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SOME PARAMETRIC CONGRUENCES INVOLVING GENERALIZED CENTRAL TRINOMIAL COEFFICIENTS

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Abstract. For $n = 0, 1, 2, \ldots$ and $b, c \in \mathbb{Z}$, the nth generalized central trinomial coefficient $T_n(b, c)$ is the coefficient of x^n in the expansion of $(x^2 + bx + c)^n$. In particular, $T_n = T_n(1, 1)$ $(n = 0, 1, 2, \ldots)$ are central trinomial coefficients. Let p be an odd prime. For any $b, c \in \mathbb{Z}$ with $p \nmid bc(b + 2c)$, we determine

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k(b, c^2)}{4^k (b + 2c)^k} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k(b, c)}{(4b)^k}$$

modulo p^2. As consequences,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{12^k} T_k \equiv \left(\frac{p}{3}\right) \frac{3^{p-1} + 3}{4} \pmod{p^2}$$

provided $p > 3$ (where $(-)$ denotes the Legendre symbol), and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k(2, -1)}{8^k} \equiv \begin{cases} 2x - p/(2x) \pmod{p^2} & \text{if } p = x^2 + 4y^2 \ (x, y \in \mathbb{Z}) \text{ and } 4 \mid x - 1, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

1. Introduction

For any $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $b, c \in \mathbb{Z}$, the generalized central trinomial coefficient $T_n(b, c)$ (cf. [12]) is defined as the coefficient of x^n in the expansion of $(x^2 + bx + c)^n$ (or the constant term in the expansion of $(x + b + c/x)^n$). By the multinomial theorem, it is clear that

$$T_n(b, c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} b^{n-2k} c^k, \tag{1.1}$$

where $\lfloor x \rfloor$ is the floor function. The generalized central trinomial coefficients have many interesting combinatorial interpretations; for example, from (1.1), it is easy to see that $T_n(b, c)$ with $b, c \in \mathbb{N}$ counts the colored lattice paths from $(0, 0)$ to $(n, 0)$ using only steps $U = (1, 1)$, $D = (1, -1)$ and $H = (1, 0)$, where H and D may have b and c colors, respectively. Note that $T_n := T_n(1, 1)$ is the nth central trinomial coefficient and $T_n(2, 1)$ is exactly the central

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binomial coefficient $\binom{2n}{n}$. The generalized central trinomial coefficients are also related to the well-known Legendre polynomials

$$P_n(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k = \sum_{k=0}^{n} \frac{\binom{n}{k}\binom{2k}{k}}{2^k}(\sqrt{x^2-1})^k(x-\sqrt{x^2-1})^{n-k} \tag{1.2}$$

(cf. [4, p. 38]) via the following identity (see [12, 16, 15]):

$$T_n(b,c) = (\sqrt{d})^n P_n\left(\frac{b}{\sqrt{d}}\right), \tag{1.3}$$

where $d = b^2 - 4c \neq 0$.

It is known that sums involving products of the binomial coefficients (e.g., $\binom{2k}{k}$, $\binom{2k}{k}^2$, $\binom{2k}{k}\binom{3k}{k}$, $\binom{2k}{k}^3$) usually have some interesting congruence properties. Since $T_n(b,c)$ is a natural extension of $\binom{2n}{n}$, Z.-W. Sun [16, 15] investigated congruences for sums involving generalized central trinomial coefficients systematically. In particular, Sun [16, Theorem 2.1] determined

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}T_k(b,c)}{m^k} \pmod{p}$$

for any $b, c, m \in \mathbb{Z}$ and odd prime p with $p \nmid m$. As a corollary, he obtained that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}T_k}{12^k} \equiv \left(\frac{6}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{64^k} \equiv \left(\frac{p}{3}\right) \pmod{p}, \tag{1.4}$$

where $(-)$ denotes the Legendre symbol. For more congruence properties of the generalized central trinomial coefficients, one may consult [2, 5, 6, 10, 16, 15].

As in [7], for any $n \in \mathbb{N}$ and $x \in \mathbb{C}$, we define

$$w_n(x) := \begin{cases} \frac{(\alpha+1)\alpha^n - (\alpha^{-1}+1)\alpha^{-n}}{\alpha - \alpha^{-1}}, & \text{if } x \neq \pm 1, \\ 2n+1, & \text{if } x = 1, \\ (-1)^n, & \text{if } x = -1, \end{cases}$$

where $\alpha = x + \sqrt{x^2-1}$.

