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p-ADIC ANALOGUES OF HYPERGEOMETRIC IDENTITIES AND THEIR APPLICATIONS

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ABSTRACT. In this paper, we confirm several conjectures of Z.-W. Sun on p-adic congruences. For example, for any odd prime p we prove that

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+2y^2 \text{ with } x,y\in\mathbb{Z},\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8,\end{cases}$$

where $A_n := \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$ $(n=0,1,2,\ldots)$ are the Apéry numbers.

1. INTRODUCTION

For $n, r \in \mathbb{N} = \{0, 1, 2, \ldots\}$, we define

$${}_{r+1}F_r\left[\begin{matrix}\alpha_0 & \alpha_1 & \cdots & \alpha_r\\ & \beta_1 & \cdots & \beta_r\end{matrix}\,\middle|\, z\right]_n := \sum_{k=0}^{n}\frac{(\alpha_0)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta_r)_k}\cdot\frac{z^k}{k!},$$

where $\alpha_0, \ldots, \alpha_r, \beta_1, \ldots, \beta_r, z \in \mathbb{C}$, and the Pochhammer symbol $(\alpha)_k$ is given by

$$(\alpha)_k := \begin{cases}\prod_{j=0}^{k-1}(\alpha+j) & \text{if } k\geq 1,\\ 1 & \text{if } k=0.\end{cases}$$

Such truncated hypergeometric series are sums of the first finite terms of the corresponding hypergeometric series. In the past decades, the arithmetic properties of the truncated hypergeometric series have been widely studied (cf. [1, 4, 7–17, 20, 21, 24, 26–29]).

The well-known Apéry numbers given by

$$A_n := \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2 \quad (n\in\mathbb{N}=\{0,1,\ldots\}),$$

were first introduced by Apéry to prove the irrationality of $\zeta(3)=\sum_{n=1}^{\infty}1/n^3$ (see [2, 19]). In 2012, Sun [22] studied finite sums involving Apéry numbers systematically and posed some

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conjectures; for example, he conjectured that for any odd prime p we have

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8. \end{cases} \tag{1.1}$$

Note that the above conjecture was also collected in [25, Conjecture 55]. We now state our first theorem.

Theorem 1.1. *For any odd prime p, the congruence* (1.1) *holds.*

Remark 1.1. In [22], Sun proved that (1.1) holds modulo any odd prime p.

In 2019, Sun [25, Conjectures 35 and 36] proposed a series of conjectural congruences involving the following polynomial in x:

$$\sum_{k=0}^{n-1}\varepsilon^k(2k+1)^{2l-1}\sum_{j=0}^{k}\binom{-x}{j}^m\binom{x-1}{k-j}^m,$$

where $\varepsilon\in\{\pm1\}$ and $n,l,m\in\mathbb{Z}^+=\{1,2,3,\ldots\}$. For any odd prime p and $m\in\{3,4,\ldots\}$, clearly

$$\sum_{k=0}^{p-1}\varepsilon^k(2k+1)\sum_{j=0}^{k}\binom{-x}{j}^m\binom{x-1}{k-j}^m=\sum_{j=0}^{p-1}\varepsilon^j\binom{-x}{j}^m\sum_{k=0}^{p-1-j}\varepsilon^k(2k+2j+1)\binom{x-1}{k}^m.$$

Denote by $\langle -x\rangle_p$ the least nonnegative residue of $-x$ modulo p. Clearly,

$$\binom{-x}{j}\equiv 0 \pmod p \quad\text{for}\quad j\in\{\langle -x\rangle_p+1,\ldots,p-1\}$$

and

$$\binom{x-1}{k}\equiv 0 \pmod p \quad\text{for}\quad k\in\{p-\langle -x\rangle_p,\ldots,p-1\}.$$

Therefore, by noting that $p-1-j\geq p-1-\langle -x\rangle_p$ for any $j\in\{0,\ldots,a\}$, we get

$$\begin{aligned}&\sum_{k=0}^{p-1}\varepsilon^k(2k+1)\sum_{j=0}^{k}\binom{-x}{j}^m\binom{x-1}{k-j}^m\\ \equiv&\sum_{j=0}^{p-1}\varepsilon^j\binom{-x}{j}^m\sum_{k=0}^{p-1}\varepsilon^k(2k+2j+1)\binom{x-1}{k}^m=(1-x)\Sigma_1+x\Sigma_2 \pmod{p^m},\end{aligned}\tag{1.2}$$

where

$$\begin{aligned}\Sigma_1:=&{}_{m+1}F_m\left[\begin{matrix}1-x & 1+\frac{1-x}{2} & 1-x & \cdots & 1-x\\ & \frac{1-x}{2} & 1 & \cdots & 1\end{matrix}\,\middle|\,(-1)^m\varepsilon\right]_{p-1}\\ &\times{}_mF_{m-1}\left[\begin{matrix}x & x & \cdots & x\\ & 1 & \cdots & 1\end{matrix}\,\middle|\,(-1)^m\varepsilon\right]_{p-1}\end{aligned}$$

and

$$\Sigma_2 := {}_{m+1}F_m\left[\begin{matrix} x & 1+\frac{x}{2} & x & \cdots & x \\ & \frac{x}{2} & 1 & \cdots & 1 \end{matrix}\middle| (-1)^m\varepsilon\right]_{p-1} \times {}_mF_{m-1}\left[\begin{matrix} 1-x & 1-x & \cdots & 1-x \\ & 1 & \cdots & 1 \end{matrix}\middle| (-1)^m\varepsilon\right]_{p-1}.$$

In view of the above, we are led to consider congruences concerning the truncated ${}_{m+1}F_m$ and ${}_mF_{m-1}$ hypergeometric series. This is the motivation of the remaining part of this paper.

Let p be an odd prime. Our results involve Morita's p-adic gamma function Γ_p (cf. [18]) which is the p-adic analogue of the classical gamma function Γ. Define $\Gamma_p(0) := 1$, and

$$\Gamma_p(n) := (-1)^n \prod_{\substack{1\le k<n \\ p\nmid k}} k \quad \text{for } n = 1, 2, 3, \dots.$$

We may regard Γ_p as a continuous function on the ring $\mathbb{Z}_p$ of p-adic integers, since $\mathbb{N}$ is a dense subset of $\mathbb{Z}_p$ with respect to p-adic norm. It follows that

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x & \text{if } p\nmid x, \\ -1 & \text{if } p \mid x. \end{cases} \tag{1.3}$$

For more properties of the p-adic gamma function, one may consult [13, 14, 17, 18].

We now state our second theorem.

Theorem 1.2. *Let p be an odd prime. Let $\alpha \in \mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p \mid p \nmid x\}$, $s = (\alpha + \langle -\alpha\rangle_p)/p$ and*

$$h_p(\alpha) = \frac{\Gamma_p\left(\frac{1+\alpha}{2}\right)\Gamma_p\left(\frac{1-3\alpha}{2}\right)}{\Gamma_p(1+\alpha)\Gamma_p(1-\alpha)\Gamma_p\left(\frac{1-\alpha}{2}\right)^2}.$$

Then the following congruence holds modulo p^3,

$${}_4F_3\left[\begin{matrix} \alpha & 1+\frac{\alpha}{2} & \alpha & \alpha \\ & \frac{\alpha}{2} & 1 & 1 \end{matrix}\middle| 1\right]_{p-1} \equiv \begin{cases} 2h_p(\alpha), & \text{if } \langle -\alpha\rangle_p \text{ is odd and } \langle -\alpha\rangle_p < \frac{2p+1}{3}, \\ (2-3s)ph_p(\alpha), & \text{if } \langle -\alpha\rangle_p \text{ is odd and } \langle -\alpha\rangle_p \ge \frac{2p+1}{3}, \\ sph_p(\alpha), & \text{if } \langle -\alpha\rangle_p \text{ is even and } \langle -\alpha\rangle_p < \frac{p+1}{3}, \\ \dfrac{(s-3s^2)p^2h_p(\alpha)}{2}, & \text{if } \langle -\alpha\rangle_p \text{ is even and } \langle -\alpha\rangle_p \ge \frac{p+1}{3}. \end{cases}$$

Remark 1.2. In 2017, He [9] studied the congruences modulo p^2 for primes $p \ge 5$ and $\alpha = 1/2, 1/3, 1/4$.

