\frac{i^2+cij+dj^2}{p})$, where $c,d\in\mathbb{Z}$; in particular he introduced

$$(c,d)_p := \left|\left(\frac{i^2+cij+dj^2}{p}\right)\right|_{1\leqslant i,j\leqslant p-1}$$

and

$$[c,d]_p := \left|\left(\frac{i^2+cij+dj^2}{p}\right)\right|_{0\leqslant i,j\leqslant p-1},$$

and proved that if $(\frac{d}{p})=1$ then

$$[c,d]_p = \begin{cases} \frac{p-1}{2}(c,d)_p & \text{if } p\nmid c^2-4d, \\ \frac{1-p}{p-2}(c,d)_p & \text{if } p\mid c^2-4d. \end{cases}$$

For any prime $p\equiv 3 \pmod 4$, Sun [9, Remark 1.3] showed that

$$\left|\frac{1}{i^2+j^2}\right|_{1\leqslant i,j\leqslant (p-1)/2} \equiv \left(\frac{2}{p}\right) \pmod p.$$

For each prime $p\equiv 5 \pmod 6$, Sun [9] conjectured that

$$2\left|\frac{1}{i^2-ij+j^2}\right|_{1\leqslant i,j\leqslant p-1}$$

is a quadratic residue modulo p. This was recently confirmed by Wu, She, and Ni [14].

Let p be an odd prime. For $b,c\in\mathbb{Z}$, Sun [11] investigated the determinant

$$D_p(b,c) = \left|(i^2+bij+cj^2)^{p-2}\right|_{1\leqslant i,j\leqslant p-1}, \tag{1.1}$$

and studied the Legendre symbol $(\frac{D_p(b,c)}{p})$. By Fermat's little theorem,

$$(i^2+bij+cj^2)^{p-2} \equiv \begin{cases} \frac{1}{i^2+bij+cj^2} \pmod p & \text{if } i^2+bij+cj^2\not\equiv 0 \pmod p, \\ 0 \pmod p & \text{if } i^2+bij+cj^2\equiv 0 \pmod p. \end{cases}$$

As pointed out in [11, (1.7)],

$$D_p(-b,c) \equiv \left(\frac{-1}{p}\right) D_p(b,c) \pmod p.$$

Thus, in view of the Wu-She-Ni result [14], if $p\equiv 2 \pmod 3$ then

$$D_p(1,1) = \left(\frac{(-1)^{(p-1)/2}D_p(-1,1)}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right) = \left(\frac{-2}{p}\right).$$

Our first purpose is to determine the Legendre symbol $(\frac{D_p(1,1)}{p})$ for any prime $p\equiv 1 \pmod 3$.

Theorem 1.1. *Let p be a prime with $p \equiv 1 \pmod 3$. Then*

$$\left(\frac{D_p(1,1)}{p}\right) = \begin{cases} 0 & \text{if } p \equiv 7 \pmod 9, \\ 1 & \text{otherwise.} \end{cases} \tag{1.2}$$

Sun [11] proved that $D_p(2,2) \equiv 0 \pmod p$ for any prime $p \equiv 3 \pmod 4$. In contrast, we obtain the following result.

Theorem 1.2. *Let p be a prime with $p \equiv 1 \pmod 4$. Then*

$$\left(\frac{D_p(2,2)}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8, \\ 0 & \text{if } p \equiv 5 \pmod 8. \end{cases} \tag{1.3}$$

In the next section, we will provide some lemmas on generalized trinomial coefficients. We are going to prove Theorems 1.1 and 1.2 in Sections 3 and 4 respectively.

2. On generalized trinomial coefficients

Let $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. The trinomial coefficients $\binom{n}{k}_2$ $(k = -n, \ldots, n)$ (cf. [1]) are defined by

$$(x + 1 + x^{-1})^n = \sum_{k=-n}^{n} \binom{n}{k}_2 x^k,$$

and the number $T_n = \binom{n}{0}_2$ is called a central trinomial coefficient.

Let $n \in \mathbb{N}$ and $b, c \in \mathbb{Z}$. We define the generalized trinomial coefficients

$$\binom{n}{k}_{b,c} \quad (k \in \mathbb{Z})$$

by

$$\left(x + b + \frac{c}{x}\right)^n = \sum_{k \in \mathbb{Z}} \binom{n}{k}_{b,c} x^k. \tag{2.1}$$

Obviously $\binom{n}{k}_{b,c} = 0$ if $|k| > n$. Note that $\binom{n}{0}_{b,c}$ is just the generalized central trinomial coefficient $T_n(b,c)$ studied in [7, 8]. Clearly,

$$\left(x + 2 + \frac{1}{x}\right)^n = \frac{(x+1)^{2n}}{x^n} = \sum_{k=-n}^{n} \binom{2n}{n+k} x^k$$

and thus $\binom{n}{k}_{2,1} = \binom{2n}{n+k}$ for all $k = -n, \ldots, n$. When $c \neq 0$, replacing x in (2.1) by c/x we get

$$\left(\frac{c}{x} + b + x\right)^n = \sum_{k \in \mathbb{Z}} \binom{n}{k}_{b,c} \left(\frac{c}{x}\right)^k = \sum_{k \in \mathbb{Z}} \binom{n}{-k} c^{-k} x^k,$$

and hence

$$\binom{n}{k}_{b,c} = \binom{n}{-k}_{b,c} c^{-k} \quad \text{for all } k \in \mathbb{Z} \tag{2.2}$$

in view of (2.1).

Let $b, c \in \mathbb{Z}$. For any $n \in \mathbb{Z}^+ = \{1, 2, 3 \dots\}$, as $(x + b + \frac{c}{x})^n$ equals

$$x\left(x+b+\frac{c}{x}\right)^{n-1} + b\left(x+b+\frac{c}{x}\right)^{n-1} + \frac{c}{x}\left(x+b+\frac{c}{x}\right)^{n-1},$$

we have the recurrence

$$\binom{n}{k}_{b,c} = \binom{n-1}{k-1}_{b,c} + b\binom{n-1}{k}_{b,c} + c\binom{n-1}{k+1}_{b,c} \tag{2.3}$$

for any $k \in \mathbb{Z}$.

