Final version for Contrib. Discrete Math.

A NEW TRIGONOMETRIC IDENTITY WITH APPLICATIONS

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Abstract. In this paper we obtain a new curious identity involving trigonometric functions. Namely, for any positive odd integer n, we prove that

$$\sum_{k=1}^{n}(-1)^k(\cot kx)\sin k(n-k)x=\frac{1-n}{2},$$

which is equivalent to the identity

$$\sum_{k=1}^{n}(-1)^k U_{n-k}(\cos kx)=-\frac{n+1}{2},$$

where $U_m(z)$ stands for the mth Chebyshev polynomial of the second kind. As a consequence, for any positive odd integer n and positive integer m, we obtain the identity

$$\sum_{k=1}^{n}(-1)^k k^{2m}B_{2m+1}\left(\frac{n-k}{2}\right)=0,$$

where $B_j(x)$ denotes the Bernoulli polynomial of degree j.

1. Introduction

Let $\mathbb{Z}^+$ denote the set of all positive integers. J.-C. Liu and F. Petrov [2, (2.11)] showed that if $\omega=e^{2\pi i/(3n+2)}$ with $n\in\mathbb{Z}^+$ then

$$\sum_{k=1}^{2n+1}\frac{(-1)^k\omega^{k(3k+1)/2}}{1-\omega^{3k}}=-\frac{n+1}{2},\tag{1.1}$$

which has the equivalent form (cf. [2, (2.17)])

$$\sum_{k=1}^{2n+1}\left(\frac{y^k}{1+y^{3k}}+\frac{(-y)^k}{1-y^{3k}}\right)=-n-1,\tag{1.2}$$

where $y=e^{2\pi i/(6n+4)}$. Motivated by this, Z.-W. Sun [3] conjectured that if $m,n\in\{2,3,\ldots\}$ and $\delta\in\{0,1\}$, then for any primitive $(m(n-\delta)-(-1)^\delta)$-th

Date: (August 29, 2022), and in revised form (February 28, 2023).
Key words and phrases. Trigonometric identity, roots of unity, Bernoulli polynomials
2020 *Mathematics Subject Classification.* Primary 05A19, 33B10; Secondary 11B68.
The first author is the corresponding author. The two authors were supported by the Natural Science Foundation of China (grants 11971222 and 12071208 respectively).

root of unity ζ, we have the identity

$$\mathrm{Re}\bigg(\sum_{k=1}^{n-1}\bigg(\frac{\zeta^k}{1+\zeta^{km}}-(-1)^{n+\delta}\frac{(-\zeta)^k}{1-\zeta^{km}}\bigg)\bigg)=(-1)^{n-1}\left\lfloor\frac{n}{2}\right\rfloor. \tag{1.3}$$

This was confirmed by Nemo and Sun in the cases $\delta=0$ and $\delta=1$ respectively; see [3] for the detailed proofs.

Inspired by the above work, we establish the following new result.

Theorem 1.1. *Let n be any positive odd integer. Then, for any complex number q with $|q|=1$ and $q^k\neq 1$ for all $k=1,\ldots,n$, we have*

$$\mathrm{Re}\bigg(\sum_{k=1}^{n}\frac{(-1)^k q^{-k(n-k)/2}}{1-q^k}\bigg)=-\frac{n+1}{4}. \tag{1.4}$$

Equivalently, we have the trigonometric identity

$$\sum_{k=1}^{n}(-1)^k U_{n-k}(\cos kx)=-\frac{n+1}{2}, \tag{1.5}$$

where x is a real number, $U_m(z)$ is the m-th Chebyshev polynomial of the second kind, defined by $U_m(\cos\theta)=(\sin{(m+1)\theta})/\sin\theta$.

Corollary 1.2. *Suppose that n is a positive odd integer and m is a positive integer. Then we have*

$$\sum_{k=1}^{n}(-1)^k k^{2m}B_{2m+1}\left(\frac{n-k}{2}\right)=0, \tag{1.6}$$

where $B_j(x)$ denotes the Bernoulli polynomial of degree j.

With the help of Theorem 1.1, we obtain the following result.