Let p be an odd prime. Mao and Pan [14] obtained a number of congruences modulo p^2 involving truncated hypergeometric identities and the p-adic gamma functions. For instance,they proved that for any $\alpha,\beta\in\mathbb{Z}_p$ with $\langle-\alpha\rangle_p\le\langle-\beta\rangle_p\le(p+\langle-\alpha\rangle_p-1)/2$ and $(\alpha-\beta+1)_{p-1}\not\equiv 0 \pmod{p^2}$ we have

$$ {}_4F_3\begin{bmatrix}\alpha & 1+\frac{\alpha}{2} & \alpha & \beta \\ & \frac{\alpha}{2} & 1 & \alpha-\beta+1\end{bmatrix}\Bigg|-1\Bigg]_{p-1}\equiv-(\alpha+\langle-\alpha\rangle_p)\cdot\frac{\Gamma_p(\alpha-\beta+1)}{\Gamma_p(1+\alpha)\Gamma_p(1-\beta)} \pmod{p^2}. $$

Letting $\beta=\alpha$ in the above congruence we get that

$$ {}_4F_3\begin{bmatrix}\alpha & 1+\frac{\alpha}{2} & \alpha & \alpha \\ & \frac{\alpha}{2} & 1 & 1\end{bmatrix}\Bigg|-1\Bigg]_{p-1}\equiv\frac{\alpha+\langle-\alpha\rangle_p}{\Gamma_p(1+\alpha)\Gamma_p(1-\alpha)} \pmod{p^2}. \tag{1.4} $$

Our next theorem is stronger than (1.4).

Theorem 1.3. *Let p be an odd prime and $\alpha\in\mathbb{Z}_p^{\times}$. Then we have*

$$ {}_4F_3\begin{bmatrix}\alpha & 1+\frac{\alpha}{2} & \alpha & \alpha \\ & \frac{\alpha}{2} & 1 & 1\end{bmatrix}\Bigg|-1\Bigg]_{p-1}\equiv\frac{\alpha+\langle-\alpha\rangle_p}{\Gamma_p(1+\alpha)\Gamma_p(1-\alpha)} \pmod{p^3}. $$

Remark 1.3. When $d\in\{2,3,4\}$, $\alpha=1/d$ and $p\equiv 1 \pmod{d}$, the result was conjectured by van Hamme [28] and confirmed by Swisher [26].

The proofs of Theorems 1.2 and 1.3 depend on the local-global theorem for p-adic supercongruences established by Pan, Tauraso and Wang [17]. Here we illustrate the local-global theorem briefly (the reader may refer to [17, Theorem 1.1] for details). For any prime $p>\binom{r+1}{2}$ the local-global theorem says that if a congruence modulo p^r holds over some r admissible hyperplanes of $\mathbb{Z}_p^n$, then it also holds over the whole $\mathbb{Z}_p^n$. In view of this, to show our theorems, we only need to prove them 'locally'.

We are going to show Theorem 1.1 in the next section. Theorems 1.2 and 1.3 will be proved in Section 3. In Section 4, we shall confirm some conjectures of Sun in [25, Conjectures 35 and 36] as applications of Theorems 1.2 and 1.3. In the last section, we will prove more conjectures of Z.-W. Sun by some known results.

2. Proof of Theorem 1.1

In order to show Theorem 1.1 we need the following lemmas.

Lemma 2.1 (Guo [6, (2.5)]). *We have the following identity*

$$ \binom{k}{i}\binom{k+i}{i}\binom{k}{j}\binom{k+j}{j}=\sum_{s=\max\{i,j\}}^{i+j}\binom{s}{i}\binom{s}{j}\binom{i+j}{s}\binom{k}{s}\binom{k+s}{s}. $$

The following result can be easily proved by induction.

Lemma 2.2. *For any $j\in\mathbb{N}$, we have*

$$ \sum_{s=j}^{2j}\binom{s}{j}^2\binom{2j}{s}\frac{(-1)^s}{2s+1}=\frac{\binom{2j}{j}^2}{(4j+1)\binom{4j}{2j}}. $$

The following lemma gives the well-known Euler's reflection formula and its p-adic analogue.

Lemma 2.3 ([18, pp. 369–371])**.** (i) *For any $z \in \mathbb{C} \setminus \mathbb{Z}$, we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{2.1}$$

(ii) *Let p be an odd prime. Then*

$$\Gamma_p(z)\Gamma_p(1-z) = (-1)^{p-\langle -z\rangle_p} \tag{2.2}$$

for all $z \in \mathbb{Z}_p$.

Lemma 2.4 ([3, Theorem 3.5.5])**.** *If $a+b=1$ and $e+f=2c+1$, then we have*

$${}_3F_2\left[\begin{matrix} a & b & c \\ & e & f \end{matrix}\middle|\, 1\right] = \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma((a+e)/2)\Gamma((a+f)/2)\Gamma((b+e)/2)\Gamma((b+f)/2)}.$$

The classical gamma function has the following Gauss multiplication formula [18, Page 371].

Lemma 2.5. *For any $m \in \mathbb{Z}^+$ and $z \in \mathbb{C}$ with $mz \notin -\mathbb{N}$, we have*

$$\prod_{0\le j<m} \Gamma\left(z+\frac{j}{m}\right) = (2\pi)^{(m-1)/2} m^{(1-2mz)/2}\Gamma(mz).$$

We obtain the following result.

Theorem 2.1. *Let p be an odd prime p. Then*

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} -\left(\frac{-1}{p}\right)\Gamma_p\left(\frac{1}{8}\right)^2\Gamma_p\left(\frac{3}{8}\right)^2 \pmod{p^2} & \text{if } p\equiv 1,3 \pmod 8, \\ 0 \pmod{p^2} & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

Remark 2.1. Theorem 2.1 is actually equivalent to Theorem 1.1. Let p be an odd prime with $p \equiv 1,3 \pmod 8$, and write $p = x^2+2y^2$ with $x, y \in \mathbb{Z}$. By [16] and [23],

$${}_4F_3\left[\begin{matrix} \frac12 & \frac14 & \frac34 \\ & 1 & 1 \end{matrix}\middle|\, 1\right]_{p-1} \equiv 4x^2-2p \pmod{p^2}.$$

From [17] we have

$${}_4F_3\left[\begin{matrix} \frac12 & \frac14 & \frac34 \\ & 1 & 1 \end{matrix}\middle|\, 1\right]_{p-1} \equiv -\left(\frac{-1}{p}\right)\Gamma_p\left(\frac18\right)^2\Gamma_p\left(\frac38\right)^2 \pmod{p^2}.$$

Combining the above, we get

$$4x^2-2p \equiv -\left(\frac{-1}{p}\right)\Gamma_p\left(\frac18\right)^2\Gamma_p\left(\frac38\right)^2 \pmod{p^2}.$$

Proof of Theorem 2.1. In view of Lemma 2.1, we have

$$\sum_{k=0}^{p-1} A_k = \sum_{k=0}^{p-1}\sum_{j=0}^{k}\binom{k+j}{j}^2\binom{k}{j}^2$$

$$=\sum_{k=0}^{p-1}\sum_{j=0}^{k}\sum_{s=j}^{2j}\binom{s}{j}^2\binom{2j}{s}\binom{k}{s}\binom{k+s}{s}$$
$$=\sum_{j=0}^{p-1}\sum_{s=j}^{2j}\binom{s}{j}^2\binom{2j}{s}\sum_{k=j}^{p-1}\binom{k+s}{s}\binom{k}{s}$$
$$=\sum_{j=0}^{p-1}\sum_{s=j}^{2j}\binom{s}{j}^2\binom{2j}{s}\frac{p}{2s+1}\binom{p+s}{s}\binom{p-1}{s},$$

where in the last step we use the identity

$$\sum_{k=s}^{n-1}\binom{k+s}{s}\binom{k}{s}=\frac{n}{2s+1}\binom{n+s}{s}\binom{n-1}{s}$$

which can be proved easily by induction on n. For each $s\in\{0,\ldots,p-1\}$, clearly

$$\mathrm{ord}_p(2s+1)\le 1 \quad\text{and}\quad \binom{p+s}{s}\binom{p-1}{s}\equiv(-1)^s \pmod{p^2}.$$