The first purpose of this paper is to establish the following parametric congruence as a generalization of (1.4).

Theorem 1.1. *Let p be an odd prime and let $b, c \in \mathbb{Z}$ with $p \nmid c(b+2c)$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}T_k(b,c^2)}{4^k(b+2c)^k} \equiv w_{(p-1)/2}\left(\frac{b-6c}{b+2c}\right) \pmod{p^2}. \tag{1.5}$$

Applying Theorem 1.1 with $b = c = 1$, we obtain the following result conjectured by Sun [16, Conjecture 2.1].

Corollary 1.1. *For any prime $p > 3$, we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k}{12^k} \equiv \left(\frac{p}{3}\right) \frac{3^{p-1}+3}{4} \pmod{p^2}. \tag{1.6}$$

Now we state our second theorem.

Theorem 1.2. *Let p be an odd prime. For any $b, c \in \mathbb{Z}$ with $p \nmid b$, we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k(b,c)}{(4b)^k} \equiv p \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{4k+1} \left(\frac{c}{b^2}\right)^k \pmod{p^2}. \tag{1.7}$$

Consequently,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k(2,-1)}{8^k} \equiv \begin{cases} 2x - p/(2x) \pmod{p^2} & \text{if } p = x^2 + 4y^2 \ (x, y \in \mathbb{Z}) \text{ and } 4 \mid x-1, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.8}$$

To deduce (1.8) from (1.7), we need the following auxiliary result.

Theorem 1.3. *Let p be an odd prime. Then*

$$p \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(-4)^k (4k+1)} \equiv \begin{cases} 2x - p/(2x) \pmod{p^2} & \text{if } p = x^2 + 4y^2 \ (x, y \in \mathbb{Z}) \text{ and } 4 \mid x-1, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.9}$$

We are going to prove Theorem 1.1 and Corollary 1.1 in the next section, and show Theorems 1.2 and 1.3 in Section 3.

2. Proofs of Theorem 1.1 and Corollary 1.1

In order to show Theorem 1.1, we need the following transformation of $T_n(b, c^2)$ which follows from (1.3) and [4, (3.136)].

Lemma 2.1. *For $n \in \mathbb{N}$ and $b, c \in \mathbb{Z}$ we have*

$$T_n(b, c^2) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} (b+2c)^{n-k} (-c)^k. \tag{2.1}$$

Proof. Denote the right-hand side of (2.1) by a_n. Via the Zeilberger algorithm (cf. [13]), we find the following recurrence :

$$-(b-2c)(b+2c)(n+1)a_n + b(2n+3)a_{n+1} - (n+2)a_{n+2} = 0 \ (n \in \mathbb{N}).$$

Note that the same recurrence holds for $T_n(b, c^2)$. Moreover, it is easy to see that $T_0(b, c^2) = a_0$ and $T_1(b, c^2) = a_1$. Thus (2.1) follows by induction on n. $\square$

Lemma 2.2. *Let $n,j\in\mathbb{N}$ with $n\ge j$. Then we have*

$$\sum_{k=j}^{n}\frac{\binom{2k}{k}\binom{k}{j}}{4^k}=\frac{n+1}{2^{2n+1}(2j+1)}\binom{n}{j}\binom{2n+2}{n+1}. \tag{2.2}$$

Proof. This can be easily proved by induction on n. $\square$

Kh. Hessami Pilehrood, T. Hessami Pilehrood and R. Tauraso [7, Theorem 2] completely determined

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}t^k}{2k+1}\pmod{p^3},$$

where p is an odd prime and t is a p-adic unit. We need their result in the modulus p case.

Lemma 2.3 (cf. [7, Theorem 2])**.** *For any odd prime p and $t\in\mathbb{Z}_p^\times$, we have*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}t^k}{2k+1}\equiv\frac{w_{(p-1)/2}(1-8t)-(-16t)^{(p-1)/2}}{p}\pmod{p}. \tag{2.3}$$

Lemma 2.4. *For any odd prime p, we have*

$$\binom{2p}{p}\equiv 2\pmod{p^2}\quad\textit{and}\quad\binom{p-1}{(p-1)/2}\equiv(-1)^{(p-1)/2}4^{p-1}\pmod{p^2}.$$

Proof. These can be verified directly for $p=3$. For $p>3$, we even have $\binom{2p}{p}\equiv 2\pmod{p^3}$ by J. Wolstenholme [18] and $\binom{p-1}{(p-1)/2}\equiv(-1)^{(p-1)/2}4^{p-1}\pmod{p^3}$ by F. Morley [11]. $\square$