Theorem 1.3. *Let $l,m,n\in\mathbb{Z}^+$ with $l\equiv m \pmod 2$ and $n\equiv 1 \pmod 2$. Then, for any primitive $(mn+l)$-th root of unity ζ, we have*

$$\mathrm{Re}\bigg(\sum_{k=1}^{n}\frac{\zeta^{k(km+l)/2}}{1-\zeta^{km}}\bigg)=-\frac{n+1}{4}. \tag{1.7}$$

Applying Theorem 1.3 with $l=1$ and $m=3$, we immediately get the following consequence.

Corollary 1.4. *Let n be a nonnegative integer and let ζ be a primitive $(6n+4)$-th root of unity. Then*

$$\sum_{k=1}^{2n+1}\frac{\zeta^{k(3k+1)/2}}{1-\zeta^{3k}}=-\frac{n+1}{2}. \tag{1.8}$$

It is interesting to compare our (1.8) with Liu and Petrov's (1.1). Actually, we first found (1.8) motivated by (1.1) and then discovered the more general Theorem 1.3 and related Theorem 1.1.

We are going to prove Theorem 1.1 in the next section, and show Corollary 1.2 and Theorem 1.3 in Section 3.

2. Proof of Theorem 1.1

Lemma 2.1. *Let n be a positive odd integer, and let z be any complex number. Then*

$$\sum_{\substack{1\leqslant k\leqslant n\\ 0\leqslant j<(n-k)/2}} (-1)^k z^{k(2j+k-n)} = 0. \tag{2.1}$$

Proof. Let σ denote the left-hand side of (2.1). Then, by changing the order of summation, we get

$$\begin{aligned}\sigma &= \sum_{j=0}^{(n-3)/2}\sum_{k=1}^{n-2j-1}(-1)^k z^{k(2j+k-n)}\\ &= \sum_{j=0}^{(n-3)/2}\sum_{l=1}^{n-2j-1}(-1)^{n-2j-l} z^{(n-2j-l)(2j+(n-2j-l)-n)}\\ &= (-1)^n\sum_{j=0}^{(n-3)/2}\sum_{l=1}^{n-2j-1}(-1)^l z^{l(2j+l-n)} = -\sigma\end{aligned}$$

and hence $\sigma = 0$. □

Proof of Theorem 1.1. Write $q = e^{2ix} = \cos 2x + i\sin 2x$ with x real, and let L denote the sum in (1.4). Then

$$\begin{aligned}L &= \sum_{k=1}^{n}(-1)^k\frac{\cos k(n-k)x - i\sin k(n-k)x}{1-\cos 2kx - i\sin 2kx}\\ &= \sum_{k=1}^{n}(-1)^k\frac{(1-\cos 2kx + i\sin 2kx)(\cos k(n-k)x - i\sin k(n-k)x)}{(1-\cos 2kx)^2-(i\sin 2kx)^2}\end{aligned}$$

and hence

$$\begin{aligned}\operatorname{Re}(L) &= \sum_{k=1}^{n}(-1)^k\frac{(1-\cos 2kx)\cos k(n-k)x + (\sin 2kx)\sin k(n-k)x}{2-2\cos 2kx}\\ &= \frac{1}{2}\sum_{k=1}^{n}(-1)^k\cos k(n-k)x + \frac{1}{2}\sum_{k=1}^{n}(-1)^k(\cot kx)\sin k(n-k)x.\end{aligned}$$

It follows that

$$\begin{aligned}2\operatorname{Re}(L) &= \sum_{k=1}^{n}(-1)^k\frac{(\sin kx)\cos k(n-k)x + (\cos kx)\sin k(n-k)x}{\sin kx}\\ &= \sum_{k=1}^{n}(-1)^k\frac{\sin k(n+1-k)x}{\sin kx} = \sum_{k=1}^{n}(-1)^k U_{n-k}(\cos kx).\end{aligned}\tag{2.2}$$

Thus (1.4) is equivalent to (1.5).

