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ARITHMETIC PROGRESSIONS REPRESENTED BY DIAGONAL TERNARY QUADRATIC FORMS

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ABSTRACT. Let $d > r \geqslant 0$ be integers. For positive integers a, b, c, if any term of the arithmetic progression $\{r + dn : n = 0, 1, 2, \ldots\}$ can be written as $ax^2+by^2+cz^2$ with $x, y, z \in \mathbb{Z}$, then the form $ax^2+by^2+cz^2$ is called (d, r)-universal. In this paper, via the theory of ternary quadratic forms we study the (d, r)-universality of some diagonal ternary quadratic forms conjectured by L. Pehlivan and K. S. Williams, and Z.-W. Sun. For example, we prove that $2x^2 + 3y^2 + 10z^2$ is $(8, 5)$-universal, $x^2 + 3y^2 + 8z^2$ and $x^2 + 2y^2 + 12z^2$ are $(10, 1)$-universal and $(10, 9)$-universal, and $3x^2 + 5y^2 + 15z^2$ is $(15, 8)$-universal.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. The Gauss-Legendre theorem on sums of three squares states that $\{x^2 + y^2 + z^2 : x, y, z \in \mathbb{Z}\} = \mathbb{N} \setminus \{4^k(8l + 7) : k, l \in \mathbb{N}\}$. A classical topic in the study of number theory asks, given a quadratic polynomial f and an integer n, how can we decide when f represents n over the integers? This topic has been extensively investigated. It is known that for any $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ the exceptional set

$$E(a, b, c) = \mathbb{N} \setminus \{ax^2 + by^2 + cz^2 : x, y, z \in \mathbb{Z}\}$$

is infinite, see, e.g., [4].

An integral quadratic form f is called *regular* if it represents each integer represented by the genus of f. L. E. Dickson [3, pp. 112-113] listed all the 102 regular ternary quadratic forms $ax^2+by^2+cz^2$ together with the explicit characterization of $E(a, b, c)$, where $1 \leqslant a \leqslant b \leqslant c \in \mathbb{Z}^+$ and $\gcd(a, b, c) = 1$. In this direction, W. C. Jagy, I. Kaplansky and A. Schiemann [7] proved that there are at most 913 regular positive definite integral ternary quadratic forms.

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By the Gauss-Legendre theorem, for any $n \in \mathbb{N}$ we can write $4n+1 = x^2+y^2+z^2$ with $x,y,z \in \mathbb{Z}$. It is also known that for any $n \in \mathbb{N}$ we can write $2n+1$ as $x^2+y^2+2z^2$ (or $x^2+2y^2+3z^2$, or $x^2+2y^2+4z^2$) with $x,y,z \in \mathbb{Z}$ (see, e.g., Kaplansky [10]). Thus, it is natural to introduce the following definition.

Definition 1.1. Let $d \in \mathbb{Z}^+$ and $r \in \{0,\ldots,d-1\}$. For $a,b,c \in \mathbb{Z}$, if any $dn+r$ with $n \in \mathbb{N}$ can be written as $ax^2+by^2+cz^2$ with $x,y,z \in \mathbb{Z}$, then we say that the ternary quadratic form $ax^2+by^2+cz^2$ is (d,r)-*universal*.

In 2008, A. Alaca, S. Alaca and K. S. Williams [1] proved that there is no binary positive definite quadratic form which can represent all nonnegative integers in a residue class.

Z.-W. Sun [16] proved that $x^2+3y^2+24z^2$ is $(6,1)$-universal. Moreover, in 2017 he [17, Remark 3.1] confirmed his conjecture that for any $n \in \mathbb{Z}^+$ and $\delta \in \{0,1\}$ we can write $6n+1$ as $x^2+3y^2+6z^2$ with $x,y,z \in \mathbb{Z}$ and $x \equiv \delta \pmod 2$. This implies that $4x^2+3y^2+6z^2$ and $x^2+12y^2+6z^2$ are $(6,1)$-universal. On August 2, 2017, Sun [18] published on OEIS his list (based on his computation) of all possible candidates of (d,r)-universal irregular ternary quadratic forms $ax^2+by^2+cz^2$ with $1 \leqslant a \leqslant b \leqslant c$ and $3 \leqslant d \leqslant 30$. For example, he conjectured that

$$x^2+3y^2+7z^2,\ x^2+3y^2+42z^2,\ x^2+3y^2+54z^2$$

are all $(6,1)$-universal, $x^2+7y^2+14z^2$ is $(7,1)$-universal and $x^2+2y^2+7z^2$ is $(7,r)$-universal for each $r=1,2,3$. In 2018 L. Pehlivan and K. S. Williams [14] also investigated such problems independently, actually they studied (d,r)-universal quadratic forms $ax^2+by^2+cz^2$ with $1 \leqslant a \leqslant b \leqslant c$ and $3 \leqslant d \leqslant 11$.

Pehlivan and Williams [14] considered the $(8,1)$-universality of $x^2+8y^2+24z^2$, $x^2+2y^2+64z^2$ and $x^2+8y^2+64z^2$ open. However, B. W. Jones and G. Pall [9] proved in 1939 that for any $n \in \mathbb{N}$ we can write

$$8n+1 = x^2+8y^2+64z^2 = x^2+2(2y)^2+64z^2$$

with $x,y,z \in \mathbb{Z}$, and hence $x^2+2y^2+64z^2$ and $x^2+8y^2+64z^2$ are indeed $(8,1)$-universal. As $8x(x+1)/2+1=(2x+1)^2$, the $(8,1)$-universality of $x^2+8y^2+24z^2$ yields $\{x(x+1)/2+y^2+3z^2:\ x,y,z \in \mathbb{Z}\} = \mathbb{N}$, which was conjectured by Sun [15] and confirmed in [5].