Proof of Theorem 1.1. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(b,c^2)}{4^k(b+2c)^k}&=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}\sum_{l=0}^{k}\binom{k}{l}\binom{2l}{l}\left(\frac{-c}{b+2c}\right)^l\\&=\sum_{l=0}^{p-1}\binom{2l}{l}\left(\frac{-c}{b+2c}\right)^l\sum_{k=l}^{p-1}\frac{\binom{2k}{k}\binom{k}{l}}{4^k}\\&=\frac{p\binom{2p}{p}}{2^{2p-1}}\sum_{l=0}^{p-1}\frac{\binom{2l}{l}\binom{p-1}{l}\left(\frac{-c}{b+2c}\right)^l}{2l+1}.\end{aligned}$$

Note that $\binom{2l}{l}/(2l+1)\equiv 0\pmod{p}$ for $l\in\{(p+1)/2,\ldots,p-1\}$ and $\binom{p-1}{l}\equiv(-1)^l\pmod{p}$ for $0\le l<p$. Thus we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(b,c^2)}{4^k(b+2c)^k}\equiv\frac{\binom{2p}{p}\binom{p-1}{(p-1)/2}^2\left(\frac{-c}{b+2c}\right)^{(p-1)/2}}{2^{2p-1}}+\frac{p\binom{2p}{p}}{2^{2p-1}}\sum_{l=0}^{(p-3)/2}\frac{\binom{2l}{l}\left(\frac{c}{b+2c}\right)^l}{2l+1}\pmod{p^2}.$$

In view of Lemma 2.4, we arrive at

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(b,c^2)}{4^k(b+2c)^k}\equiv\left(\frac{-16c}{b+2c}\right)^{(p-1)/2}+\frac{p}{4^{p-1}}\sum_{l=0}^{(p-3)/2}\frac{\binom{2l}{l}\left(\frac{c}{b+2c}\right)^l}{2l+1}\pmod{p^2}.$$

Then we complete the proof by Lemma 2.3 and Fermat's little theorem. □

Proof of Corollary 1.1. To show (1.6), it remains to prove

$$w_{(p-1)/2}\left(-\frac{5}{3}\right)\equiv\left(\frac{p}{3}\right)\frac{3^{p-1}+3}{4}\pmod{p^2}.\tag{2.4}$$

It is easy to see that

$$w_{(p-1)/2}\left(-\frac{5}{3}\right)=\frac{(-1)^{(p-1)/2}}{4}\left((1/3)^{(p-1)/2}+3\times 3^{(p-1)/2}\right).$$

From [8, p. 51], we know that $a^{(p-1)/2}\equiv(\frac{a}{p})\pmod p$ for any integer $a\not\equiv 0\pmod p$. Thus we may write $3^{(p-1)/2}$ as $(\frac{3}{p})(1+ph)$, where h is a p-adic integer. In view of this,

$$3^{p-1}=(3^{(p-1)/2})^2\equiv 1+2ph\pmod{p^2}.$$

By the above and with the help of the law of quadratic reciprocity (cf. [8]), we get

$$\begin{aligned}w_{(p-1)/2}\left(-\frac{5}{3}\right)&=\frac{(-1)^{(p-1)/2}}{4}\left(\frac{1}{(\frac{3}{p})(1+ph)}+3\left(\frac{3}{p}\right)(1+ph)\right)\\&\equiv\frac{(-1)^{(p-1)/2}}{4}\left(\frac{3}{p}\right)(4+2ph)\\&\equiv\frac{3^{p-1}+3}{4}\left(\frac{3}{p}\right)\left(\frac{-1}{p}\right)\\&=\left(\frac{p}{3}\right)\frac{3^{p-1}+3}{4}\pmod{p^2}.\end{aligned}$$

This proves (2.4). □

3. Proofs of Theorems 1.2 and 1.3

To show Theorem 1.3, we need the following identity due to Kummer (cf. [1, p. 126]).

Lemma 3.1. *For any $a,b\in\mathbb{C}$ with $a,a-b,a/2-b\notin\{-1,-2,-3,\ldots\}$, we have*

$$\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k(b)_k}{(1)_k(a-b+1)_k}=\frac{\Gamma(a-b+1)\Gamma(\frac{a}{2}+1)}{\Gamma(a+1)\Gamma(\frac{a}{2}-b+1)},$$

where $(x)_k=\prod_{0\le j<k}(x+j)$ is the Pochhammer symbol and $\Gamma(\cdot)$ stands for the Gamma function.