Lemma 2.1. *Let p be an odd prime, and let $b, c \in \mathbb{Z}$. For $k \in \{-p+2, \dots, p-2\}$, we have*

$$\begin{aligned}&(4c-b^2)\binom{p-2}{k}_{b,c}\\ \equiv&\begin{cases}\binom{p-1}{-1}_{b,c} + c\binom{p-1}{1}_{b,c} - b \pmod p & \text{if } k = 0,\\ (k+1)\binom{p-1}{k-1}_{b,c} - (k-1)c\binom{p-1}{k+1}_{b,c} \pmod p & \text{if } 0 < |k| \leqslant p-2.\end{cases}\end{aligned} \tag{2.4}$$

Proof. For the sake of convenience, for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ we simply write $\begin{bmatrix} n \\ k \end{bmatrix}$ for $\binom{n}{k}_{b,c}$.

Taking derivatives of both sides of the identity

$$\sum_{k=-p}^{p} \begin{bmatrix} p \\ k \end{bmatrix} x^k = \left(x+b+\frac{c}{x}\right)^p, \tag{2.5}$$

we get

$$\sum_{k=-p}^{p} \begin{bmatrix} p \\ k \end{bmatrix} k x^{k-1} = p\left(x+b+\frac{c}{x}\right)^{p-1}\left(1-\frac{c}{x^2}\right). \tag{2.6}$$

Taking derivatives of both sides of (2.6), we obtain

$$\begin{aligned}&\sum_{k=-p}^{p} \begin{bmatrix} p \\ k \end{bmatrix} k(k-1)x^{k-2}\\ =&p(p-1)\left(x+b+\frac{c}{x}\right)^{p-2}\left(1-\frac{c}{x^2}\right)^2 + p\left(x+b+\frac{c}{x}\right)^{p-1}\frac{2c}{x^3}.\end{aligned} \tag{2.7}$$

For each $k = -p, \ldots, p$, comparing the coefficients of x^{k-1} on both sides of (2.6) we get

$$\frac{k}{p}\begin{bmatrix}p\\k\end{bmatrix} = \begin{bmatrix}p-1\\k-1\end{bmatrix} - c\begin{bmatrix}p-1\\k+1\end{bmatrix}; \tag{2.8}$$

similarly, comparing the coefficients of x^{k-2} on both sides of (2.7) we obtain

$$\begin{aligned}&\frac{k(k-1)}{p}\begin{bmatrix}p\\k\end{bmatrix} - 2c\begin{bmatrix}p-1\\k+1\end{bmatrix}\\ &= (p-1)\left(\begin{bmatrix}p-2\\k-2\end{bmatrix} - 2c\begin{bmatrix}p-2\\k\end{bmatrix} + c^2\begin{bmatrix}p-2\\k+2\end{bmatrix}\right).\end{aligned} \tag{2.9}$$

Let $k \in \{-p, \ldots, p\}$. With the aid of the recurrence (2.3), we have

$$\begin{aligned}&\begin{bmatrix}p-2\\k-2\end{bmatrix} - 2c\begin{bmatrix}p-2\\k\end{bmatrix} + c^2\begin{bmatrix}p-2\\k+2\end{bmatrix}\\ =&\left(\begin{bmatrix}p-1\\k-1\end{bmatrix} - b\begin{bmatrix}p-2\\k-1\end{bmatrix} - c\begin{bmatrix}p-2\\k\end{bmatrix}\right) - 2c\begin{bmatrix}p-2\\k\end{bmatrix}\\ &+ c\left(\begin{bmatrix}p-1\\k+1\end{bmatrix} - \begin{bmatrix}p-2\\k\end{bmatrix} - b\begin{bmatrix}p-2\\k+1\end{bmatrix}\right)\\ =&\begin{bmatrix}p-1\\k-1\end{bmatrix} + c\begin{bmatrix}p-1\\k+1\end{bmatrix} - 4c\begin{bmatrix}p-2\\k\end{bmatrix} - b\left(\begin{bmatrix}p-2\\k-1\end{bmatrix} + c\begin{bmatrix}p-2\\k+1\end{bmatrix}\right)\\ =&\begin{bmatrix}p-1\\k-1\end{bmatrix} + c\begin{bmatrix}p-1\\k+1\end{bmatrix} - 4c\begin{bmatrix}p-2\\k\end{bmatrix} - b\left(\begin{bmatrix}p-1\\k\end{bmatrix} - b\begin{bmatrix}p-2\\k\end{bmatrix}\right)\\ =&\begin{bmatrix}p-1\\k-1\end{bmatrix} - b\begin{bmatrix}p-1\\k\end{bmatrix} + c\begin{bmatrix}p-1\\k+1\end{bmatrix} + (b^2-4c)\begin{bmatrix}p-2\\k\end{bmatrix}\\ =&\begin{bmatrix}p-1\\k-1\end{bmatrix} - \left(\begin{bmatrix}p\\k\end{bmatrix} - \begin{bmatrix}p-1\\k-1\end{bmatrix} - c\begin{bmatrix}p-1\\k+1\end{bmatrix}\right)\\ &+ c\begin{bmatrix}p-1\\k+1\end{bmatrix} + (b^2-4c)\begin{bmatrix}p-2\\k\end{bmatrix}\\ =&-\begin{bmatrix}p\\k\end{bmatrix} + 2\begin{bmatrix}p-1\\k-1\end{bmatrix} + 2c\begin{bmatrix}p-1\\k+1\end{bmatrix} + (b^2-4c)\begin{bmatrix}p-2\\k\end{bmatrix}.\end{aligned}$$