The first part and part (ii) with $i \in \{2,3\}$ of the following result were conjectured by Pehlivan and Williams [14], as well as Sun [18].

Theorem 1.1. (i) *The form* $2x^2+3y^2+10z^2$ *is* $(8,5)$*-universal.*

(ii) *Let* $n\in\mathbb{Z}^+$, $\delta\in\{1,9\}$ *and* $i\in\{1,2,3\}$. *Then* $10n+\delta=x_1^2+2x_2^2+3x_3^2$ *for some* $(x_1,x_2,x_3)\in\mathbb{Z}^3$ *with* $2\mid x_i$.

In the spirit of Sun [19], it is easy to see that Theorem 1.1(i) has the following equivalent form: For any $n\in\mathbb{N}$ there are $x,y,z\in\mathbb{Z}$ and $\delta\in\{0,1\}$ such that $n=x(x+1)+3y(y+1)/2+5z(z+\delta)$.

Kaplansky [10] showed that there are at most 23 positive definite integral ternary quadratic forms that can represent all positive odd integers (19 for sure and 4 plausible candidates, see also Jagy [6] for further progress). Using one of the 19 forms, we obtain the following result originally conjectured by Sun [18].

Theorem 1.2. *The forms* $x^2+3y^2+14z^2$ *and* $2x^2+3y^2+7z^2$ *are both* $(14,7)$*-universal.*

Now we turn to study Sun's conjectural $(15,r)$-universality of some positive definite integral ternary quadratic forms.

Theorem 1.3. (i) *For any* $n\in\mathbb{N}$ *and* $i\in\{1,2,3\}$, *there exists* $(x_1,x_2,x_3)\in\mathbb{Z}^3$ *with* $3\mid x_i$ *such that* $15n+5=2x_1^2+3x_2^2+5x_3^2$.

(ii) *The form* $x^2+y^2+15z^2$ *is* $(15,5r)$*-universal for* $r=1,2$, *and* $3x^2+3y^2+5z^2$ *is* $(15,5)$*-universal.*

(iii) *For any* $r=1,2$, *both* $x^2+y^2+30z^2$ *and* $2x^2+3y^2+5z^2$ *are* $(15,5r)$*-universal. Also, the forms* $x^2+6y^2+15z^2$ *and* $3x^2+3y^2+10z^2$ *are* $(15,10)$*-universal.*

(iv) *The form* $x^2+2y^2+15z^2$ *is* $(15,3r)$*-universal for each* $r=1,2,3,4$, *and the form* $3x^2+5y^2+10z^2$ *is* $(15,3r)$*-universal for* $r=1,4$.

Theorem 1.4. (i) *The form* $x^2+3y^2+5z^2$ *is* $(15,3r)$*-universal for each* $r=1,2,3,4$. *Also,* $x^2+5y^2+15z^2$ *is* $(15,3r)$*-universal for* $r=2,3$.

(ii) *The form* $x^2+3y^2+15z^2$ *is* $(15,r)$*-universal for each* $r\in\{1,7,13\}$. *Also, the form* $x^2+15y^2+30z^2$ *is* $(15,r)$*-universal for* $r=1,4$, *and the form* $x^2+10y^2+15z^2$ *is* $(15,r)$*-universal for all* $r\in\{4,11,14\}$.

(iii) *The form* $3x^2+5y^2+6z^2$ *is* $(15,r)$*-universal for each* $r\in\{8,11,14\}$. *Also,* $3x^2+5y^2+15z^2$ *and* $3x^2+5y^2+30z^2$ *are both* $(15,8)$*-universal.*

Remark 1.1. Our proof of Theorem 1.4 relies heavily on the genus theory of quadratic forms as well as the Siegel-Minkowski formula.

We will give a brief overview of the theory of ternary quadratic forms in the next section, and show Theorem 1.1-1.4 in Sections 3-5 respectively.

2. Some preparations

Let

$$f(x,y,z)=ax^2+by^2+cz^2+ryz+szx+txy \tag{2.1}$$

be a positive definite ternary quadratic form with integral coefficients. Its associated matrix is

$$A=\begin{pmatrix}2a & t & s\\ t & 2b & r\\ s & r & 2c\end{pmatrix}.$$

The discriminant of f is defined by $d(f):=\det(A)/2$.

The following lemma is a fundamental result on integral representations of quadratic forms (cf. [2, pp.129]).

Lemma 2.1. *Let f be a nonsingular integral quadratic form and let m be a nonzero integer represented by f over the real field $\mathbb{R}$ and the ring $\mathbb{Z}_p$ of p-adic integers for each prime p. Then m is represented by some form f^* over $\mathbb{Z}$ with f^* in the same genus of f.*

Now, we introduce some standard notations in the theory of quadratic forms which can be found in [2, 11, 12]. For the positive definite ternary quadratic form f given by (2.1) and $n\in\mathbb{N}$, we write

$$r(n,f):=|\{(x,y,z)\in\mathbb{Z}^3: f(x,y,z)=n\}|$$

(where $|S|$ denotes the cardinality of a set S), and let

$$r(n,\mathrm{gen}(f)):=\Bigg(\sum_{f^*\in\mathrm{gen}(f)}\frac{1}{|\mathrm{Aut}(f^*)|}\Bigg)^{-1}\sum_{f^*\in\mathrm{gen}(f)}\frac{r(n,f^*)}{|\mathrm{Aut}(f^*)|},$$

where the summation is over a set of representatives of the classes in $\mathrm{gen}(f)$, and $\mathrm{Aut}(f^*)$ is the group of integral isometries of f^*.