We also need Morita's p-adic Gamma function Γ_p (cf. [14]), where p is an odd prime. Recall that $\Gamma_p(0) := 1$ and

$$\Gamma_p(n) := (-1)^n \prod_{\substack{1\le k<n\\ p\nmid k}} k \quad \text{for } n = 1, 2, 3, \ldots .$$

Let $\mathbb{Z}_p$ denote the ring of all p-adic integers. The definition of Γ_p can be extended to $\mathbb{Z}_p$ since $\mathbb{N}$ is a dense subset of $\mathbb{Z}_p$ with respect to the p-adic norm. It follows that

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } x \not\equiv 0 \pmod p, \\ -1, & \text{if } x \equiv 0 \pmod p. \end{cases}$$

It is known (cf. [14, p. 369]) that for any $x \in \mathbb{Z}_p$ we have

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\langle -x\rangle_p - 1}, \tag{3.1}$$

where $\langle x\rangle_p$ is the least nonnegative residue of x modulo p. It is also known (cf. [17]) that for $\alpha, t \in \mathbb{Z}_p$ we have

$$\Gamma_p(\alpha+tp) \equiv \Gamma_p(\alpha)\left(1 + tp(\Gamma_p'(0) + H_{p-1-\langle -\alpha\rangle_p})\right) \pmod{p^2}, \tag{3.2}$$

where $H_n = \sum_{k=1}^n 1/k$ denotes the nth harmonic number.

Proof of Theorem 1.3. It is easy to verify that

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(-4)^k(4k+1)} = \sum_{k=0}^{(p-1)/2} \frac{(\frac12)_k(\frac14)_k(-1)^k}{(1)_k(\frac54)_k}. \tag{3.3}$$

We first assume that $p \equiv 1 \pmod 4$. In this case, $\mathrm{ord}_p(p/(5/4)_k) \ge 0$ for all k among $0, 1, \ldots, (p-1)/2$, where $\mathrm{ord}_p(\cdot)$ stands for the p-adic order. It is easy to verify that

$$\begin{aligned} p&\sum_{k=0}^{(p-1)/2} \frac{(\frac{1-p}{2})_k(\frac{1-2p}{4})_k(-1)^k}{(1)_k(\frac54)_k} \\ &\equiv p\sum_{k=0}^{(p-1)/2} \frac{(\frac12)_k(\frac14)_k(-1)^k}{(1)_k(\frac54)_k}\left(1 - \frac{p}{2}\sum_{j=0}^{k-1}\frac{1}{1/2+j} - \frac{p}{2}\sum_{j=0}^{k-1}\frac{1}{1/4+j}\right) \pmod{p^2} \end{aligned}$$

and

$$\begin{aligned} p&\sum_{k=0}^{(p-1)/2} \frac{(\frac{2-p}{4})_k(\frac{1-p}{4})_k(-1)^k}{(1)_k(\frac54)_k} \\ &\equiv p\sum_{k=0}^{(p-1)/2} \frac{(\frac12)_k(\frac14)_k(-1)^k}{(1)_k(\frac54)_k}\left(1 - \frac{p}{4}\sum_{j=0}^{k-1}\frac{1}{1/2+j} - \frac{p}{4}\sum_{j=0}^{k-1}\frac{1}{1/4+j}\right) \pmod{p^2}. \end{aligned}$$

On the other hand, by Lemma 3.1 we have

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{(\frac{1-p}{2})_k(\frac{1-2p}{4})_k(-1)^k}{(1)_k(\frac54)_k}&=\lim_{t\to1}\sum_{k=0}^{\infty}\frac{(\frac{1-tp}{2})_k(\frac{1-2tp}{4})_k(-1)^k}{(1)_k(\frac54)_k}\\ =\lim_{t\to1}\frac{\Gamma(\frac54)\Gamma(\frac{5-tp}{4})}{\Gamma(\frac{3-tp}{2})\Gamma(\frac{4+tp}{4})}&=\frac{\Gamma(\frac54)\Gamma(\frac{p-1}{2})}{\Gamma(\frac{p-1}{4})\Gamma(\frac{4+p}{4})}\lim_{t\to1}\frac{\sin(\frac{3-tp}{2}\pi)}{\sin(\frac{5-tp}{4}\pi)}=(-1)^{(p-1)/4}\frac{2\Gamma(\frac54)\Gamma(\frac{p-1}{2})}{\Gamma(\frac{p-1}{4})\Gamma(\frac{4+p}{4})},\end{aligned}$$

where we have used the well-known formula $\Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)$ (cf. [14, p. 371]). Also,

$$\sum_{k=0}^{(p-1)/2}\frac{(\frac{2-p}{4})_k(\frac{1-p}{4})_k(-1)^k}{(1)_k(\frac54)_k}=\frac{\Gamma(\frac54)\Gamma(\frac{10-p}{8})}{\Gamma(\frac{6-p}{4})\Gamma(\frac{8+p}{8})}.$$