Combining this with (2.8) and (2.9), we get

$$\begin{aligned}&(k-1)\left(\begin{bmatrix}p-1\\k-1\end{bmatrix} - c\begin{bmatrix}p-1\\k+1\end{bmatrix}\right) - 2cp\begin{bmatrix}p-1\\k+1\end{bmatrix}\\ &= (1-p)\left(\begin{bmatrix}p\\k\end{bmatrix} - 2\begin{bmatrix}p-1\\k-1\end{bmatrix} - (b^2-4c)\begin{bmatrix}p-2\\k\end{bmatrix}\right)\end{aligned}$$

and hence

$$(k+1)\begin{bmatrix}p-1\\k-1\end{bmatrix}-(k-1)c\begin{bmatrix}p-1\\k+1\end{bmatrix}\equiv\begin{bmatrix}p\\k\end{bmatrix}-(b^2-4c)\begin{bmatrix}p-2\\k\end{bmatrix}\ (\mathrm{mod}\ p). \tag{2.10}$$

Since

$$\sum_{k=-p}^{p}\begin{bmatrix}p\\k\end{bmatrix}x^{p+k}=\big(x^2+bx+c\big)^p$$
$$\equiv x^{2p}+b^px^p+c^p\equiv x^{2p}+bx^p+c\ (\mathrm{mod}\ p),$$

we see that

$$\begin{bmatrix}p\\k\end{bmatrix}\equiv\begin{cases}b\ (\mathrm{mod}\ p) & \text{if } k=0,\\ 1\ (\mathrm{mod}\ p) & \text{if } k=p,\\ c\ (\mathrm{mod}\ p) & \text{if } k=-p,\\ 0\ (\mathrm{mod}\ p) & \text{if } k\in\{\pm1,\ldots,\pm(p-1)\}.\end{cases} \tag{2.11}$$

Combining this with (2.10), we immediately obtain the desired (2.4). □

Lemma 2.2. *Let p be an odd prime, and let $b,c\in\mathbb{Z}$. Then*

$$\begin{aligned}&(x^2+bx+c)^{p-2}-c^{p-2}\\ \equiv&\binom{p-2}{1}_{b,c}x^{p-1}+\binom{p-2}{0}_{b,c}x^{p-2}\\ &+\sum_{1<k<p-1}\left(\binom{p-2}{k}_{b,c}+c^{p-1-k}\binom{p-2}{p-1-k}_{b,c}\right)x^{k-1}\ (\mathrm{mod}\ p).\end{aligned} \tag{2.12}$$

Proof. In view of (2.2), we have

$$\begin{aligned}
&(x^2+bx+c)^{p-2} - \binom{p-2}{0}_{b,c} x^{p-2} \\
=& \sum_{\substack{k=-(p-2)\\ k\neq 0}}^{p-2} \binom{p-2}{k}_{b,c} x^{p-2+k} \\
=& \sum_{k=1}^{p-2} \left(\binom{p-2}{k}_{b,c} x^{p-2+k} + \binom{p-2}{-k}_{b,c} x^{p-2-k} \right) \\
=& \sum_{k=1}^{p-2} \left(\binom{p-2}{k}_{b,c} x^{p-2+k} + \binom{p-2}{k}_{b,c} c^k x^{p-2-k} \right) \\
=& \sum_{k=1}^{p-2} \left(\binom{p-2}{k}_{b,c} x^{p-2+k} + \binom{p-2}{p-1-k}_{b,c} c^{p-1-k} x^{k-1} \right) \\
=& \sum_{k=1}^{p-2} \left(\binom{p-2}{k}_{b,c} x^{p-1} + \binom{p-2}{p-1-k}_{b,c} c^{p-1-k} \right) x^{k-1}.
\end{aligned}$$

Note that

$$\binom{p-2}{1}_{b,c} x^{p-1} + \binom{p-2}{p-1-1}_{b,c} c^{p-1-1} = \binom{p-2}{1}_{b,c} x^{p-1} + c^{p-2}.$$

Therefore, from the above we get the desired (2.12). □

For convenience, for an assertion A we set

$$[A] = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2.1. *Let p be an odd prime. If $p \equiv 1 \pmod 3$, then*

$$\begin{aligned}
(x^2+x+1)^{p-2} \equiv 1 + \frac{2}{3}x^{p-1} - \frac{1}{3}x^{p-2} \\
+ \sum_{k=2}^{p-2} \left(k\left(\frac{k}{3}\right) + [3 \mid k-1] - \frac{1}{3} \right) x^{k-1} \pmod p.
\end{aligned} \tag{2.13}$$

Proof. By (2.11), we have

$$\binom{p}{0}_2 = 1, \text{ and } \binom{p}{k}_2 = 0 \text{ for } k = 1, \ldots, p-1.$$

Combining this with (2.3), we see that

$$\binom{p-1}{k-1}_2+\binom{p-1}{k}_2+\binom{p-1}{k+1}_2\equiv 0 \pmod p$$

for all $k=1,\ldots,p-1$. In view of this and the easy equalities

$$\binom{p-1}{p}_2=0 \quad\text{and}\quad \binom{p-1}{p-1}_2=1,$$

by induction we obtain that

$$\binom{p-1}{p-k}_2\equiv\left(\frac{k}{3}\right) \pmod p \quad \text{for all } k=0,1,\ldots,p. \tag{2.14}$$

Note that

$$T_p=\binom{p-1}{0}_2\equiv\left(\frac{p}{3}\right)3^{p-1} \pmod{p^2} \tag{2.15}$$

as proved by Cao and Sun [2].