We also need the following result obtained from the Siegel-Minkowski formula and the knowledge of local densities.

Lemma 2.2. ([20, Lemma 4.1]) *Let f be a positive ternary quadratic form with discriminant $d(f)$. Let $m\in\{1,2\}$ and suppose that m is represented by $\mathrm{gen}(f)$. Then, for each prime $p\nmid 2md(f)$, we have*

$$\frac{r(mp^2,\mathrm{gen}(f))}{r(m,\mathrm{gen}(f))}=p+1-\left(\frac{-md(f)}{p}\right), \tag{2.2}$$

where $(\frac{\cdot}{p})$ is the Legendre symbol.

3. Proof of Theorem 1.1

Lemma 3.1. *For any $n \in \mathbb{Z}^+$ and $\delta \in \{1, 9\}$, we can write $10n + \delta = x^2 + 2y^2 + 3z^2$ with $x, y, z \in \mathbb{Z}$ and $y^2 + z^2 \neq 0$.*

Proof. By [3, pp.112–113] we can write $10n + \delta = x^2 + 2y^2 + 3z^2$ with $x, y, z \in \mathbb{Z}$; if $10n + \delta$ is not a square then $y^2 + z^2$ is obviously nonzero.

Now suppose that $10n + \delta = m^2$ for some $m \in \mathbb{N}$. As $n > 0$, we have $m > 1$.

Case 1. m has a prime factor $p > 3$.

In this case, by Lemma 2.2 we have

$$r(p^2, x^2 + 2y^2 + 3z^2) = 2\left(p + 1 - \left(\frac{-6}{p}\right)\right).$$

Hence, $r(m^2, x^2 + 2y^2 + 3z^2) \geqslant r(p^2, x^2 + 2y^2 + 3z^2) > 2$. Thus, for some $(r, s, t) \in \mathbb{Z}^3$ with $s^2 + t^2 \neq 0$ we have $10n + \delta = m^2 = r^2 + 2s^2 + 3t^2$.

Case 2. $10n + \delta = m^2 = 3^{2k}$ with $k \in \mathbb{Z}^+$.

In this case,

$$10n + \delta = 3^{2k} = (2 \times 3^{k-1})^2 + 2 \times (3^{k-1})^2 + 3 \times (3^{k-1})^2.$$

In view of the above, we have completed the proof. □

Lemma 3.2. *If $n = 2x^2 + 3y^2 > 0$ with $x, y \in \mathbb{Z}$ and $5 \mid n$, then we can write $n = 2u^2 + 3v^2$ with $u, v \in \mathbb{Z}$ and $5 \nmid uv$.*

Proof. We use induction on $k = \mathrm{ord}_5(\gcd(x, y))$, the 5-adic order of the greatest common divisor of x and y.

When $k = 0$, the desired result holds trivially.

Now let $k \geqslant 1$ and assume the desired result for smaller values of k. Write $x = 5^k x_0$ and $y = 5^k y_0$, where x_0 and y_0 are integers not all divisible by 5. Then $x_0 + 6y_0$ or $x_0 - 6y_0$ is not divisible by 5. Hence we may choose $\varepsilon \in \{\pm 1\}$ such that $5 \nmid x_0 + 6\varepsilon y_0$. Set $x_1 = 5^{k-1}(x_0 + 6\varepsilon y_0)$ and $y_1 = 5^{k-1}(4x_0 - \varepsilon y_0)$. Then $\mathrm{ord}_5(\gcd(x_1, y_1)) = k - 1$. Note that

$$5^{2k}(2x_0^2 + 3y_0^2) = 5^{2k-2}(2(x_0 + 6\varepsilon y_0)^2 + 3(4x_0 - \varepsilon y_0)^2) = 2x_1^2 + 3y_1^2.$$

So, applying the induction hypothesis we immediately obtain the desired result. □

Proof of Theorem 1.1. (i) It is easy to see that $8n + 5$ can be represented by the genus of $f(x, y, z) = 2x^2 + 3y^2 + 10z^2$. There are two classes in the genus of f, and the one not containing f has the representative $g(x,y,z) = 3x^2+5y^2+5z^2+2yz-2zx+2xy$. It is easy to verify the following identity:

$$f\left(\frac{x}{2}+y-z,\ y+z,\ \frac{x}{2}\right) = g(x,y,z). \tag{3.1}$$

Suppose that $8n+5 = g(x,y,z)$ for some $x,y,z \in \mathbb{Z}$. Then

$$1 \equiv 8n+5 = g(x,y,z) \equiv 3x^2+(y+z)^2+2x(y-z) \pmod 4.$$

Hence $y \not\equiv z \pmod 2$ and $2 \mid x$. In light of the identity (3.1), $8n+5$ is represented by f over $\mathbb{Z}$.

By Lemma 2.1 and the above, $8n+5$ can be represented by $2x^2+3y^2+10z^2$ over $\mathbb{Z}$.

(ii) Let $h(x,y,z) = x^2+2y^2+3z^2$. By [3, pp.112–113], we can write $10n+\delta = h(x,y,z)$ for some $x,y,z \in \mathbb{Z}$.

We claim that there are $u,v,w \in \mathbb{Z}$ with $u-2v+4w \equiv 0 \pmod 5$ such that $10n+\delta = h(u,v,w)$. Here we handle the case $\delta = 1$. (The case $\delta = 9$ can be handled similarly.)

Case 1. $x^2 \equiv -1 \pmod 5$.