Therefore, by Lemma 2.2 we have

$$\sum_{k=0}^{p-1}A_k\equiv p\sum_{j=0}^{p-1}\sum_{s=j}^{\min\{p-1,2j\}}\binom{s}{j}^2\binom{2j}{s}\frac{(-1)^s}{2s+1}$$
$$=p\sum_{j=0}^{(p-1)/2}\sum_{s=j}^{2j}\binom{s}{j}^2\binom{2j}{s}\frac{(-1)^s}{2s+1}+p\sum_{j=(p+1)/2}^{p-1}\sum_{s=j}^{p-1}\binom{s}{j}^2\binom{2j}{s}\frac{(-1)^s}{2s+1}$$
$$=p\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}^2}{(4j+1)\binom{4j}{2j}}+p\sum_{j=(p+1)/2}^{p-1}\sum_{s=j}^{p-1}\binom{s}{j}^2\binom{2j}{s}\frac{(-1)^s}{2s+1}$$
$$=p\cdot{}_3F_2\left[\begin{matrix}\frac12 & \frac12 & \frac12\\ & \frac34 & \frac54\end{matrix}\,\middle|\,1\right]_{(p-1)/2}+p\sum_{j=(p+1)/2}^{p-1}\sum_{s=j}^{p-1}\binom{s}{j}^2\binom{2j}{s}\frac{(-1)^s}{2s+1} \pmod{p^2}.$$

It is easy to see that $\mathrm{ord}_p(2s+1)=0$ and

$$\binom{2j}{s}\binom{s}{j}=\binom{2j}{j}\binom{j}{s-j}\equiv 0 \pmod p$$

provided that $s,j\in\{(p+1)/2,\ldots,p-1\}$. Hence we have

$$\sum_{k=0}^{p-1}A_k\equiv p\cdot{}_3F_2\left[\begin{matrix}\frac12 & \frac12 & \frac12\\ & \frac34 & \frac54\end{matrix}\,\middle|\,1\right]_{(p-1)/2} \pmod{p^2}.$$

Clearly, at most one of $\langle -3/4\rangle_p$ and $\langle -5/4\rangle_p$ is smaller than $(p-1)/2$. Thus

$$\frac{p}{(1)_k\left(\frac{3}{4}\right)_k\left(\frac{5}{4}\right)_k}\in\mathbb{Z}_p\quad\text{for all } k=0,1,\ldots,\frac{p-1}{2}.$$

In view of Lemmas 2.4 and 2.5, and noting that $((1+p)/2)_k((1-p)/2)_k\equiv(1/2)_k^2 \pmod{p^2}$, we obtain further that

$$\begin{aligned}\sum_{k=0}^{p-1}A_k&\equiv p\cdot{}_3F_2\left[\begin{matrix}\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\&\frac{3}{4}&\frac{5}{4}\end{matrix}\middle|\,1\right]_{\frac{p-1}{2}}\\&\equiv p\cdot{}_3F_2\left[\begin{matrix}\frac{1-p}{2}&\frac{1+p}{2}&\frac{1}{2}\\&\frac{3}{4}&\frac{5}{4}\end{matrix}\middle|\,1\right]_{\frac{p-1}{2}}\\&=\frac{p\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{5-2p}{8}\right)\Gamma\left(\frac{7-2p}{8}\right)\Gamma\left(\frac{5+2p}{8}\right)\Gamma\left(\frac{7+2p}{8}\right)}\\&=\frac{p\Gamma\left(\frac{3}{8}\right)\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)\Gamma\left(\frac{9}{8}\right)}{\Gamma\left(\frac{5-2p}{8}\right)\Gamma\left(\frac{7-2p}{8}\right)\Gamma\left(\frac{5+2p}{8}\right)\Gamma\left(\frac{7+2p}{8}\right)}\pmod{p^2}.\end{aligned}$$

Case 1. $p\equiv 1 \pmod 8$.

In this case, we have

$$\frac{\Gamma\left(\frac{3}{8}\right)}{\Gamma\left(\frac{5-2p}{8}\right)}=\frac{\Gamma_p\left(\frac{3}{8}\right)}{\Gamma_p\left(\frac{5-2p}{8}\right)}\cdot(-1)^{(p-1)/4},\quad\frac{\Gamma\left(\frac{5}{8}\right)}{\Gamma\left(\frac{7-2p}{8}\right)}=\frac{\Gamma_p\left(\frac{5}{8}\right)}{\Gamma_p\left(\frac{7-2p}{8}\right)}\cdot(-1)^{(p-1)/4},$$
$$\frac{\Gamma\left(\frac{7}{8}\right)}{\Gamma\left(\frac{5+2p}{8}\right)}=\frac{\Gamma_p\left(\frac{7}{8}\right)}{\Gamma_p\left(\frac{5+2p}{8}\right)}\cdot(-1)^{(p-1)/4},\quad\frac{\Gamma\left(\frac{9}{8}\right)}{\Gamma\left(\frac{7+2p}{8}\right)}=\frac{\Gamma_p\left(\frac{9}{8}\right)}{\frac{p}{8}\cdot\Gamma_p\left(\frac{7+2p}{8}\right)}\cdot(-1)^{(p-1)/4}.$$

It follows that

$$\begin{aligned}\sum_{k=0}^{p-1}A_k&\equiv\frac{8\Gamma_p\left(\frac{3}{8}\right)\Gamma_p\left(\frac{5}{8}\right)\Gamma_p\left(\frac{7}{8}\right)\Gamma_p\left(\frac{9}{8}\right)}{\Gamma_p\left(\frac{5-2p}{8}\right)\Gamma_p\left(\frac{7-2p}{8}\right)\Gamma_p\left(\frac{5+2p}{8}\right)\Gamma_p\left(\frac{7+2p}{8}\right)}\\&\equiv-\frac{\Gamma_p\left(\frac{3}{8}\right)\Gamma_p\left(\frac{1}{8}\right)}{\Gamma_p\left(\frac{5}{8}\right)\Gamma_p\left(\frac{7}{8}\right)}\pmod{p^2}.\end{aligned}$$

With the aid of (2.2), we have

$$\frac{1}{\Gamma_p\left(\frac{5}{8}\right)\Gamma_p\left(\frac{7}{8}\right)}=\Gamma_p\left(\frac{3}{8}\right)\Gamma_p\left(\frac{1}{8}\right)(-1)^{2p-(5p-5)/8-(7p-7)/8}=\Gamma_p\left(\frac{3}{8}\right)\Gamma_p\left(\frac{1}{8}\right).$$

Therefore

$$\sum_{k=0}^{p-1}A_k\equiv-\Gamma_p\left(\frac{1}{8}\right)^2\Gamma_p\left(\frac{3}{8}\right)^2\pmod{p^2}.$$

Case 2. $p\equiv 3 \pmod 8$.

By similar arguments as in Case 1, we arrive at

$$\frac{\Gamma\left(\frac{3}{8}\right)}{\Gamma\left(\frac{5+2p}{8}\right)}=\frac{\Gamma_p\left(\frac{3}{8}\right)}{\frac{p}{8}\Gamma_p\left(\frac{5+2p}{8}\right)}(-1)^{(p+1)/4},\quad \frac{\Gamma\left(\frac{5}{8}\right)}{\Gamma\left(\frac{7+2p}{8}\right)}=\frac{\Gamma_p\left(\frac{5}{8}\right)}{\Gamma_p\left(\frac{7+2p}{8}\right)}(-1)^{(p+1)/4},$$
$$\frac{\Gamma\left(\frac{7}{8}\right)}{\Gamma\left(\frac{5-2p}{8}\right)}=\frac{\Gamma_p\left(\frac{7}{8}\right)}{\Gamma_p\left(\frac{5-2p}{8}\right)}(-1)^{(p+1)/4},\quad \frac{\Gamma\left(\frac{9}{8}\right)}{\Gamma\left(\frac{7-2p}{8}\right)}=\frac{\Gamma_p\left(\frac{9}{8}\right)}{\Gamma_p\left(\frac{7-2p}{8}\right)}(-1)^{(p+1)/4}.$$

Therefore

$$\sum_{k=0}^{p-1}A_k\equiv-\frac{\Gamma_p\left(\frac{3}{8}\right)\Gamma_p\left(\frac{1}{8}\right)}{\Gamma_p\left(\frac{5}{8}\right)\Gamma_p\left(\frac{7}{8}\right)}\pmod{p^2}.$$

On the other hand,

$$\frac{1}{\Gamma_p\left(\frac{5}{8}\right)\Gamma_p\left(\frac{7}{8}\right)}=\Gamma_p\left(\frac{3}{8}\right)\Gamma_p\left(\frac{1}{8}\right)(-1)^{2p-(7p-5)/8-(5p-7)/8}=-\Gamma_p\left(\frac{3}{8}\right)\Gamma_p\left(\frac{1}{8}\right).$$

Therefore

$$\sum_{k=0}^{p-1}A_k\equiv\Gamma_p\left(\frac{1}{8}\right)^2\Gamma_p\left(\frac{3}{8}\right)^2\pmod{p^2}.$$

The remaining cases $p\equiv 5,7 \pmod 8$ can be proved similarly. So our proof of Theorem 2.1 is completed. □

3. Proofs of Theorems 1.2 and 1.3

Theorem 1.2 is actually a p-aidc analogue of the following ${}_4F_3$ identity.