By Lemma 2.1, (2.2) and (2.14), we have

$$\begin{aligned}3\binom{p-2}{0}_2&\equiv\binom{p-1}{-1}_2+\binom{p-1}{1}_2-1=2\binom{p-1}{1}_2-1\\&\equiv 2\left(\frac{p-1}{3}\right)-1 \pmod p.\end{aligned}$$

Combining Lemma 2.1 and (2.14), we see that for each $k=1,\ldots,p-2$ we have

$$\begin{aligned}3\binom{p-2}{k}_2&\equiv(k+1)\binom{p-1}{k-1}_2-(k-1)\binom{p-1}{k+1}_2\\&\equiv(k+1)\left(\frac{p-k+1}{3}\right)-(k-1)\left(\frac{p-k-1}{3}\right) \pmod p\end{aligned}$$

and

$$\begin{aligned}
&3\left(\binom{p-2}{k}_2+\binom{p-2}{p-1-k}_2\right)\\
\equiv\ &(k+1)\left(\frac{p-k+1}{3}\right)-(k-1)\left(\frac{p-k-1}{3}\right)\\
&+(p-1-k+1)\left(\frac{p-(p-1-k)+1}{3}\right)\\
&-(p-1-k-1)\left(\frac{p-(p-1-k)-1}{3}\right)\\
\equiv\ &(k+1)\left(\frac{p-k+1}{3}\right)-(k-1)\left(\frac{p-k-1}{3}\right)\\
&-k\left(\frac{k+2}{3}\right)+(k+2)\left(\frac{k}{3}\right)\pmod p.
\end{aligned}$$

Now we suppose that $p\equiv 1\pmod 3$. By the last paragraph,

$$\binom{p-2}{0}_2\equiv-\frac{1}{3}\pmod p, \tag{2.16}$$

and for each $k=1,\ldots,p-2$ we have

$$\begin{aligned}
&3\left(\binom{p-2}{k}_2+\binom{p-2}{p-1-k}_2\right)\\
\equiv\ &(k+1)\left(\frac{-k-1}{3}\right)-(k-1)\left(\frac{-k}{3}\right)-k\left(\frac{k-1}{3}\right)+(k+2)\left(\frac{k}{3}\right)\\
=\ &(2k+1)\left(\frac{k}{3}\right)-\left(\frac{k+1}{3}\right)-k\left(\left(\frac{k+1}{3}\right)+\left(\frac{k-1}{3}\right)\right)\\
=\ &(3k+1)\left(\frac{k}{3}\right)-\left(\frac{k+1}{3}\right)\pmod p.
\end{aligned}$$

Applying Lemma 2.2 with $b=c=1$, we see that

$$\begin{aligned}
(x^2+x+1)^{p-2}\equiv\ &1+\binom{p-2}{1}_2x^{p-1}+\binom{p-2}{0}_2x^{p-2}\\
&+\sum_{k=2}^{p-2}\left[\binom{p-2}{k}_2+\binom{p-2}{p-1-k}_2\right]x^{k-1}\pmod p.
\end{aligned}$$

By Lemma 2.1 and (2.15), we have

$$3\binom{p-2}{1}_2\equiv 2\binom{p-1}{0}_2\equiv 2\left(\frac{p}{3}\right)=2\pmod p.$$

Combining this with (2.16) and the last paragraph, we obtain the desired (2.13). □

3. Proof of Theorem 1.1

We need the following known lemma [5, Lemma 10] on determinants.

Lemma 3.1. *Let R be a commutative ring with identity, and let $P(x) = \sum_{i=0}^{n-1} a_i x^i \in R[x]$. Then we have*

$$\det\left[P(X_iY_j)\right]_{1\leqslant i<j\leqslant n} = a_0a_1\cdots a_{n-1}\prod_{1\leqslant i<j\leqslant n}(X_i-X_j)(Y_i-Y_j).$$

We also need the following known lemma (cf. [10, Theorem 1.1]).

Lemma 3.2. *Let p be an odd prime. For each p-adic integer x, let $\{x\}_p$ denote the least nonnegative residue of x modulo p. Define*

$$\mathrm{Inv}_p := \#\{(i,j) : 1\leqslant i<j\leqslant p-1 \text{ and } \{i^{-1}\}_p > \{j^{-1}\}_p\},$$

where $\#S$ denotes the cardinality of a set S. Then we have

$$\mathrm{Inv}_p \equiv \frac{p+1}{2} \pmod 2.$$

Proof of Theorem 1.1. Recall that

$$D_p(1,1) = \left|(i^2+ij+j^2)^{p-2}\right|_{1\leqslant i,j\leqslant p-1}.$$

By Corollary 2.1, we have

$$(x^2+x+1)^{p-2} \equiv \frac{2}{3}(x^{p-1}-1)+F(x), \tag{3.1}$$

where

$$F(x) = \frac{5}{3}-\frac{1}{3}x^{p-2}+\sum_{k=2}^{p-2}\left(k\left(\frac{k}{3}\right)+[3\mid k-1]-\frac{1}{3}\right)x^{k-1} \pmod p.$$

By Fermat's little theorem and (3.1), for any $i,j=1,\ldots,p-1$ we have

$$\frac{(i^2+ij+j^2)^{p-2}}{j^{2(p-2)}} = \left(\frac{i^2}{j^2}+\frac{i}{j}+1\right)^{p-2} \equiv F\left(\frac{i}{j}\right) \pmod p,$$

and hence

$$\left(\frac{D_p(1,1)}{p}\right) = \left(\frac{|F(i/j)|_{1\leqslant i,j\leqslant p-1}}{p}\right).$$

By Lemma 3.1, we have

$$\left|F\left(\frac{i}{j}\right)\right|_{1\leqslant i,j\leqslant p-1} = -\frac{5}{9}\prod_{k=2}^{p-2}\left(k\left(\frac{k}{3}\right)+[3\mid k-1]-\frac{1}{3}\right) \times \prod_{1\leqslant i<j\leqslant p-1}(i-j)\left(\frac{1}{i}-\frac{1}{j}\right).$$