It is easy to see that $y^2 \equiv 0 \pmod 5$ and $z^2 \equiv -1 \pmod 5$, or $y^2 \equiv 1 \pmod 5$ and $z^2 \equiv 0 \pmod 5$. When $y^2 \equiv 0 \pmod 5$ and $z^2 \equiv -1 \pmod 5$, without loss of generality we may assume that $z \equiv x \pmod 5$ (otherwise, we may replace z by $-z$). If $y^2 \equiv 1 \pmod 5$ and $z^2 \equiv 0 \pmod 5$, then we simply assume $y \equiv -2x \pmod 5$ without loss of generality. Note that our choice of y and z meets the requirement $x-2y+4z \equiv 0 \pmod 5$.

Case 2. $x^2 \equiv 0 \pmod 5$.

Clearly, we have $y^2 \equiv -1 \pmod 5$ and $z^2 \equiv 1 \pmod 5$. Without loss of generality, we may assume that $y \equiv 2z \pmod 5$ and hence $x-2y+4z \equiv 0 \pmod 5$.

Case 3. $x^2 \equiv 1 \pmod 5$.

Apparently, we have $y^2 \equiv z^2 \pmod 5$. By Lemmas 3.1 and 3.2, we may simply assume that $5 \nmid yz$. When $y^2 \equiv z^2 \equiv x^2 \equiv 1 \pmod 5$, without loss of generality we may assume that $x \equiv y \equiv -z \pmod 5$. If $y^2 \equiv z^2 \equiv (2x)^2 \equiv -1 \pmod 5$, then we may assume that $y \equiv z \equiv 2x \pmod 5$ without any loss of generality. So, in this case our choice of y and z also meets the requirement $x-2y+4z \equiv 0 \pmod 5$.

In view of the above analysis, we may simply assume $x-2y+4z \equiv 0 \pmod 5$ without any loss of generality. Note that $h(x,y,z) = h(x^*,y^*,z^*)$, where

$$
\begin{aligned}
z^* =& \frac{x-2y+4z}{5} \not\equiv z \pmod 2,\\
x^* =& 2y-z+2z^* \not\equiv x \pmod 2,\\
y^* =& y-3z+3z^* \not\equiv y \pmod 2.
\end{aligned}
$$

So we have the desired result in part (ii) of Theorem 1.1. □

4. Proofs of Theorems 1.2-1.3

Proof of Theorem 1.2. By [10], we can write $2n+1 = F(r,s,t)$ with $r,s,t \in \mathbb{Z}$, where $F(x,y,z) = x^2+3y^2+2yz+5z^2$. Since

$$(2r-3t)^2 + 3(r+2t)^2 + 14s^2 = 7F(r,s,t)$$

and

$$2(s+3t)^2 + 3(2s-t)^2 + 7r^2 = 7F(r,s,t),$$

we see that $7(2n+1)$ is represented by the form $x^2+3y^2+14z^2$ as well as the form $2x^2+3y^2+7z^2$. □

Proof of Theorem 1.3. (i) By [3, pp.112–113], we may write $3n+1 = r^2+s^2+6t^2$ with $r,s,t \in \mathbb{Z}$. One may easily verify the following identities:

$$
\begin{aligned}
5(r^2+s^2+6t^2) &= 2(r\pm 3t)^2 + 3(r\mp 2t)^2 + 5s^2\\
&= 2(s\pm 3t)^2 + 3(s\mp 2t)^2 + 5r^2.
\end{aligned}
$$

As exactly one of r and s is divisible by 3, one of the the four numbers $r\pm 2t$ and $s\pm 2t$ is a multiple of 3. This proves part (i) of Theorem 1.3.

(ii) Let $r \in \{1,2\}$. By [3, pp.112–113], for some $x,y,z \in \mathbb{Z}$ we have $3n+r = x^2+y^2+3z^2$. Hence

$$15n+5r = 5(x^2+y^2+3z^2) = (x+2y)^2 + (2x-y)^2 + 15z^2.$$

By [3, pp.112–113], we may write $3n+1 = u^2+3v^2+3w^2$ with $u,v,w \in \mathbb{Z}$. Thus

$$15n+5 = 5(u^2+3v^2+3w^2) = 3(v+2w)^2 + 3(2v-w)^2 + 5u^2.$$

(iii) Let $r \in \{1,2\}$. By [3, pp.112–113], there are $u,v,w \in \mathbb{Z}$ such that $3n+r = r^2+s^2+6t^2$. There are two classes in the genus of the form $x^2+y^2+30z^2$, and the one not containing $x^2+y^2+30z^2$ has a representative

$2x^2+3y^2+5z^2$. Since

$$\begin{aligned}15n+5r&=5(x^2+y^2+6z^2)\\&=(x+2y)^2+(2x-y)^2+30z^2\\&=2(x+3z)^2+3(x-2z)^2+5y^2,\end{aligned}$$

we see that $15n+5r$ is represented by $x^2+y^2+30z^2$ as well as $2x^2+3y^2+5z^2$.

By [3, pp.112–113], we can write $3n+2=2u^2+3v^2+3w^2$ with $u,v,w\in\mathbb{Z}$. There are two classes in the genus of $x^2+6y^2+15z^2$, and the one not containing $x^2+6y^2+15z^2$ has a representative $3x^2+3y^2+10z^2$. As

$$\begin{aligned}15n+10&=5(2u^2+3v^2+3w^2)\\&=(2u+3v)^2+6(u-v)^2+15w^2\\&=3(u+2v)^2+3(2v-w)^2+10u^2,\end{aligned}$$

we see that $15n+10$ is represented by $x^2+6y^2+15z^2$ as well as $3x^2+3y^2+10z^2$.