Lemma 3.1 ([3, p. 182, 25(a)]). *For any $\alpha,\beta,\gamma\in\mathbb{C}$ with $\Re((1+\alpha)/2-\beta-\gamma)>0$, we have*

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}\alpha & 1+\frac{\alpha}{2} & \beta & \gamma\\ & \frac{\alpha}{2} & 1+\alpha-\beta & 1+\alpha-\gamma\end{matrix}\middle|\,1\right]\\ =&\frac{\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}-\beta-\gamma\right)}{\Gamma(1+\alpha)\Gamma(1+\alpha-\beta-\gamma)\Gamma\left(\frac{1+\alpha}{2}-\beta\right)\Gamma(\frac{1+\alpha}{2}-\gamma)}.\end{aligned}\tag{3.1}$$

Proof of Theorem 1.2. Let $a=\langle-\alpha\rangle_p$.

(i) We now handle the case $2\nmid a$ and $a<\frac{2p+1}{3}$ in details. Set

$$\begin{aligned}\Psi(x,y,z):=&{}_4F_3\left[\begin{matrix}-a+x & 1+\frac{-a+x}{2} & -a+y & -a+z\\ & \frac{-a+x}{2} & 1+x-y & 1+x-z\end{matrix}\middle|\,1\right]_{p-1}\\ &-\frac{2\Gamma_p(1+x-y)\Gamma_p(1+x-z)\Gamma_p\left(\frac{1-a+x}{2}\right)\Gamma_p\left(\frac{1+3a+x-2y-2z}{2}\right)}{\Gamma_p(1-a+x)\Gamma_p(1+a+x-y-z)\Gamma_p\left(\frac{1+a+x-2y}{2}\right)\Gamma_p\left(\frac{1+a+x-2z}{2}\right)}.\end{aligned}$$

It is easy to see that

$${}_4F_3\left[\begin{matrix}\alpha & 1+\frac{\alpha}{2} & \alpha & \alpha\\ & \frac{\alpha}{2} & 1 & 1\end{matrix}\middle|\,1\right]_{p-1}\equiv 2g_p(\alpha)\pmod{p^3}$$

is equivalent to

$$\Psi(sp,sp,sp)\equiv 0 \pmod{p^3}, \tag{3.2}$$

where $s=(\alpha+a)/p$.

For $p=3,5$ we can verify (3.2) for any $1\le\alpha\le p^3$ numerically.

Now we assume that $p\ge 7$. In view of the local-global theorem from [17], we only need to show that

$$\Psi(rp,sp,tp)\equiv 0 \pmod{p^3} \tag{3.3}$$

provided that $r,s,t\in\mathbb{Z}_p$ and at least one of r,s,t is zero. We first show that

$$\Psi(0,sp,tp)=0 \tag{3.4}$$

for each $s,t\in\mathbb{Z}_p$. In fact, we may assume that $sp,tp,(s+t)p\in\mathbb{Q}\setminus\mathbb{Z}$ (since any $x\in\mathbb{Z}\cap\mathbb{Z}_p$ can be approximated by a sequence of p-adic integers $(x_m)_{m\ge0}$ in $(\mathbb{Q}\setminus\mathbb{Z})\cap\mathbb{Z}_p$). In light of (3.1),

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix}-a & 1+\frac{-a}{2} & -a+sp & -a+tp\\ & \frac{-a}{2} & 1-sp & 1-tp\end{matrix}\,\middle|\,1\right]_{p-1}\\
=&\lim_{z\to0}{}_4F_3\left[\begin{matrix}-a+z & 1+\frac{-a+z}{2} & -a+sp & -a+tp\\ & \frac{-a+z}{2} & 1+z-sp & 1+z-tp\end{matrix}\,\middle|\,1\right]\\
=&\frac{\Gamma(1-sp)}{\Gamma\left(\frac{1+a-2sp}{2}\right)}\cdot\frac{\Gamma(1-tp)}{\Gamma\left(\frac{1+a-2tp}{2}\right)}\cdot\frac{\Gamma\left(\frac{1+3a-2sp-2tp}{2}\right)}{\Gamma(1+a-sp-tp)}\cdot\lim_{z\to0}\frac{\Gamma\left(\frac{1-a+z}{2}\right)}{\Gamma(1-a+z)}.
\end{aligned}$$

Since a is odd and $a<(2p+1)/3$, by (1.3) we have

$$\frac{\Gamma(1-sp)}{\Gamma\left(\frac{1+a-2sp}{2}\right)}=\prod_{j=1}^{\frac{a-1}{2}}\frac{1}{j+sp}=(-1)^{\frac{a-1}{2}}\frac{\Gamma_p(1-sp)}{\Gamma_p\left(\frac{1+a-2sp}{2}\right)},$$

$$\frac{\Gamma(1-tp)}{\Gamma\left(\frac{1+a-2tp}{2}\right)}=\prod_{j=1}^{\frac{a-1}{2}}\frac{1}{j+tp}=(-1)^{\frac{a-1}{2}}\frac{\Gamma_p(1-tp)}{\Gamma_p\left(\frac{1+a-2tp}{2}\right)},$$

$$\frac{\Gamma\left(\frac{1+3a-2sp-2tp}{2}\right)}{\Gamma(1+a-sp-tp)}=\prod_{j=1}^{\frac{a-1}{2}}(a+j-sp-tp)=(-1)^{\frac{a-1}{2}}\frac{\Gamma_p\left(\frac{1+3a-2sp-2tp}{2}\right)}{\Gamma_p(1+a-sp-tp)}.$$

In view of (2.1),

$$\Gamma\left(\frac{1-a+z}{2}\right)\Gamma\left(\frac{1+a-z}{2}\right)=\frac{\pi}{\sin\pi\frac{1-a+z}{2}}$$

and

$$\Gamma(1-a+z)\Gamma(a-z)=\frac{\pi}{\sin\pi(1-a+z)}.$$

Furthermore,

$$\lim_{z\to0}\frac{\Gamma\left(\frac{1-a+z}{2}\right)}{\Gamma(1-a+z)}=\lim_{z\to0}\frac{\Gamma(a-z)}{\Gamma(\frac{1+a-z}{2})}\cdot\frac{\sin\pi(1-a+z)}{\sin\pi\frac{1-a+z}{2}}$$

$$= 2(-1)^{\frac{a-1}{2}} \frac{\Gamma_p(a)}{\Gamma_p(\frac{1+a}{2})} \cdot \frac{\cos \pi(a-1)}{\cos \pi \frac{a-1}{2}} = \frac{2\Gamma_p(a)}{\Gamma_p(\frac{1+a}{2})}.$$

In view of (2.2), we have

$$\frac{2\Gamma_p(a)}{\Gamma_p(\frac{1+a}{2})} = (-1)^{\frac{a-1}{2}} \frac{2\Gamma_p\left(\frac{1-a}{2}\right)}{\Gamma_p(1-a)}.$$

By the above, we arrive at

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}-a & 1+\frac{-a}{2} & -a+sp & -a+tp\\ & \frac{-a}{2} & 1-sp & 1-tp\end{matrix}\middle|\, 1\right]_{p-1}\\ =&\frac{\Gamma_p(1-sp)}{\Gamma_p\left(\frac{1+a-2sp}{2}\right)} \cdot \frac{\Gamma_p(1-tp)}{\Gamma_p\left(\frac{1+a-2tp}{2}\right)} \cdot \frac{\Gamma_p\left(\frac{1+3a-2sp-2tp}{2}\right)}{\Gamma_p(1+a-sp-tp)} \cdot \frac{2\Gamma_p\left(\frac{1-a}{2}\right)}{\Gamma_p(1-a)}.\end{aligned}$$

This proves (3.4).