In view of Lemma 3.2,

$$\prod_{1\leqslant i<j\leqslant p-1}(i-j)\left(\frac{1}{i}-\frac{1}{j}\right) = (-1)^{\mathrm{Inv}_p}\prod_{1\leqslant i<j\leqslant p-1}(i-j)^2 = (-1)^{(p+1)/2}\prod_{j=2}^{p-1}((j-1)!)^2. \tag{3.2}$$

Thus

$$\begin{aligned}&\left|F\left(\frac{i}{j}\right)\right|_{1\leqslant i,j\leqslant p-1}\\ \equiv\ &(-1)^{(p+1)/2+1}\cdot\frac{5}{9}\prod_{r=0}^{2}\prod_{\substack{k=2\\ k\equiv r\ (\mathrm{mod}\ 3)}}^{p-2}\left(k\left(\frac{k}{3}\right)+[3\mid k-1]-\frac{1}{3}\right)\times\prod_{j=1}^{p-2}(j!)^2\\ =\ &\frac{(-1)^{(p+1)/2}}{(-3)^{(p-1)/3}}\prod_{i=0}^{(p-4)/3}\left((3i+1)+\frac{2}{3}\right)\left(-(3i+2)-\frac{1}{3}\right)\times\prod_{j=1}^{p-2}(j!)^2\\ =\ &\frac{(-1)^{(p+1)/2+(p-1)/3}}{3^{p-1}}\prod_{i=0}^{\frac{p-4}{3}}(9i+5)(-9i-7)\times\prod_{j=1}^{p-2}(j!)^2 \pmod p.\end{aligned}$$

As

$$9\left(\frac{p-4}{3}-i\right)+7=3(p-4)+7-9i=3p-5-9i\equiv-(9i+5) \pmod p$$

for any $i=0,\ldots,(p-4)/3$, by the above and Lemma 3.2 we have

$$\left|F\left(\frac{i}{j}\right)\right|_{1\leqslant i,j\leqslant p-1}\equiv(-1)^{(p+1)/2+(p-1)/3}\prod_{i=0}^{(p-4)/3}(9i+5)^2\times\prod_{j=1}^{p-2}(j!)^2 \pmod p.$$

For $0\leqslant i\leqslant(p-4)/3$, clearly $9i+5\leqslant 3p-7<3p$, and $9i+5\neq p$ since $p\equiv 1 \pmod 3$. Note that

$$9i+5=2p \text{ for some } i=0,\ldots,\frac{p-4}{3} \iff p\equiv-2\equiv 7 \pmod 9.$$

Thus

$$\left(\frac{|F(i/j)|_{1\leqslant i,j\leqslant p-1}}{p}\right)=\left(\frac{-1}{p}\right)^{(p+1)/2+(p-1)/3}\times\begin{cases}0 & \text{if } p\equiv 7 \pmod 9,\\ 1 & \text{otherwise.}\end{cases}$$

Therefore

$$\left(\frac{D_p(1,1)}{p}\right)=\left(\frac{|F(i/j)|_{1\leqslant i,j\leqslant p-1}}{p}\right)=\begin{cases}0 & \text{if } p\equiv 7 \pmod 9,\\ 1 & \text{otherwise.}\end{cases}$$

This concludes our proof of Theorem 1.1. □

4. Proof of Theorem 1.2

Let $A,\ B\in\mathbb{Z}$. The Lucas sequence $u_n=u_n(A,B)$ $(n\in\mathbb{N})$ is defined as follows:

$$u_0=0,\ u_1=1,\ \text{and}\ u_{n+1}=Au_n-Bu_{n-1}\ \text{for}\ n=1,2,3,\ldots.$$

Let α and β be the two roots of the quadratic equation $x^2-Ax+B=0$. By Binet's formula,

$$(\alpha-\beta)u_n=\alpha^n-\beta^n\quad\text{for all } n\in\mathbb{N}.$$

The following lemma is well-known (see, e.g., [6, Lemma 2.3]).

Lemma 4.1. *Let $A,B\in\mathbb{Z}$, and let p be an odd prime. Then*

$$u_p(A,B)\equiv\left(\frac{A^2-4B}{p}\right)\pmod p. \tag{4.1}$$

Provided $p\nmid B$, we also have

$$u_{p-(\frac{A^2-4B}{p})}(A,B)\equiv 0\pmod p. \tag{4.2}$$

Lemma 4.2. *Let $b,c\in\mathbb{Z}$, and let p be an odd prime. Then*

$$\binom{p-1}{p-k}_{b,c}\equiv u_k(-b,c)\pmod p\quad\text{for all } k=0,1,\ldots,p-1. \tag{4.3}$$

Proof. Obviously,

$$\binom{p-1}{p}_{b,c}=0=u_0(-b,c)\ \text{and}\ \binom{p-1}{p-1}_{b,c}=1=u_1(-b,c).$$

Now let $k\in\{1,\ldots,p-2\}$, and assume that $\binom{p-1}{p-j}_{b,c}\equiv u_j(b,c)\pmod p$ for all $j=0,\ldots,k$. By (2.3), we have

$$\binom{p}{p-k}_{b,c}=\binom{p-1}{p-k-1}_{b,c}+b\binom{p-1}{p-k}_{b,c}+c\binom{p-1}{p-k+1}_{b,c}.$$

Since

$$\binom{p}{p-k}_{b,c} \equiv 0 \pmod p$$

by (2.11), we have

$$\begin{aligned}\binom{p-1}{p-k-1}_{b,c} &\equiv -b\binom{p-1}{p-k}_{b,c} - c\binom{p-1}{p-k+1}_{b,c}\\ &\equiv -bu_k(-b,c) - cu_{k-1}(-b,c) = u_{k+1}(-b,c) \pmod p.\end{aligned}$$