Combining the above we obtain

$$p\sum_{k=0}^{(p-1)/2}\frac{(\frac12)_k(\frac14)_k(-1)^k}{(1)_k(\frac54)_k}\equiv\sigma_1-\sigma_2\pmod{p^2},$$

where

$$\sigma_1:=\frac{2p\Gamma(\frac54)\Gamma(\frac{10-p}{8})}{\Gamma(\frac{6-p}{4})\Gamma(\frac{8+p}{8})}\quad\text{and}\quad\sigma_2:=(-1)^{(p-1)/4}\frac{2p\Gamma(\frac54)\Gamma(\frac{p-1}{2})}{\Gamma(\frac{p-1}{4})\Gamma(\frac{4+p}{4})}.$$

By [9], we have

$$H_{\lfloor p/2\rfloor}\equiv-2q_p(2)\pmod p\quad\text{and}\quad H_{\lfloor p/4\rfloor}\equiv-3q_p(2)\pmod p,$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$. It is easy to see that

$$\frac{\Gamma(\frac{10-p}{8})}{\Gamma(\frac{8+p}{8})}=(-1)^{(p-1)/4}\frac{8\Gamma_p(\frac{10-p}{8})}{p\Gamma_p(\frac{8+p}{8})}.$$

Thus, by (3.1) and (3.2) we have

$$\begin{aligned}\sigma_1=&\frac{16\Gamma_p(\frac54)\Gamma_p(\frac54-\frac p8)}{\Gamma_p(\frac32-\frac p4)\Gamma_p(1+\frac p8)}\\ \equiv&-\frac{16\Gamma_p(\frac54)^2}{\Gamma_p(\frac32)}\left(1-\frac p8H_{\lfloor p/4\rfloor}+\frac p4H_{\lfloor p/2\rfloor}\right)\\ \equiv&-2\Gamma_p\left(\frac14\right)^2\Gamma_p\left(\frac12\right)\left(1-\frac p8q_p(2)\right)\pmod{p^2}.\end{aligned}$$

Similarly, it is not hard to find that

$$\frac{\Gamma(\frac54)}{\Gamma(\frac{4+p}{4})}=(-1)^{(p-1)/4}\frac{4\Gamma_p(\frac54)}{p\Gamma_p(\frac{4+p}{4})}.$$

Thus, by (3.1) and (3.2) we arrive at

$$\begin{aligned}\sigma_2 &= (-1)^{(p-1)/4}\frac{8\Gamma_p(\frac{5}{4})\Gamma_p(\frac{p-1}{2})}{\Gamma_p(\frac{p-1}{4})\Gamma_p(\frac{4+p}{4})}\\ &\equiv (-1)^{(p+3)/4}\frac{8\Gamma_p(\frac{5}{4})\Gamma_p(-\frac{1}{2})}{\Gamma_p(-\frac{1}{4})}\left(1+\frac{p}{2}H_{\lfloor p/2\rfloor}-\frac{p}{4}H_{\lfloor p/4\rfloor}\right)\\ &\equiv -\Gamma_p\left(\frac{1}{4}\right)^2\Gamma_p\left(\frac{1}{2}\right)\left(1-\frac{p}{4}q_p(2)\right) \pmod{p^2}.\end{aligned}$$

In view of the above, we have

$$p\sum_{k=0}^{(p-1)/2}\frac{(\frac{1}{2})_k(\frac{1}{4})_k(-1)^k}{(1)_k(\frac{5}{4})_k}\equiv -\Gamma_p\left(\frac{1}{4}\right)^2\Gamma_p\left(\frac{1}{2}\right) \pmod{p^2}.$$

By [3], if $p=x^2+4y^2$ $(x,y\in\mathbb{Z})$ with $x\equiv 1 \pmod 4$ then

$$-\Gamma_p\left(\frac{1}{4}\right)^2\Gamma_p\left(\frac{1}{2}\right)\equiv 2x-\frac{p}{2x} \pmod{p^2}.$$

Thus, with aid of (3.3), we have (1.9) in the case $p\equiv 1 \pmod 4$.