Now we turn to show

$$\Psi(rp, 0, tp) = 0 \tag{3.5}$$

for any $r, t \in \mathbb{Z}_p$. Also, we may assume that $rp, tp, rp - tp \in \mathbb{Q} \setminus \mathbb{Z}$. By (3.1), we have

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}-a+rp & 1+\frac{-a+rp}{2} & -a & -a+tp\\ & \frac{-a+rp}{2} & 1+rp & 1+rp-tp\end{matrix}\middle|\, 1\right]_{p-1}\\ =&{}_4F_3\left[\begin{matrix}-a+rp & 1+\frac{-a+rp}{2} & -a & -a+tp\\ & \frac{-a+rp}{2} & 1+rp & 1+rp-tp\end{matrix}\middle|\, 1\right]\\ =&\frac{\Gamma(1+rp)}{\Gamma(1-a+rp)} \cdot \frac{\Gamma(1+rp-tp)}{\Gamma(1+a+rp-tp)} \cdot \frac{\Gamma\left(\frac{1-a+rp}{2}\right)}{\Gamma\left(\frac{1+a+rp}{2}\right)} \cdot \frac{\Gamma\left(\frac{1+3a+rp}{2}-tp\right)}{\Gamma\left(\frac{1+a+rp}{2}-tp\right)}\\ =&\frac{rp \cdot \Gamma_p(1+rp)}{\Gamma_p(1-a+rp)} \cdot \frac{\Gamma_p(1+rp-tp)}{\Gamma_p(1+a+rp-tp)} \cdot \frac{\Gamma_p\left(\frac{1-a+rp}{2}\right)}{\frac{1}{2}rp \cdot \Gamma_p\left(\frac{1+a+rp}{2}\right)} \cdot \frac{\Gamma_p\left(\frac{1+3a+rp}{2}-tp\right)}{\Gamma_p\left(\frac{1+a+rp}{2}-tp\right)}.\end{aligned}$$

Thus (3.5) holds. Due to the symmetry reason, we also have $\Psi(rp, sp, 0) = 0$ for any $r, s \in \mathbb{Z}_p$.

(ii) For the other cases, we set

$$\begin{aligned}\Phi(x,y,z) :=& {}_4F_3\left[\begin{matrix}-a+x & 1+\frac{-a+x}{2} & -a+y & -a+z\\ & \frac{-a+x}{2} & 1+x-y & 1+x-z\end{matrix}\middle|\, 1\right]_{p-1}\\ &- f(x,y,z)\frac{\Gamma_p(1+x-y)\Gamma_p(1+x-z)\Gamma_p\left(\frac{1-a+x}{2}\right)\Gamma_p\left(\frac{1+3a+x-2y-2z}{2}\right)}{\Gamma_p(1-a+x)\Gamma_p(1+a+x-y-z)\Gamma_p\left(\frac{1+a+x-2y}{2}\right)\Gamma_p\left(\frac{1+a+x-2z}{2}\right)},\end{aligned}$$

where

$$f(x,y,z) = \begin{cases} 2p+x-2y-2z & \text{if } a \text{ is odd and } a \geq \frac{2p+1}{3},\\ x & \text{if } a \text{ is even and } a < \frac{p+1}{3},\\ x \cdot \dfrac{p+x-2y-2z}{2} & \text{if } a \text{ is even and } a \geq \frac{p+1}{3}.\end{cases}$$

We only need to show that $\Phi(rp,sp,tp)=0$ for any $r,s,t\in\mathbb{Z}_p$ with $0\in\{r,s,t\}$. This can be proved by similar arguments, so we leave it to the reader as an exercise.

In view of the above, we have completed our proof of Theorem 1.2. □

The following identity is due to Whipple.

Lemma 3.2 ([30, (5.1)]). *For any $\alpha,\beta,\gamma\in\mathbb{C}$ with $\beta-\alpha,\gamma-\alpha\notin\mathbb{Z}^+$, we have*

$$ {}_4F_3\left[\begin{matrix}\alpha & 1+\frac{\alpha}{2} & \beta & \gamma\\ & \frac{\alpha}{2} & 1+\alpha-\beta & 1+\alpha-\gamma\end{matrix}\,\middle|\,-1\right]=\frac{\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)}{\Gamma(1+\alpha)\Gamma(1+\alpha-\beta-\gamma)}. $$

Proof of Theorem 1.3. Let $a=\langle-\alpha\rangle_p$. Set

$$ \begin{aligned}\Omega(x,y,z):={}&{}_4F_3\left[\begin{matrix}-a+x & 1+\frac{-a+x}{2} & -a+y & -a+z\\ & \frac{-a+x}{2} & 1+x-y & 1+x-z\end{matrix}\,\middle|\,-1\right]_{p-1}\\ &-\frac{x\Gamma_p(1+x-y)\Gamma_p(1+x-z)}{\Gamma_p(1-a+x)\Gamma_p(1+a+x-y-z)}.\end{aligned} $$

As in the proof of Theorem 1.2, it suffices to show that

$$ \Omega(rp,sp,tp)\equiv 0 \pmod{p^3} \tag{3.6} $$

provided $r,s,t\in\mathbb{Z}_p$ and $0\in\{r,s,t\}$. Again, we may assume $p\geq 7$.

We first consider the case $r=0$. Also, assume that $sp,tp,(s+t)p\in\mathbb{Q}\setminus\mathbb{Z}$. By Lemma 3.2,

$$ \begin{aligned}&{}_4F_3\left[\begin{matrix}-a & 1+\frac{-a}{2} & -a+sp & -a+tp\\ & \frac{-a}{2} & 1-sp & 1-tp\end{matrix}\,\middle|\,-1\right]_{p-1}\\ &={}_4F_3\left[\begin{matrix}-a & 1+\frac{-a}{2} & -a+sp & -a+tp\\ & \frac{-a}{2} & 1-sp & 1-tp\end{matrix}\,\middle|\,-1\right]\\ &=\frac{\Gamma(1-sp)\Gamma(1-tp)}{\Gamma(1-a)\Gamma(1+a-sp-tp)}.\end{aligned} $$

Since $\alpha\in\mathbb{Z}_p^\times$, we have $a-1\in\{1,\ldots,p-2\}$. By (2.1) we know that $1/\Gamma(-n)=0$ for any nonnegative integer n. Thus $\Omega(0,sp,tp)=0$.

Below we consider the case $s=0$. Assume that $rp,(r-t)p\in\mathbb{Q}\setminus\mathbb{Z}$. With the help of Lemma 3.2, we get

$$ \begin{aligned}&{}_4F_3\left[\begin{matrix}-a+rp & 1+\frac{-a+rp}{2} & -a & -a+tp\\ & \frac{-a+rp}{2} & 1+rp & 1+rp-tp\end{matrix}\,\middle|\,-1\right]_{p-1}\\ &={}_4F_3\left[\begin{matrix}-a+rp & 1+\frac{-a+rp}{2} & -a & -a+tp\\ & \frac{-a}{2} & 1+rp & 1+rp-tp\end{matrix}\,\middle|\,-1\right]\\ &=\frac{\Gamma(1+rp)\Gamma(1+rp-tp)}{\Gamma(1-a+rp)\Gamma(1+a+rp-tp)}.\end{aligned} $$

By (1.3), we have

$$\frac{\Gamma(1+rp)}{\Gamma(1-a+rp)}=(-1)^a\frac{rp\Gamma_p(1+rp)}{\Gamma_p(1-a+rp)}$$

and

$$\frac{\Gamma(1+rp-tp)}{\Gamma(1+a+rp-tp)}=(-1)^a\frac{\Gamma_p(1+rp-tp)}{\Gamma_p(1+a+rp-tp)}.$$

Thus $\Omega(rp,0,tp)=0$. By symmetry, we also have $\Omega(rp,sp,0)=0$.

Combining the above we get (3.6). The proof of Theorem 1.3 is now complete. □

4. Applications of Theorems 1.2 and 1.3

Sun [25, Conjecture 35] posed many conjectural congruences, here we confirm [25, Conjecture 35] partially.