By the above, we have proved (4.3) by induction. □

Lemma 4.3. *Let p be an odd prime, and let $b,c\in\mathbb{Z}$ with $p\nmid c(b^2-4c)$. Let*

$$U(k) = \binom{p-2}{k}_{b,c} + c^{p-1-k}\binom{p-2}{p-1-k}_{b,c}. \tag{4.4}$$

(i) *If $U(k)\equiv 0 \pmod p$ for some $k\in\{2,\ldots,p-2\}$, then*

$$\left(\frac{D_p(b,c)}{p}\right) = 0.$$

(ii) *If $U(k)\not\equiv 0 \pmod p$ for all $2\leqslant k\leqslant p-2$, then*

$$\begin{aligned}&\left(\frac{c}{p}\right)^{(p-1)(p-3)/8}\left(\frac{D_p(b,c)}{p}\right)\\ &=\left(\frac{4c-b^2+2c(\frac{b^2-4c}{p})}{p}\right)\left(\frac{2cu_{p-1}(-b,c)-b}{p}\right)\left(\frac{U(p-2)U(\frac{p-1}{2})}{p}\right).\end{aligned} \tag{4.5}$$

Proof. Recall that

$$D_p(b,c) = \left|(i^2+bij+cj^2)^{p-2}\right|_{1\leqslant i,j\leqslant p-1}.$$

By Lemma 2.2, we have

$$(x^2+bx+c)^{p-2} \equiv \binom{p-2}{1}_{b,c}(x^{p-1}-1) + G(x), \tag{4.6}$$

where

$$\begin{aligned}G(x) = c^{p-2} + \binom{p-2}{1}_{b,c} + \binom{p-2}{0}_{b,c} x^{p-2}\\ + \sum_{1<k<p-1}\left(\binom{p-2}{k}_{b,c} + c^{p-1-k}\binom{p-2}{p-1-k}_{b,c}\right)x^{k-1}.\end{aligned}$$

By Fermat's little theorem and (4.6), for any $i,j=1,\ldots,p-1$, we have

$$\frac{(i^2+bij+cj^2)^{p-2}}{j^{2(p-2)}}=\left(\frac{i^2}{j^2}+\frac{bi}{j}+c\right)^{p-2}\equiv G\left(\frac{i}{j}\right)\pmod p.$$

Thus

$$\left(\frac{D_p(b,c)}{p}\right)=\left(\frac{|G(i/j)|_{1\leqslant i,j\leqslant p-1}}{p}\right).$$

In view of Lemma 3.1 and the equality (3.2),

$$\begin{aligned}&\left|G\left(\frac{i}{j}\right)\right|_{1\leqslant i,j\leqslant p-1}\\
=&\left(c^{p-2}+\binom{p-2}{1}_{b,c}\right)\binom{p-2}{0}_{b,c}\prod_{k=2}^{p-2}U(k)\times\prod_{1\le i<j\le p-1}(i-j)\left(\frac1i-\frac1j\right)\\
=&\left(c^{p-2}+\binom{p-2}{1}_{b,c}\right)\binom{p-2}{0}_{b,c}U(p-2)U\left(\frac{p-1}{2}\right)\\
&\times\prod_{k=2}^{\frac{p-3}{2}}U(k)U(p-1-k)\times(-1)^{\frac{p+1}{2}}\prod_{j=1}^{p-2}(j!)^2\end{aligned}$$

and hence

$$\begin{aligned}\left(\frac{D_p(b,c)}{p}\right)=&\left(\frac{(c^{p-2}+\binom{p-2}{1}_{b,c})\binom{p-2}{0}_{b,c}}{p}\right)\left(\frac{U(p-2)U((p-1)/2)}{p}\right)\\
&\times\prod_{k=2}^{(p-3)/2}\left(\frac{U(k)U(p-1-k)}{p}\right).\end{aligned}\tag{4.7}$$

(i) If $U(k)\equiv 0\pmod p$ for some $2\leqslant k\leqslant p-2$, then by (4.7) we immediately have $(\frac{D_p(b,c)}{p})=0$.

(ii) Now suppose that $U(k)\not\equiv 0\pmod p$ for any $2\leqslant k\leqslant p-2$. For each $k=1,\ldots,p-1$, with the aid of (4.4) we get

$$\begin{aligned}U(p-k-1)&=c^k\binom{p-2}{k}_{b,c}+\binom{p-2}{p-1-k}_{b,c}\\
&\equiv c^k\binom{p-2}{k}_{b,c}+c^{p-1}\binom{p-2}{p-1-k}_{b,c}\equiv c^kU(k)\pmod p.\end{aligned}$$

Thus

$$\prod_{k=2}^{(p-3)/2}\left(\frac{U(k)U(p-1-k)}{p}\right)=\prod_{k=2}^{(p-3)/2}\left(\frac{c^kU(k)^2}{p}\right)$$
$$=\left(\frac{c}{p}\right)^{\sum_{k=2}^{(p-3)/2}k}=\left(\frac{c}{p}\right)^{(p-1)(p-3)/8-1},$$

and hence (4.7) has the following equivalent form:

$$\begin{aligned}&\left(\frac{D_p(b,c)}{p}\right)\left(\frac{c}{p}\right)^{(p-1)(p-3)/8}\\&=\left(\frac{(1+c\binom{p-2}{1}_{b,c})\binom{p-2}{0}_{b,c}}{p}\right)\left(\frac{U(p-2)U((p-1)/2)}{p}\right).\end{aligned}\tag{4.8}$$

By Lemmas 2.1 and 4.1, the equality (2.2) and the congruence (4.1), we have

$$\begin{aligned}&\left(1+c\binom{p-2}{1}_{b,c}\right)\binom{p-2}{0}_{b,c}\\
\equiv&\left(1+\frac{2c}{4c-b^2}\binom{p-1}{0}_{b,c}\right)\frac{1}{4c-b^2}\left(\binom{p-1}{-1}_{b,c}+c\binom{p-1}{1}_{b,c}-b\right)\\
=&\left(1+\frac{2c}{4c-b^2}\binom{p-1}{0}_{b,c}\right)\frac{1}{4c-b^2}\left(2c\binom{p-1}{1}_{b,c}-b\right)\\
\equiv&\left(\frac{1}{4c-b^2}\right)^2\big((4c-b^2)+2cu_p(-b,c)\big)\left(2cu_{p-1}(-b,c)-b\right)\\
\equiv&\left(\frac{1}{4c-b^2}\right)^2\left((4c-b^2)+2c\left(\frac{b^2-4c}{p}\right)\right)\left(2cu_{p-1}(-b,c)-b\right)\pmod p.\end{aligned}$$

Combining this with (4.8), we immediately get the desired (4.5).