(iv) Let $r\in\{1,2,3,4\}$. By [3, pp.112–113], there are $x,y,z\in\mathbb{Z}$ such that $5n+r=x^2+2y^2+5z^2$. Hence

$$15n+3r=3(x^2+2y^2+5z^2)=(x-2y)^2+2(x+y)^2+15z^2.$$

Now let $r\in\{1,4\}$. By [3, pp.112–113], there are $u,v,w\in\mathbb{Z}$ such that $5n+r=u^2+5v^2+10w^2$. Thus,

$$15n+3r=3(u^2+5v^2+10w^2)=3u^2+5(v-2w)^2+10(v+w)^2.$$

In view of the above, we have completed the proof of Theorem 1.3. □

5. Proof of Theorem 1.4

Proof of Theorem 1.4(i). Let $r\in\{1,2,3,4\}$. It is easy to see that $15n+3r$ can be represented by $f_1(x,y,z)=x^2+3y^2+5z^2$ locally. There are two classes in the genus of f_1, and the one not containing f_1 has a representative $f_2(x,y,z)=x^2+2y^2+8z^2-2yz$. One may easily verify the following identities:

$$f_1\left(\frac{x-y-z}{3}-2z,\ \frac{x-y-z}{3}+y,\ \frac{x-y-z}{3}+z\right)=f_2(x,y,z), \tag{5.1}$$

$$f_1\left(\frac{x+y+z}{3}+2z,\ \frac{x+y+z}{3}-y,\ \frac{x+y+z}{3}-z\right)=f_2(x,y,z). \tag{5.2}$$

Suppose that $15n+3r=f_2(x,y,z)$ with $x,y,z\in\mathbb{Z}$. Then

$$x^2-(y+z)^2\equiv f_2(x,y,z)\equiv 0 \pmod 3,$$

and hence $(x-y-z)/3$ or $(x+y+z)/3$ is an integer. Therefore, by (5.1), (5.2) and Lemma 2.1, we obtain that $x^2+3y^2+5z^2$ is $(15,3r)$-universal.

Now let $r\in\{2,3\}$. One can easily verify that $15n+3r$ is represented by the genus of $g_1(x,y,z)=x^2+5y^2+15z^2$. There are two classes in the genus of g_1, and the one not containing g_1 has a representative $g_2(x,y,z)=4x^2+4y^2+5z^2+2xy$. It is easy to verify the identity

$$g_1\left(y+\frac{x+y\mp 5z}{3},\ x-\frac{x+y\pm z}{3},\ \frac{x+y\pm z}{3}\right)=g_2(x,y,z). \tag{5.3}$$

If $15n+3r=g_2(x,y,z)$ with $x,y,z\in\mathbb{Z}$, then

$$(x+y)^2-z^2\equiv g_2(x,y,z)\equiv 0 \pmod 3)$$

Thus, with the help of (5.3) and Lemma 2.1, we obtain the desired result. □

Lemma 5.1. (Oh [13]) *Let V be a positive definite ternary quadratic space over $\mathbb{Q}$. For any isometry $T\in O(V)$ of infinite order,*

$$V_T=\{x\in V: \text{there is a positive integer } k \text{ such that } T^k(x)=x\}$$

is a subspace of V of dimension one, and $T(x)=\det(T)x$ for any $x\in V_T$.

Remark 5.1. Unexplained notations of quadratic space can be found in [2, 11, 12].

Lemma 5.2. *Let $n\in\mathbb{N}$ and $r\in\{1,7,13\}$. If we can write $15n+r=f_2(x,y,z)=3x^2+4y^2+4z^2+2yz$ with $x,y,z\in\mathbb{Z}$, then there are $u,v,w\in\mathbb{Z}$ with $u+2v-2w\not\equiv 0 \pmod 3$ such that $15n+r=f_2(u,v,w)$.*

Proof. Suppose that every integral solution of the equation $f_2(x,y,z)=15n+r$ satisfies $x+2y-2z\equiv 0 \pmod 3$. We want to deduce a contradiction.

Let

$$T=\begin{pmatrix}1/3 & 2/3 & -2/3\\ -2/3 & 2/3 & 1/3\\ 2/3 & 1/3 & 2/3\end{pmatrix},$$

and let V be the quadratic space corresponding to f_2. Since

$$f_2\left(\frac{x+2y-2z}{3},-x+z+\frac{x+2y-2z}{3},x+y-\frac{x+2y-2z}{3}\right)=f_2(x,y,z), \tag{5.4}$$

we have $T\in O(V)$. One may easily verify that the order of T is infinite and the space V_T defined in Lemma 5.1 coincides with $\{(0,t,t):t\in\mathbb{Q}\}$. As $15n+r\neq f_2(0,t,t)$ for any $t\in\mathbb{Z}$, we have $15n+r=f_2(x_0,y_0,z_0)$ for some

$(x_0,y_0,z_0)\in\mathbb{Z}^3\setminus V_T$. Clearly, the set $\{T^k(x_0,y_0,z_0):k\geqslant 0\}$ is infinite and its elements are solutions to the equation $f_2(x,y,z)=15n+r$. This leads a contradiction since the number of integral representations of any integer by a positive quadratic forms is finite. $\square$

Lemma 5.3. (Jagy [8]) *If $n=2x^2+2xy+3y^2$ $(x,y\in\mathbb{Z})$ is a positive integer divisible by 3, then there are $u,v\in\mathbb{Z}$ with $3\nmid uv$ such that $n=2u^2+2uv+3v^2$.*

The following lemma is a known result, see, e.g., [8, 9, 13, 17].