Theorem 4.1 ([25, Conjecture 35]). (i) *Let $p>3$ be a prime. For any $x\in\mathbb{Z}_p$ with $3x\not\equiv 1,2 \pmod p$, we have*

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-x}{j}^3\binom{x-1}{k-j}^3\equiv 0 \pmod{p^2}. \tag{4.1}$$

For any $x\in\mathbb{Z}_p$ with $x\equiv 1/3 \pmod p$, we have

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-x}{j}^3\binom{x-1}{k-j}^3\equiv x+\frac{p\left(\frac{p}{3}\right)-1}{3} \pmod{p^2}. \tag{4.2}$$

(ii) *Let p be an odd prime. If $p\not\equiv 5 \pmod 8$, then*

$$\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-1/4}{j}^3\binom{-3/4}{k-j}^3\equiv p^2 \pmod{p^3}. \tag{4.3}$$

If $p\equiv 5,7 \pmod 8$, then

$$\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/2}{j}^3\binom{-1/2}{k-j}^3\equiv 0 \pmod{p^3}. \tag{4.4}$$

Remark 4.1. Sun [25, Conjecture 35] also conjectured that

$$\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/3}{j}^3\binom{-2/3}{k-j}^3\equiv 0 \pmod{p^3}.$$

for any odd prime $p\equiv 2 \pmod 3$. in view of the method we prove Theorem 4.1, we are led to evaluate

$$\sum_{k=0}^{p-1}\binom{-1/3}{k}^3 \quad\text{and}\quad \sum_{k=0}^{p-1}\binom{-2/3}{k}^3$$

modulo p^2.

To show Theorem 4.1 we need the following known results.

Lemma 4.1 ([17, Theorem 5.1])**.** *Suppose that p is an odd prime and $\alpha \in \mathbb{Z}_p^\times$. Let $s = (\alpha + \langle -\alpha \rangle_p)/p$, and*

$$g_p(\alpha) = \frac{\Gamma_p(1+\frac{1}{2}\alpha)\Gamma_p(1-\frac{3}{2}\alpha)}{\Gamma_p(1+\alpha)\Gamma_p(1-\alpha)\Gamma_p(1-\frac{1}{2}\alpha)^2}.$$

Then we have the following congruence modulo p^3:

$${}_3F_2\left[\begin{matrix}\alpha & \alpha & \alpha \\ & 1 & 1\end{matrix}\middle| 1\right]_{p-1} \equiv \begin{cases} 2g_p(\alpha) & \text{if } \langle -\alpha \rangle_p \text{ is even and } \langle -\alpha \rangle_p < 2p/3, \\ p(2-3s)g_p(\alpha) & \text{if } \langle -\alpha \rangle_p \text{ is even and } \langle -\alpha \rangle_p \geq 2p/3, \\ psg_p(\alpha) & \text{if } \langle -\alpha \rangle_p \text{ is odd and } \langle -\alpha \rangle_p < p/3, \\ \dfrac{p^2 s(1-3s)g_p(\alpha)}{2} & \text{if } \langle -\alpha \rangle_p \text{ is odd and } \langle -\alpha \rangle_p \geq p/3. \end{cases}$$

Lemma 4.2 ([17, Corollary 8.1])**.** *Let p be an odd prime. If $p \equiv 1, 3 \pmod 8$, then*

$${}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1\end{matrix}\middle| -1\right]_{p-1} \equiv -\Gamma_p\left(\frac{1}{8}\right)^2 \Gamma_p\left(\frac{3}{8}\right)^2 \pmod{p^3}.$$

If $p \equiv 5, 7 \pmod 8$, then

$${}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1\end{matrix}\middle| -1\right]_{p-1} \equiv \frac{3p^2}{64} \cdot \Gamma_p\left(\frac{1}{8}\right)^2 \Gamma_p\left(\frac{3}{8}\right)^2 \pmod{p^3}.$$

Proof of Theorem 4.1. By (1.2) with $m = 3$, we have

$$\sum_{k=0}^{p-1} \varepsilon^k (2k+1) \sum_{j=0}^{k} \binom{-x}{j}^3 \binom{x-1}{k-j}^3 \equiv (1-x)\Sigma_1 + x\Sigma_2 \pmod{p^3}, \tag{4.5}$$

where

$$\Sigma_1 := {}_3F_2\left[\begin{matrix}x & x & x \\ & 1 & 1\end{matrix}\middle| -\varepsilon\right]_{p-1} \cdot {}_4F_3\left[\begin{matrix}1-x & 1+\frac{1-x}{2} & 1-x & 1-x \\ & \frac{1-x}{2} & 1 & 1\end{matrix}\middle| -\varepsilon\right]_{p-1}$$

and

$$\Sigma_2 := {}_3F_2\left[\begin{matrix}1-x & 1-x & 1-x \\ & 1 & 1\end{matrix}\middle| -\varepsilon\right]_{p-1} \cdot {}_4F_3\left[\begin{matrix}x & 1+\frac{x}{2} & x & x \\ & \frac{x}{2} & 1 & 1\end{matrix}\middle| -\varepsilon\right]_{p-1}.$$

(i) Denote $\langle -x \rangle_p$ by a. We first consider (4.1). Here we only handle the case $2 \mid a$ since the case $2 \nmid a$ can be confirmed similarly. As a is even, $\langle x-1 \rangle_p = p-1-a$ is also even. Obviously, (4.1) holds if $p \mid a$. So we assume that $3x \not\equiv 0, 1, 2 \pmod p$. Below we divide this into three subcases.

Case 1. $(p+1)/3 \leq a < 2p/3$.

In this case, $p/3 - 1 < \langle x-1 \rangle_p \leq (2p-4)/3$. Since $3x \not\equiv 1, 2 \pmod p$ and $p > 3$, we know that $\langle x-1 \rangle_p \neq (p-2)/3, (p-1)/3, p/3$. Thus $(p+1)/3 \leq \langle x-1 \rangle_p \leq (2p-4)/3$. Then (4.1) follows from Theorem 1.2 and Lemma 4.1.

Case 2. $a < (p+1)/3$.

Observe that $\langle x-1\rangle_p > (2p-4)/3$. Since $3x \not\equiv 0,1,2 \pmod p$, we have $\langle x-1\rangle_p \neq (2p-3)/3, (2p-2)/3, (2p-1)/3$, and hence $\langle x-1\rangle_p \geq 2p/3$. Therefore (4.1) follows.

Case 3. $a \geq 2p/3$.

Clearly, $\langle x-1\rangle_p \leq p/3-1$. Thus (4.1) follows from Theorem 1.2 and Lemma 4.1 immediately.

Now we turn to (4.2). We first assume that $p \equiv 1 \pmod 6$. In this case, $a = (p-1)/3$ is even and $\langle x-1\rangle_p = p-1-a = (2p-2)/3$. Thus, by Theorem 1.2 and Lemma 4.1, we have

$$(1-x)\Sigma_1 \equiv 0 \pmod{p^2}$$

and

$$\begin{aligned}x\Sigma_2 &\equiv 2x(x+a)\cdot\frac{\Gamma_p\left(1+\frac{1-x}{2}\right)\Gamma_p\left(1-\frac{3(1-x)}{2}\right)\Gamma_p\left(\frac{1+x}{2}\right)\Gamma_p\left(\frac{1-3x}{2}\right)}{\Gamma_p(2-x)\Gamma_p(x)\Gamma_p\left(1-\frac{1-x}{2}\right)^2\Gamma_p(1+x)\Gamma_p(1-x)\Gamma_p\left(\frac{1-x}{2}\right)^2}\\
&= (x+a)\Gamma_p\left(\frac{1-x}{2}\right)\Gamma_p\left(\frac{1+x}{2}\right)\Gamma_p\left(\frac{3x-1}{2}\right)\Gamma_p\left(\frac{3-3x}{2}\right)\\
&= (x+a)(-1)^{\frac{2(p-1)}{3}} = x+a = x+\frac{p-1}{3}.\end{aligned}$$

So, when $p \equiv 1 \pmod 6$ we have (4.2) since $\left(\frac{p}{3}\right) = 1$.

Below we suppose that $p \equiv 5 \pmod 6$. Now, $a = (2p-1)/3$ is odd and $\langle x-1\rangle_p = p-1-a = (p-2)/3$. Also, by Theorem 1.2 and Lemma 4.1, we have

$$(1-x)\Sigma_1 \equiv 0 \pmod{p^2}$$

and

$$\begin{aligned}x\Sigma_2 &\equiv 2x(1-x+\langle x-1\rangle_p)\cdot\frac{\Gamma_p\left(1+\frac{1-x}{2}\right)\Gamma_p\left(1-\frac{3(1-x)}{2}\right)\Gamma_p\left(\frac{1+x}{2}\right)\Gamma_p\left(\frac{1-3x}{2}\right)}{\Gamma_p(2-x)\Gamma_p(x)\Gamma_p\left(1-\frac{1-x}{2}\right)^2\Gamma_p(1+x)\Gamma_p(1-x)\Gamma_p\left(\frac{1-x}{2}\right)^2}\\
&= (p-a-x)\Gamma_p\left(\frac{1-x}{2}\right)\Gamma_p\left(\frac{1+x}{2}\right)\Gamma_p\left(\frac{3x-1}{2}\right)\Gamma_p\left(\frac{3-3x}{2}\right)\\
&= (p-a-x)(-1)^{\frac{p-2}{3}} = x+\frac{-p-1}{3}.\end{aligned}$$

As $\left(\frac{p}{3}\right) = -1$, this proves (4.2).