In view of the above, we have completed our proof of Lemma 4.3. □

Proof of Theorem 1.2. In view of Binet's formula, for any $k\in\mathbb{N}$ we have

$$u_k:=u_k(-2,2)=\frac{(-1+i)^k-(-1-i)^k}{2i}$$

and thus

$$u_k=(-4)^{\lfloor\frac{k}{4}\rfloor}\times\begin{cases}0&\text{if }k\equiv0\pmod 4,\\1&\text{if }k\equiv1\pmod 4,\\-2&\text{if }k\equiv2\pmod 4,\\2&\text{if }k\equiv3\pmod 4,\end{cases}\tag{4.9}$$

which can also be proved easily by induction. By Lemma 4.2,

$$\binom{p-1}{p-k}_{2,2} \equiv u_k \pmod p \quad \text{for all } k=0,1,\ldots,p-1. \tag{4.10}$$

Let $k\in\{2,\ldots,p-2\}$, and

$$U(k)=\binom{p-2}{k}_{2,2}+2^{p-1-k}\binom{p-2}{p-1-k}_{2,2}.$$

By Lemma 2.1,

$$4\binom{p-2}{k}_{2,2}\equiv (k+1)\binom{p-1}{k-1}_{2,2}-2(k-1)\binom{p-1}{k+1}_{2,2} \pmod p$$

and

$$\begin{aligned}4\binom{p-2}{p-1-k}_{2,2}&\equiv (p-k)\binom{p-1}{p-2-k}_{2,2}-2(p-2-k)\binom{p-1}{p-k}_{2,2}\\&\equiv -k\binom{p-1}{p-2-k}_{2,2}+(2k+4)\binom{p-1}{p-k}_{2,2} \pmod p.\end{aligned}$$

Thus, by the above, we have

$$\begin{aligned}4U(k)\equiv{}&(k+1)\binom{p-1}{k-1}_{2,2}-2(k-1)\binom{p-1}{k+1}_{2,2}\\&+2^{p-1-k}\left((2k+4)\binom{p-1}{p-k}_{2,2}-k\binom{p-1}{p-2-k}_{2,2}\right).\end{aligned}$$

Therefore, with the aid of (4.10), we get

$$\begin{aligned}4U(k)\equiv{}&(k+1)u_{p-k+1}-2(k-1)u_{p-k-1}\\&+2^{-k}((2k+4)u_k-ku_{k+2}) \pmod p.\end{aligned} \tag{4.11}$$

Now we handle the case $p\equiv 5 \pmod 8$. Applying (4.11) with $k=(p-1)/2$ and noting $(\frac{2}{p})=-1$, we obtain

$$\begin{aligned}4U\left(\frac{p-1}{2}\right)&\equiv \frac{p+1}{2}u_{(p+3)/2}-2\times\frac{p-3}{2}u_{(p+1)/2-1}\\&\quad+2^{-(p-1)/2}\left((p+3)u_{(p-1)/2}-\frac{p-1}{2}u_{(p+3)/2}\right)\\&\equiv \frac12 u_{(p+3)/2}+3u_{(p-1)/2}+\left(\frac{2}{p}\right)\left(3u_{(p-1)/2}+\frac12 u_{(p+3)/2}\right)\\&\equiv 0 \pmod p.\end{aligned}$$

Combining this with Lemma 4.3, we see that

$$\left(\frac{D_p(2,2)}{p}\right)=0.$$

Below we assume that $p\equiv 1 \pmod 8$ and write $p=8q+1$ with $q\in\mathbb{Z}^+$. Let $k\in\{2,\ldots,p-2\}$, and write $k=4s+r$ with $s\in\mathbb{N}$ and $r\in\{0,1,2,3\}$. We want to show that $U(k)\not\equiv 0 \pmod p$.

Case 1. $r=0$.

In this case, by (4.11) and (4.9) we have

$$\begin{aligned}4U(k)\equiv&(k+1)u_{p-k+1}-2(k-1)u_{p-k-1}+2^{-k}((2k+4)u_k-ku_{k+2})\\ \equiv&-2(k+1)(-4)^{\lfloor\frac{p-k+1}{4}\rfloor}+2^{-k}\cdot 2k(-4)^{\lfloor\frac{k}{4}\rfloor}\\ =&-2(-4)^{2q-s}(k+1)+2^{-4s+1}(-4)^sk\\ \equiv&-2\left(\frac{2}{p}\right)(-4)^{-s}(k+1)+2^{-4s+1}(-4)^sk\\ \equiv&-2(-4)^{-s}(k+1)+2^{-4s+1}(-4)^sk \pmod p.\end{aligned}$$

So

$$(-4)^{s+1}U(k)\equiv 2(k+1)-2^{-4s+1}(-4)^{2s}k\equiv 2 \pmod p,$$

and hence $U(k)\not\equiv 0 \pmod p$.

Case 2. $r=1$.