Lemma 5.4. *If $n=x^2+y^2$ $(x,y\in\mathbb{Z})$ is a positive integer divisible by 5, then $n=u^2+v^2$ for some $u,v\in\mathbb{Z}$ with $5\nmid uv$.*

Proof of Theorem 1.4(ii). (a) For each $r\in\{1,7,13\}$, it is easy to see that $15n+r$ can be represented by the genus of $f_1(x,y,z)=x^2+3y^2+15z^2$. There are two classes in the genus of $f_1(x,y,z)$, and the one not containing f_1 has a representative $f_2(x,y,z)=3x^2+4y^2+4z^2+2yz$. One may easily verify the following identities:

$$f_1\left(x-y+z,\ \frac{x-2y}{3}-z,\ \frac{x+y}{3}\right)=f_2(x,y,z), \tag{5.5}$$

$$f_1\left(x+y-z,\ \frac{x-2z}{3}-y,\ \frac{x+z}{3}\right)=f_2(x,y,z). \tag{5.6}$$

Suppose that $15n+r=f_2(x,y,z)$ for some $x,y,z\in\mathbb{Z}$. As $3\nmid y$ or $3\nmid z$, when $3\nmid x$ we may assume that $(x+y)(x+z)\equiv 0\pmod 3$ (otherwise we may replace x by $-x$) without loss of generality. If $3\mid x$ and $y\not\equiv z\pmod 3$, then $3\mid yz$ and hence $(x+y)(x+z)\equiv 0\pmod 3$. In the remaining case $3\mid x$ and $y\equiv z\pmod 3$, we have $x+2y-2z\equiv 0\pmod 3$; however, we may apply Lemma 5.2 to choose integers $u,v,w\in\mathbb{Z}$ so that $15n+r=f_2(u,v,w)$ and $u+2v-2w\not\equiv 0\pmod 3$.

In view of the above analysis, there always exist $u,v,w\in\mathbb{Z}$ with $(u+v)(u+w)\equiv 0\pmod 3$ such that $15n+r=f_2(u,v,w)$. With the help of (5.5), (5.6), and Lemma 2.1, we obtain the $(15,r)$-universality of $x^2+3y^2+15z^2$.

(b) Let $r\in\{1,4\}$. One can easily verify that $15n+r$ can be represented by $g_1(x,y,z)=x^2+15y^2+30z^2$ locally. There are two classes in the genus of g_1, and the one not containing g_1 has a representative $g_2(x,y,z)=6x^2+9y^2+10z^2-6xy$.

Suppose that $15n+r=g_2(x,y,z)$ with $x,y,z\in\mathbb{Z}$. Clearly $3\nmid z$. Since $15n+r\ne 10z^2$, by Lemma 5.3 we may assume that x and y are not all divisible by 3. Thus we just need to consider the following two cases.

Case b1. $3\nmid x$.

When this occurs, without loss of generality, we may assume that $x\equiv -z \pmod 3$ (otherwise we may replace z be $-z$). In view of the identity

$$g_1\left(x-3y,\ \frac{x-2z}{3},\ \frac{x+z}{3}\right)=g_2(x,y,z), \tag{5.7}$$

there are $x^*,y^*,z^*\in\mathbb{Z}$ such that $15n+\delta=g_1(x^*,y^*,z^*)$.

Case b2. $3\mid x$ and $3\nmid y$

In this case, with the help of the identity

$$g_2(x-y,\ -y,\ z)=g_2(x,y,z),$$

we return to Case b1 since $x-y\not\equiv 0 \pmod 3$.

Now applying Lemma Lem2.1 we immediately obtain the $(15,r)$-universality of $x^2+5y^2+30z^2$.

(c) For any $r\in\{4,11,14\}$, it is easy to verify that $15n+r$ can be represented by $h_1(x,y,z)=x^2+10y^2+15z^2$ locally. There are two classes in the genus of h_1, and the one not containing h_1 has a representative $h_2(x,y,z)=5x^2+5y^2+6z^2$.

Suppose that $15n+r=h_2(x,y,z)$ with $x,y,z\in\mathbb{Z}$. Since $15n+r\ne 6z^2$, by Lemma 5.4 and the symmetry of x and y, we simply assume $5\nmid y$ without loss of generality. We claim that we may adjust the signs of x,y,z to satisfy the congruence $(2x+y+2z)(x-2y-2z)\equiv 0 \pmod 5$.

Case c1. $x^2\equiv y^2 \pmod 5$.

Without loss of generality, we may assume $x\equiv y\equiv z \pmod 5$ if $x^2\equiv y^2\equiv z^2 \pmod 5$, and $x\equiv -y\equiv -2z \pmod 5$ if $x^2\equiv y^2\equiv -z^2 \pmod 5$. So our claim holds in this case.

Case c2. $x^2\equiv -y^2 \pmod 5$.

If $x^2\equiv -y^2\equiv z^2 \pmod 5$, without loss of generality, we may assume that $x\equiv -2y\equiv z \pmod 5$. If $x^2\equiv -y^2\equiv -z^2 \pmod 5$, we may assume that $x\equiv -2y\equiv 2z \pmod 5$ without loss of any generality. Thus x,y,z satisfy the desired congruence in our claim.

Case c3. $x^2\equiv 0 \pmod 5$.

If $y^2\equiv z^2 \pmod 5$, we may assume that $y\equiv -z \pmod 5$. If $y^2\equiv -z^2 \pmod 5$, without loss of generality, we may assume that $y\equiv -2z \pmod 5$. Clearly, our claim also holds in this case.