Now we consider the remaining case $p\equiv 3 \pmod 4$. Note that $\mathrm{ord}_p(4k+1)=0$ for $k=0,1,\ldots,(p-1)/2$. Therefore, by Lemma 3.1 we have

$$\begin{aligned}p\sum_{k=0}^{(p-1)/2}\frac{(-1)^k(\frac{1}{2})_k(\frac{1}{4})_k}{(1)_k(\frac{5}{4})_k} &\equiv p\sum_{k=0}^{(p-1)/2}\frac{(-1)^k(\frac{1-p}{2})_k(\frac{1-2p}{4})_k}{(1)_k(\frac{5}{4})_k}\\ &= p\lim_{u\to(1-p)/2}\sum_{k=0}^{(p-1)/2}\frac{(-1)^k(u)_k(u-\frac{1}{4})_k}{(1)_k(\frac{5}{4})_k}\\ &= p\lim_{u\to(1-p)/2}\frac{\Gamma(\frac{5}{4})\Gamma(\frac{u}{2}+1)}{\Gamma(u+1)\Gamma(\frac{5}{4}-\frac{u}{2})}\\ &= \frac{p\Gamma(\frac{5}{4})\Gamma(\frac{5-p}{4})}{\Gamma(\frac{4+p}{4})}\lim_{u\to(1-p)/2}\frac{1}{\Gamma(u+1)}=0 \pmod{p^2},\end{aligned}$$

where in the last step we have used the fact that

$$\lim_{u\to n}\frac{1}{\Gamma(-u)}=\frac{1}{\pi}\lim_{u\to n}\Gamma(1+u)\sin((u+1)\pi)=0 \quad \text{for each } n\in\mathbb{N}.$$

Combining this with (3.3), we find that (1.9) also holds in the case $p\equiv 3 \pmod 4$.

By the above, we have completed the proof of Theorem 1.3. □

Proof of Theorem 1.2. In view of (1.1) and Lemma 2.2, we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(b,c)}{(4b)^k}&=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{2j}\binom{2j}{j}\left(\frac{c}{b^2}\right)^j\\&=\sum_{j=0}^{(p-1)/2}\binom{2j}{j}\left(\frac{c}{b^2}\right)^j\sum_{k=2j}^{p-1}\frac{\binom{2k}{k}\binom{k}{2j}}{4^k}\\&=\frac{p\binom{2p}{p}}{2^{2p-1}}\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{p-1}{2j}}{4j+1}\left(\frac{c}{b^2}\right)^j.\end{aligned}$$

If $p\equiv 3\pmod 4$, then $p\nmid(4j+1)$ for all $j=0,1,\ldots,(p-1)/2$. In this case, by Lemma 2.4 and Fermat's little theorem, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(b,c)}{(4b)^k}\equiv p\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}}{4j+1}\left(\frac{c}{b^2}\right)^j\pmod{p^2}.$$

Now suppose $p\equiv 1\pmod 4$. Then, by Lemma 2.4, we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(b,c)}{(4b)^k}&=\frac{p\binom{2p}{p}}{2^{2p-1}}\sum_{\substack{0\le j\le(p-1)/2\\ j\ne(p-1)/4}}\frac{\binom{2j}{j}\binom{p-1}{2j}}{4j+1}\left(\frac{c}{b^2}\right)^j+\frac{\binom{2p}{p}\binom{(p-1)/2}{(p-1)/4}\binom{p-1}{(p-1)/2}}{2^{2p-1}}\left(\frac{c}{b^2}\right)^{(p-1)/4}\\&\equiv p\sum_{\substack{0\le j\le(p-1)/2\\ j\ne(p-1)/4}}\frac{\binom{2j}{j}}{4j+1}\left(\frac{c}{b^2}\right)^j+\binom{(p-1)/2}{(p-1)/4}\left(\frac{c}{b^2}\right)^{(p-1)/4}\\&=p\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}}{4j+1}\left(\frac{c}{b^2}\right)^j\pmod{p^2}.\end{aligned}$$

Combining the above, we have proved (1.7).

In light of Theorem 1.3,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(2,-1)}{8^k}\equiv p\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{(-4)^k(4k+1)}\pmod{p^2}.$$

Combining this with (1.7), we immediately obtain (1.8). This ends our proof. □

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