In this case, by (4.11) and (4.9) we get

$$\begin{aligned}4U(k)\equiv\ &(k+1)(-4)^{\lfloor\frac{p-k+1}{4}\rfloor}-4(k-1)(-4)^{\lfloor\frac{p-k-1}{4}\rfloor}\\ &+2^{-k}((2k+4)(-4)^{\lfloor\frac{k}{4}\rfloor}-2k(-4)^{\lfloor\frac{k+2}{4}\rfloor})\\ \equiv\ &(k+1)(-4)^{2q-s}-4(k-1)(-4)^{2q-s-1}\\ &+2^{-4s-1}((2k+4)(-4)^s-2k(-4)^s)\\ \equiv\ &2(-4)^{2q-s}k+2^{-4s-1+2}(-4)^s\\ \equiv\ &\left(\frac{2}{p}\right)2(-4)^{-s}k+2^{-4s+1}(-4)^s\\ \equiv\ &2(-4)^{-s}k+2^{-4s+1}(-4)^s \pmod p,\end{aligned}$$

and hence

$$-(-4)^{s+1}U(k)\equiv 2k+2^{-4s+1}(-4)^{2s}\equiv 2k+2\not\equiv 0 \pmod p.$$

Therefore $U(k)\not\equiv 0 \pmod p$.

Case 3. $r=2$.

In light of (4.11) and (4.9), we have

$$\begin{aligned}4U(k)\equiv&\ -2(k-1)(-4)^{\lfloor\frac{p-k-1}{4}\rfloor}\times(-2)+2^{-k}(2k+4)(-4)^{\lfloor\frac{k}{4}\rfloor}\times(-2)\\ \equiv&\ 4(k-1)(-4)^{2q-s-1}-2^{-4s-2}(4k+8)(-4)^{s}\\ \equiv&\ -\left(\frac{2}{p}\right)(-4)^{-s}(k-1)-2^{-4s}(-4)^{s}(k+2)\\ \equiv&\ -(-4)^{-s}(k-1)-2^{-4s}(-4)^{s}(k+2)\ (\mathrm{mod}\ p),\end{aligned}$$

and hence

$$(-4)^{s+1}U(k)\equiv(k-1)+2^{-4s}(-4)^{2s}(2+k)\equiv 2k+1\ (\mathrm{mod}\ p).$$

Note that $2k+1\neq p$ since $p\equiv 1\ (\mathrm{mod}\ 8)$ and $k\equiv 2\ (\mathrm{mod}\ 4)$. Therefore $U(k)\not\equiv 0\ (\mathrm{mod}\ p)$.

Case 4. $r=3$.

By (4.11) and (4.9), we have

$$\begin{aligned}4U(k)\equiv&\ 2(k+1)(-4)^{\lfloor\frac{p-k+1}{4}\rfloor}-2(k-1)(-4)^{\lfloor\frac{p-k-1}{4}\rfloor}\\ &+2^{-k}(4(2+k)(-4)^{\lfloor\frac{k}{4}\rfloor}-k(-4)^{\lfloor\frac{k+2}{4}\rfloor})\\ =&\ 2(k+1)(-4)^{2q-s-1}-2(k-1)(-4)^{2q-s-1}\\ &+2^{-4s-3}(4(2+k)(-4)^{s}-k(-4)^{s+1})\\ \equiv&\ -(-4)^{2q-s}+2^{-4s-1}(-4)^{s}(2k+2)\\ \equiv&\ -\left(\frac{2}{p}\right)(-4)^{-s}+2^{-4s}(-4)^{s}(k+1)\\ \equiv&\ -(-4)^{-s}+2^{-4s}(-4)^{s}(k+1)\ (\mathrm{mod}\ p).\end{aligned}$$

So

$$(-4)^{s+1}U(k)\equiv 1-2^{-4s}(-4)^{2s}(k+1)=-k\ (\mathrm{mod}\ p),$$

and hence $U(k)\not\equiv 0\ (\mathrm{mod}\ p)$.

By the above analysis, $U(k)\not\equiv 0\ (\mathrm{mod}\ p)$ for each $k=2,3,\ldots p-2$. Note that

$$4\times 2^2-2^2+2\times 2\left(\frac{2^2-4\times 2}{p}\right)=4+4\left(\frac{-4}{p}\right)=8,$$

and $(\frac{8}{p})=(\frac{2}{p})=1$ since $p\equiv 1\ (\mathrm{mod}\ 8)$. Thus, by Lemma 4.3(ii), we have

$$\left(\frac{D_p(2,2)}{p}\right)=\left(\frac{2u_{p-1}(-2,2)-1}{p}\right)\left(\frac{U(p-2)U((p-1)/2)}{p}\right).\tag{4.12}$$

Clearly,

$$u_{p-1}=u_{p-\left(\frac{(-2)^2-4\times 2}{p}\right)}\equiv 0\ (\mathrm{mod}\ p)$$

by (4.2). Thus

$$\left(\frac{2u_{p-1}(-2,2)-1}{p}\right)=\left(\frac{-1}{p}\right)=1.$$

In view of (4.11) and Lemma 4.1, we see that

$$\begin{aligned}4U(p-2)&\equiv -u_3+3u_1+2^{p-1}u_p\\&\equiv -2+3\times 1+\left(\frac{(-2)^2-4\times 2}{p}\right)=2 \pmod p\end{aligned}$$

and

$$\begin{aligned}4U\left(\frac{p-1}{2}\right)&=\frac{p+1}{2}u_{(p+3)/2}-2\left(\frac{p-1}{2}-1\right)u_{(p-1)/2}\\&\quad+2^{-(p-1)/2}\left((p+3)u_{(p-1)/2}-\frac{p-1}{2}u_{(p+3)/2}\right)\\&\equiv\frac{1}{2}\left(1+\left(\frac{2}{p}\right)\right)u_{(p+3)/2}+3\left(1+\left(\frac{2}{p}\right)\right)u_{(p-1)/2}\\&\equiv u_{(p+3)/2}+6u_{(p-1)/2}=-2\times(-4)^{\lfloor\frac{p+1}{8}\rfloor} \pmod p.\end{aligned}$$

(Note that we use (4.9) in the last step.)

Combining the last paragraph with (4.12), we immediately obtain that

$$\left(\frac{D_p(2,2)}{p}\right)=1.$$

This concludes our proof of Theorem 1.2. □

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