In view of the above analysis, there are $x,y,z\in\mathbb{Z}$ with $2x+y+2z\equiv 0 \pmod 5$ or $x-2y-2z\equiv 0 \pmod 5$ such that $15n+r=h_2(x,y,z)$. One may easily verify the following identities

$$h_1\left(x-2y,\ \frac{2x+y+2z}{5}-z,\ \frac{2x+y+2z}{5}\right)=h_2(x,y,z), \tag{5.8}$$

$$h_1\left(x+2y,\ \frac{x-2y-2z}{5}+z,\ \frac{x-2y-2z}{5}\right)=h_2(x,y,z). \tag{5.9}$$

With the help of (5.8), (5.9) and Lemma 2.1, the $(15,r)$-universality of $x^2+10y^2+15z^2$ is valid. □

Lemma 5.5. *Let $n\in\mathbb{N}$ and $g_2(x,y,z)=2x^2+8y^2+15z^2-2xy$. Assume that $15n+8=g_2(x,y,z)$ for some $x,y,z\in\mathbb{Z}$ with $y^2+z^2\neq 0$. Then $15n+8=g_2(u,v,w)$ for some $u,v,w\in\mathbb{Z}$ with $3\nmid v+w$.*

Proof. Suppose that every integral solution of the equation $g_2(x,y,z)=15n+8$ satisfies $y+z\equiv 0 \pmod 3$. We want to deduce a contradiction.

Let

$$T=\begin{pmatrix}1 & -2/3 & -2/3\\ 0 & -1/3 & -4/3\\ 0 & 2/3 & -1/3\end{pmatrix}$$

and let V be the quadratic space corresponding to g_2. Since

$$g_2\left(x+\frac{-2y-2z}{3},\ \frac{-y-4z}{3},\ \frac{2y-z}{3}\right)=g_2(x,y,z).$$

We have $T\in O(V)$. One may easily verify that the order of T is infinite and the space V_T defined in Lemma 5.1 coincides with $\{(t,0,0):t\in\mathbb{Q}\}$. By the assumption in the lemma, we have $15n+8=g_2(x_0,y_0,z_0)$ for some $(x_0,y_0,z_0)\in\mathbb{Z}^3\setminus V_T$. Note that the set $\{T^k(x_0,y_0,z_0):k\geqslant 0\}$ is infinite and all elements of this set are solutions to the equation $g_2(x,y,z)=15n+8$. This leads to a contradiction since the number of integral representations of any integer by a positive quadratic forms is finite. □

Proof of Theorem 1.4(iii). (a) Let $r\in\{8,11,14\}$. It is easy to see that $15n+r$ can be represented by $f_1(x,y,z)=3x^2+5y^2+6z^2$ locally. There are two classes in the genus of f_1, and the one not containing f_1 has a representative $f_2(x,y,z)=2x^2+6y^2+9z^2+6yz$.

Suppose that $15n+r=f_2(x,y,z)$ for some $x,y,z\in\mathbb{Z}$. Then $3\nmid x$.

Case 1. $3\nmid y$.

In this case, without loss of generality we may assume that $x \equiv -y \pmod 3$ (otherwise we may replace x by $-x$). In view of the identity

$$f_1\left(\frac{2x-y}{3}-z,\ -y,\ \frac{x+y}{3}+z\right)=f_2(x,y,z), \tag{5.10}$$

there are $x^*, y^*, z^* \in \mathbb{Z}$ such that $15n+\delta = f_1(x^*, y^*, z^*)$.

Case 2. $y^2+z^2 \neq 0$ and $3 \mid y$.

When this occurs, by Lemma 5.3 we may simply assume that $3 \nmid z$. With the help of the identity

$$f_2(x,\ y+z,\ -z)=f_2(x,y,z),$$

we return to Case 1.

Case 3. $15n+r=2m^2$ for some $m \in \mathbb{N}$.

When $15n+r=2m^2=2\times 2^{2k}$ with $k \geqslant 1$, we have

$$15n+r=2\times 2^{2k}=3\times(2^{k-1})^2+5\times(2^{k-1})^2+6\times 0^2.$$

Now suppose that m has a prime factor $p>5$. By Lemma 2.2 we have

$$r(2p^2,f_1)+r(2p^2,f_2)=2\left(p+1-\left(\frac{-5}{p}\right)\right)>10. \tag{5.11}$$

Clearly $r(2p^2,f_1)>5$ or $r(2p^2,f_2)>5$. When $r(2p^2,f_1)>5$, the number $2m^2$ can be represented by f_1 over $\mathbb{Z}$ since $r(2m^2,f_1)\geqslant r(2p^2,f_1)$. When $r(2p^2,f_2)>5$, there are $u,v,w\in\mathbb{Z}$ with $v^2+w^2\neq 0$ such that $f_2(u,v,w)=15n+r$. By Lemma 5.3, we return to Case 1 or Case 2.

In view of the above, by applying Lemma 2.1 we get the $(15,r)$-universality of $3x^2+5y^2+6z^2$.

(b) It is easy to see that $15n+8$ can be represented by the genus of $g_1(x,y,z)=3x^2+5y^2+15z^2$. There are two classes in the genus of g_1, and the one not containing g_1 has a representative $g_2(x,y,z)=2x^2+8y^2+15z^2-2xy$.

Suppose that the equation $15n+8=g_2(x,y,z)$ is solvable over $\mathbb{Z}$. We claim that there are $u,v,w\in\mathbb{Z}$ with $(u+w)(u-v-w)\equiv 0 \pmod 3$ such that $15n+8=g_2(u,v,w)$.