(ii) We now turn to prove (4.3) for $p \not\equiv 5 \pmod 8$. We just handle the case $p \equiv 1 \pmod 8$ since the remaining case $p \equiv 3 \pmod 4$ can be handled similarly. In view of Theorem 1.2 and Lemma 4.1.

$$\begin{aligned}{}_3F_2\begin{bmatrix}\frac14 & \frac14 & \frac14\\ & 1 & 1\end{bmatrix}\,\Bigg|\, 1\Bigg]_{p-1} &\equiv \frac{2\Gamma_p\left(\frac98\right)\Gamma_p\left(\frac58\right)}{\Gamma_p\left(\frac54\right)\Gamma_p\left(\frac34\right)\Gamma_p\left(\frac78\right)^2} \pmod{p^3},\\
{}_3F_2\begin{bmatrix}\frac34 & \frac34 & \frac34\\ & 1 & 1\end{bmatrix}\,\Bigg|\, 1\Bigg]_{p-1} &\equiv \frac{-\frac{p}{4}\cdot\Gamma_p\left(\frac{11}{8}\right)\Gamma_p\left(-\frac18\right)}{\Gamma_p\left(\frac74\right)\Gamma_p\left(\frac14\right)\Gamma_p\left(\frac58\right)^2} \pmod{p^3},\end{aligned}$$

$$
{}_4F_3\begin{bmatrix} \frac{1}{4} & \frac{9}{8} & \frac{1}{4} & \frac{1}{4} \\ & \frac{1}{8} & 1 & 1 \end{bmatrix} 1\Bigg]_{p-1} \equiv \frac{p}{4}\cdot\frac{\Gamma_p\left(\frac{5}{8}\right)\Gamma_p\left(\frac{1}{8}\right)}{\Gamma_p\left(\frac{5}{4}\right)\Gamma_p\left(\frac{3}{4}\right)\Gamma_p\left(\frac{3}{8}\right)^2} \pmod{p^3},
$$
$$
{}_4F_3\begin{bmatrix} \frac{3}{4} & \frac{11}{8} & \frac{3}{4} & \frac{3}{4} \\ & \frac{3}{8} & 1 & 1 \end{bmatrix} 1\Bigg]_{p-1} \equiv -\frac{15p^2}{16}\cdot\frac{\Gamma_p\left(\frac{7}{8}\right)\Gamma_p\left(-\frac{5}{8}\right)}{\Gamma_p\left(\frac{7}{4}\right)\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{1}{8}\right)^2} \pmod{p^3}.
$$

Then, by (2.2) and (4.5), we deduce that

$$
\sum_{k=0}^{p-1}(-1)^k(2k+1)\sum_{j=0}^{k}\binom{-1/4}{j}^3\binom{-3/4}{k-j}^3 \equiv \frac{3}{4}\cdot p^2+\frac{1}{4}\cdot p^2 = p^2 \pmod{p^3}.
$$

This proves (4.3).

(iii) Now we turn to prove (4.4) for $p\equiv 5,7 \pmod 8$. From (4.5), we have

$$
\begin{aligned}
&\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/2}{j}^3\binom{-1/2}{k-j}^3 \\
&\equiv {}_3F_2\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{bmatrix} -1\Bigg]_{p-1}\cdot {}_4F_3\begin{bmatrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 \end{bmatrix} -1\Bigg]_{p-1} \pmod{p^3}.
\end{aligned} \tag{4.6}
$$

By Theorem 1.3,

$$
{}_4F_3\begin{bmatrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 \end{bmatrix} -1\Bigg]_{p-1} \equiv 0 \pmod{p}. \tag{4.7}
$$

As $p\equiv 5,7 \pmod 8$, we have

$$
{}_3F_2\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{bmatrix} -1\Bigg]_{p-1} \equiv 0 \pmod{p^2} \tag{4.8}
$$

by Lemma 4.2. Combining (4.6), (4.7) and (4.8), we immediately get the desired (4.4).

In view of the above, we have completed the proof of Theorem 4.1. □

5. Other related congruences

In the previous two sections, we established p-adic analogues of two hypergeometric identities and used them to solve some congruences conjectured by Sun. Our following theorem confirms [25, Conjecture 36] partially.

Theorem 5.1 ([25, Conjecture 36])**.** (i) *For each prime $p>3$, we have*

$$
\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/6}{j}^4\binom{-5/6}{k-j}^4 \equiv 0 \pmod{p^2}. \tag{5.1}
$$

(ii) *Let p be an odd prime. If $p\equiv 3 \pmod 4$, then*

$$\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/4}{j}^4\binom{-3/4}{k-j}^4\equiv 0 \pmod{p^2},\tag{5.2}$$

and

$$\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/2}{j}^5\binom{-1/2}{k-j}^5\equiv 0 \pmod{p^3}.\tag{5.3}$$

If $p\equiv 5 \pmod 6$, then

$$\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/6}{j}^6\binom{-5/6}{k-j}^6\equiv 0 \pmod{p^2}.\tag{5.4}$$

Proof. (a) Let p be a prime with $p\equiv 3 \pmod 4$. By (1.2), we know that

$$\begin{aligned}&\sum_{k=0}^{p-1}(2k+1)\sum_{j=0}^{k}\binom{-1/2}{j}^5\binom{-1/2}{k-j}^5\\ &\equiv {}_6F_5\left[\begin{matrix}\frac12 & \frac54 & \frac12 & \frac12 & \frac12 & \frac12\\ & \frac14 & 1 & 1 & 1 & 1\end{matrix}\middle| -1\right]_{p-1}\cdot {}_5F_4\left[\begin{matrix}\frac12 & \frac12 & \frac12 & \frac12 & \frac12\\ & 1 & 1 & 1 & 1\end{matrix}\middle| -1\right]_{p-1} \pmod{p^5}.\end{aligned}$$

van Hamme [28] conjectured that

$${}_6F_5\left[\begin{matrix}\frac12 & \frac54 & \frac12 & \frac12 & \frac12 & \frac12\\ & \frac14 & 1 & 1 & 1 & 1\end{matrix}\middle| -1\right]_{p-1}\equiv 0 \pmod{p^3}.$$

This was confirmed by Liu [11] who showed further that if $p\neq 3$ then

$${}_6F_5\left[\begin{matrix}\frac12 & \frac54 & \frac12 & \frac12 & \frac12 & \frac12\\ & \frac14 & 1 & 1 & 1 & 1\end{matrix}\middle| -1\right]_{p-1}\equiv -\frac{p^3}{16}\Gamma_p\left(\frac14\right) \pmod{p^4}.$$

This congruence remains ture if we replace the modulus p^4 by p^5, as conjectured by Liu [11] and confirmed by Wang [29]. In view of the above, (5.3) does hold.

(b) Now let $p>3$ be a prime. We want to prove (5.1).

Case 1. $p\equiv 1 \pmod 6$.

In this case, $\langle -1/6\rangle_p=(p-1)/6<p/2$ and $\langle -5/6\rangle_p=(5p-5)/6>p/2$. By [14, Theorem 2.22], we obtain

$${}_5F_4\left[\begin{matrix}\frac16 & \frac{13}{12} & \frac16 & \frac16 & \frac16\\ & \frac{1}{12} & 1 & 1 & 1\end{matrix}\middle| 1\right]_{p-1}\equiv 0 \pmod p \quad\text{and}\quad {}_5F_4\left[\begin{matrix}\frac56 & \frac{17}{12} & \frac56 & \frac56 & \frac56\\ & \frac{5}{12} & 1 & 1 & 1\end{matrix}\middle| 1\right]_{p-1}\equiv 0 \pmod{p^2}.$$

Thus, in view of (1.2) it suffices to show that

$${}_4F_3\left[\begin{matrix}\frac56 & \frac56 & \frac56 & \frac56\\ & 1 & 1 & 1\end{matrix}\middle| 1\right]_{p-1}\equiv 0 \pmod p.$$

Recall the well-known Karlsson-Minton summation formula (cf. [5, p. 18]):

$$ {}_{r+1}F_r\left[\begin{matrix} a & b_1+m_1 & \cdots & b_r+m_r \\ & b_1 & \cdots & b_r \end{matrix}\middle|\, 1\right] = 0, \tag{5.5} $$

provided that $m_1, m_2, \ldots, m_r$ are nonnegative integers with $\Re(-a) > m_1+\cdots+m_r$. Combining this with (5.5), we get

$$ {}_4F_3\left[\begin{matrix} \frac{5}{6} & \frac{5}{6} & \frac{5}{6} & \frac{5}{6} \\ & 1 & 1 & 1 \end{matrix}\middle|\, 1\right]_{p-1} \equiv {}_4F_3\left[\begin{matrix} \frac{5-5p}{6} & \frac{5+p}{6} & \frac{5+p}{6} & \frac{5+p}{6} \\ & 1 & 1 & 1 \end{matrix}\middle|\, 1\right] = 0 \pmod p $$

since $(p-1)/2 < (5p-5)/6$.