Case 1. $3 \mid x$.

Clearly, $3\nmid y$. If $3\nmid z$, without loss of generality we may assume that $z\equiv -y \pmod 3$ (otherwise we replace z by $-z$). Then $(u,v,w)=(x,y,z)$ meets our purpose.

Case 2. $3\nmid x$ and $y^2+z^2\neq 0$.

In this case, by Lemma 5.5 there are $x', y', z' \in \mathbb{Z}$ with $3 \nmid y' + z'$ such that $15n + 8 = g_2(x', y', z')$. If $x' \equiv y' \pmod 3$, then by using the identity

$$g_2(x - y,\ -y,\ z) = g_2(x, y, z),$$

we return to Case 1. If $x' \not\equiv y' \pmod 3$, then $3 \mid y'$ and $3 \nmid z'$ since $(x' + y')^2 \equiv 1 \pmod 3$ and $3 \nmid x'$. Without loss of generality, we may assume that $x' \equiv -z' \pmod 3$. So $(u, v, w) = (x', y', z')$ meets our purpose.

Case 3. $15n + 8 = 2m^2$ with $m \in \mathbb{N}$.

If $m = 2^k$ for some $k \in \mathbb{Z}^+$, then

$$2m^2 = 3 \times (2^{k-1})^2 + 5 \times (2^{k-1})^2 + 15 \times 0^2.$$

Now suppose that m has a prime factor $p > 5$. By Lemma 2.2, we have

$$r(2p^2, g_1) + r(2p^2, g_2) = 2\left(p + 1 - \left(\frac{-2}{p}\right)\right) > 10. \tag{5.12}$$

Clearly, $r(2p^2, g_1) > 5$ or $r(2p^2, g_2) > 5$. When $r(2p^2, g_1) > 5$, we have $r(2m^2, g_1) \geq r(2p^2, g_1) > 5$. If $r(2p^2, g_2) > 5$, then there exist $x_0, y_0, z_0 \in \mathbb{Z}$ with $y_0^2 + z_0^2 \neq 0$ such that $15n + 8 = g_2(x_0, y_0, z_0)$. So we are reduced to previous cases.

In view of the proved claim, the $(15, 8)$-universality of g_1 follows from Lemma 2.1 and the identities

$$g_1\left(\frac{x - 5z}{3} - y,\ -y + z,\ -\frac{x + z}{3}\right) = g_2(x, y, z),$$

$$g_1\left(\frac{x - y - z}{3} + y + 2z,\ y - z,\ \frac{x - y - z}{3}\right) = g_2(x, y, z).$$

(c) One may easily verify that $15n+8$ can be represented by $h_1(x, y, z) = 3x^2 + 5y^2 + 30z^2$ locally. There are two classes in the genus of h_1, and the one not containing h_1 has a representative $h_2(x, y, z) = 2x^2 + 15y^2 + 15z^2$.

Suppose that the equation $15n + 8 = h_2(x, y, z)$ for some $x, y, z \in \mathbb{Z}$. In light of Lemma 5.4, we may assume $5 \nmid y$ if $y^2 + z^2 > 0$. We claim that there are $u, v, w \in \mathbb{Z}$ with $(u - v + 2w)(u - 2v + w) \equiv 0 \pmod 5$ such that $15n + 8 = h_2(u, v, w)$. This holds trivially if $y^2 + z^2 > 0$ and $5 \mid z$. Below we discuss the remaining cases.

Case 1. $y^2 \equiv \varepsilon z^2 \pmod 5$ and $y^2 + z^2 \neq 0$, where $\varepsilon \in \{\pm 1\}$.

If $y^2 \equiv z^2 \equiv x^2 \pmod 5$, then we may assume that $x \equiv y \equiv z \pmod 5$. If $y^2 \equiv z^2 \equiv -x^2 \pmod 5$, without loss of generality we may assume that $x \equiv -2y \equiv 2z \pmod 5$. So, $(u, v, w) = (x, y, z)$ meets our requirement in the case $\varepsilon = 1$. The case $\varepsilon = -1$ can be handled similarly.

Case 2. $15n + 8$ is twice a square, say $2m^2$ with $m \in \mathbb{Z}^+$.

When $m=2^k$ with $k\in\mathbb{Z}^+$, we have

$$15n+8=2\times 2^{2k}=3\times(2^{k-1})^2+5\times(2^{k-1})^2+30\times 0^2.$$

Now assume that m has a prime factor $p>5$. By Lemma 2.2, we have

$$2r(2p^2,h_1)+r(2p^2,h_2)=2\left(p+1-\left(\frac{-1}{p}\right)\right)>10. \tag{5.13}$$

Clearly $r(2p^2,h_1)\ge 4$ or $r(2p^2,h_2)\ge 4$. If $r(2p^2,h_1)\ge 4$, then $r(2m^2,h_1)\ge r(2p^2,h_1)\ge 4$. When $r(2p^2,h_2)\ge 4$, there exist $u,v,w\in\mathbb{Z}$ with $v^2+w^2\ne 0$ such that $h_2(u,v,w)=2m^2$. Thus we are reduced to Case 1.

In view of the proved claim and the identities

$$\begin{aligned}h_2(x,y,z)=&h_1\left(2y+z,\ \frac{2x+3y-z}{5}-z,\ \frac{x-y+2z}{5}\right)\\=&h_1\left(y+2z,\ \frac{2x+y-3z}{5}+y,\ \frac{x-2y+z}{5}\right),\end{aligned}$$

by applying Lemma 2.1 we obtain the $(15,8)$-universality of $3x^2+5y^2+30z^2$. □

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