Case 2. $p \equiv 5 \pmod 6$.

In this case, $\langle -1/6\rangle_p = (5p-1)/6 > p/2$ and $\langle -5/6\rangle_p = (p-5)/6 < p/2$. Thus, by [14, Theorem 2.22], we have

$$ {}_5F_4\left[\begin{matrix} \frac{1}{6} & \frac{13}{12} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & \frac{1}{12} & 1 & 1 & 1 \end{matrix}\middle|\, 1\right]_{p-1} \equiv 0 \pmod{p^2} \quad \text{and} \quad {}_5F_4\left[\begin{matrix} \frac{5}{6} & \frac{17}{12} & \frac{5}{6} & \frac{5}{6} & \frac{5}{6} \\ & \frac{5}{12} & 1 & 1 & 1 \end{matrix}\middle|\, 1\right]_{p-1} \equiv 0 \pmod p. $$

Furthermore, by (5.5) we have

$$ {}_4F_3\left[\begin{matrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & 1 & 1 & 1 \end{matrix}\middle|\, 1\right]_{p-1} \equiv {}_4F_3\left[\begin{matrix} \frac{1-5p}{6} & \frac{1+p}{6} & \frac{1+p}{6} & \frac{1+p}{6} \\ & 1 & 1 & 1 \end{matrix}\middle|\, 1\right] = 0 \pmod p $$

since $(p-5)/2 < (5p-1)/6$. Combining this with (1.2), we obtain (5.1).

(c) For any prime $p \equiv 3 \pmod 4$, we can prove (5.2) by the method in the proof of (5.1).

Now turn to prove (5.4) for any prime $p \equiv 5 \pmod 6$. As $\langle -1/6\rangle_p = (5p-1)/5 > 2p/3$ and $\langle -5/6\rangle_p = (p-5)/6 < p/3$, by [14, Theorems 2.17 and 2.20] we have

$$ {}_7F_6\left[\begin{matrix} \frac{1}{6} & \frac{13}{12} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & \frac{1}{12} & 1 & 1 & 1 & 1 & 1 \end{matrix}\middle|\, 1\right]_{p-1} \equiv 0 \pmod{p^2} $$

and

$$ {}_7F_6\left[\begin{matrix} \frac{5}{6} & \frac{17}{12} & \frac{5}{6} & \frac{5}{6} & \frac{5}{6} & \frac{5}{6} & \frac{5}{6} \\ & \frac{5}{12} & 1 & 1 & 1 & 1 & 1 \end{matrix}\middle|\, 1\right]_{p-1} \equiv 0 \pmod p. $$

From (5.5), we deduce that

$$ {}_6F_5\left[\begin{matrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & 1 & 1 & 1 & 1 & 1 \end{matrix}\middle|\, 1\right]_{p-1} \equiv 0 \pmod p. $$

Combining this with (1.2), we immediately obtain the desired (5.4).

In view of the above, we have completed the proof of Theorem 5.1. □

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References

[1] S. Ahlgren and K. Ono, *A Gaussian hypergeometric series evaluation and Apéry number congruences*, J. Reine Angew. Math. **518** (2000), 187–212.

[2] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque **61** (1979), 11–13.

[3] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia Math. Appl. 71, Cambridge University Press, Cambridge, 1999.

[4] A. Deines, J.G. Fuselier, L. Long, H. Swisher and F.-T. Tu, *Hypergeometric series, truncated hypergeometric series, and Gaussian hypergeometric functions*, Directions in number theory, pp. 125–159, Assoc. Women Math. Ser., 3, Springer, 2016.

[5] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia Math. Appl. 96, Cambridge University Press, Cambridge, 2004.

[6] V.J.W. Guo, *Some congruences involving powers of Legendre polynomials*, Integral Transforms Spec. Funct. **26** (2015), 660–666.

[7] B. He, *Some congruences on truncated hypergeometric series*, Proc. Amer. Math. Soc. **143** (2015), no. 12, 5173–5180.

[8] B. He, *Supercongruences and truncated hypergeometric series*, Proc. Amer. Math. Soc. **145** (2017), 501–508.

[9] B. He, *Congruences concerning truncated hypergeometric series*, Proc. Roy. Soc. Edinburgh Sect. A **147** (2017), no. 3, 599–613.

[10] J.-C. Liu, *Congruences for truncated hypergeometric series $_2F_1$*, Bull. Aust. Math. Soc. **96** (2017), 14–23.

[11] J.-C. Liu, *On Van Hamme's (A.2) and (H.2) supercongruences*, J. Math. Anal. Appl. **471** (2019), 613–622.

[12] L. Long, *Hypergeometric evaluation identities and supercongruences*, Pacific J. Math. **249** (2011), no. 2, 405–418.

[13] L. Long and R. Ramakrishna, *Some supercongruences occurring in truncated hypergeometric series*, Adv. Math. **290** (2016), 773–808.

[14] G.-S. Mao and H. Pan, *p-Adic analogues of hypergeometric identities*, preprint, arXiv:1703.01215.

[15] E. Mortenson, *Supercongruences between truncated $_2F_1$ hypergeometric functions and their Gaussian analogs*, Trans. Amer. Math. Soc. **355** (2003), 987–1007.

[16] E. Mortenson, *Supercongruences for truncated $_{n+1}F_n$ hypergeometric series with applications to certain weight three newforms*, Proc. Amer. Math. Soc. **133** (2005), 321–330.

[17] H. Pan, R. Tauraso and C. Wang, *A local-global theorem for p-adic supercongruences*, J. reine angew. Math. **790** (2022), 53–83.

[18] A.M. Robert, *A Course in p-adic Analysis*, Graduate Texts in Mathematics, 198. Springer, New York, 2000.

[19] N.J.A. Sloane, Sequence A005259 in OEIS, http://oeis.org/A005259.

[20] Z.-W. Sun, *On congruences related to central binomal coefficients*, J. Number Theory **131** (2011), 2219–2238.

[21] Z.-W. Sun, *Super congruences and Euler numbers*, Sci. China Math. **54** (2011), no. 12, 2509–2535.

[22] Z.-W. Sun, *On sums of Apéry polynomials and related congruences*, J. Number Theorey **132** (2012), no. 11, 2673–2699.

[23] Z.-W. Sun, *On sums involving products of three binomial coefficients*, Acta Arith. **156** (2012), no. 2, 123–141.

[24] Z.-W. Sun, *Supecongruences involving products of two binomial coefficients*, Finite Fields Appl. **22** (2013), 24–44.

[25] Z.-W. Sun, *Open conjectures on congruences*, Nanjing Univ. J. Math. Biquarterly **36** (2019), no. 1, 1–99.

[26] H. Swisher, *On the supercongruence conjectures of van Hamme*, Res. Math. Sci. **2** (2015), Art. 18, 21 pp.

[27] R. Tauraso, *Supercongruences for a truncated hypergeometric series*, Integers **12** (2012), Paper No. A45, 12 pp.

[28] L. van Hamme, *Some conjectures concerning partial sums of generalized hypergeometric series*, Lect. Notes Pure Appl. Math. **192** (1997), 223–236.

[29] C. Wang, *Proof of a congruence concerning truncated hypergeometric series* ${}_6F_5$, preprint, arXiv:1812.10324.

[30] F.J.W. Whipple, *On well-poised series, generalized hypergeometric series having parameters in pairs, each pair with the same sum*, Proc. London Math. Soc. **24** (1926), 247–263.

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