Set $z = e^{ix}$. Then

$$\begin{aligned}2\,\mathrm{Re}(L) =& \sum_{k=1}^{n}(-1)^k \frac{z^{k(n+1-k)} - z^{-k(n+1-k)}}{z^k - z^{-k}}\\ =& \sum_{k=1}^{n}(-1)^k \sum_{j=0}^{n-k}(z^k)^j (z^{-k})^{n-k-j} = \sum_{k=1}^{n}(-1)^k \sum_{j=0}^{n-k} z^{k(2j+k-n)}.\end{aligned}$$

For each $k = 1, \ldots, n$, clearly

$$\sum_{(n-k)/2<j\leqslant n-k} z^{k(2j+k-n)} = \sum_{0\leqslant s<(n-k)/2} z^{k(2(n-k-s)+k-n)} = \sum_{0\leqslant s<(n-k)/2} z^{-k(2s+k-n)}.$$

Thus

$$\begin{aligned}2\,\mathrm{Re}(L) =& \sum_{\substack{k=1\\ 2\mid n-k}}^{n} (-1)^k z^{k(2(n-k)/2+k-n)}\\ &+ \sum_{k=1}^{n}(-1)^k \sum_{0\leqslant j<(n-k)/2} \left(z^{k(2j+k-n)} + z^{-k(2j+k-n)}\right)\\ =& \sum_{r=0}^{(n-1)/2} (-1)^{n-2r} + \sum_{\substack{1\leqslant k\leqslant n\\ 0\leqslant j<(n-k)/2}} (-1)^k\left(z^{k(2j+k-n)} + z^{-k(2j+k-n)}\right).\end{aligned}$$

Combining this with Lemma 2.1, we obtain that

$$2\,\mathrm{Re}(L) = (-1)^n \frac{n+1}{2} = -\frac{n+1}{2}$$

and hence (1.4) follows.

The proof of Theorem 1.1 is now complete. □

3. Proofs of Corollary 1.2 and Theorem 1.3

Recall that the Bernoulli numbers $B_0, B_1, B_2, \ldots$ are given by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \quad (0 < |x| < 2\pi).$$

Proof of Corollary 1.2. Note that

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^k \cos k(n-k)x\\ &= (-1)^n + \sum_{k=1}^{(n-1)/2} \left((-1)^k + (-1)^{n-k}\right)\cos k(n-k)x = -1\end{aligned}$$

since n is odd. Combining this with (2.2), we see that (1.5) has the following equivalent form:

$$\sum_{k=1}^{n}(-1)^k(\cot kx)\sin k(n-k)x = \frac{1-n}{2}. \tag{3.1}$$

It is well known that

$$\cot x = \sum_{j=0}^{\infty}\frac{(-1)^j 2^{2j}B_{2j}x^{2j-1}}{(2j)!} \quad (0<|x|<\pi)$$

(cf. [1, p. 232]) and

$$\sin x = \sum_{j=0}^{\infty}\frac{(-1)^j x^{2j+1}}{(2j+1)!}.$$

So, by (3.1) we have

$$\frac{1-n}{2} = \sum_{k=1}^{n}(-1)^k x^{2m}\sum_{j=0}^{m}\frac{(-1)^j 2^{2j}B_{2j}k^{2j-1}}{(2j)!}\cdot\frac{(-1)^{m-j}(k(n-k))^{2m-2j+1}}{(2m-2j+1)!}$$

whenever $0<|x|<\pi/n$. Comparing the coefficients of x^{2m} in the both sides of the above equality, we obtain

$$\sum_{k=1}^{n}(-1)^k k^{2m}\sum_{j=0}^{m}\frac{2^{2j}B_{2j}}{(2j)!}\cdot\frac{(n-k)^{2m-2j+1}}{(2m-2j+1)!} = 0,$$

which is equivalent to the desired identity (1.6). □

Proof of Theorem 1.3. Clearly $L = mn + l$ is even. For $k = 1,\dots,n$, we have

$$\zeta^{k(km+l)/2} = \zeta^{k(L-m(n-k))/2} = (-1)^k\zeta^{-mk(n-k)/2}.$$

Thus

$$\sum_{k=1}^{n}\frac{\zeta^{k(km+l)/2}}{1-\zeta^{km}} = \sum_{k=1}^{n}\frac{(-1)^k(\zeta^m)^{-k(n-k)/2}}{1-(\zeta^m)^k}.$$

Note that

$$L_0 := \frac{L}{\gcd(L,m)} > \frac{mn}{\gcd(L,m)} \geqslant n$$

and $q=\zeta^m$ is a primitive L_0-th root of unity. Applying Theorem 1.1 we see that the real part of

$$\sum_{k=1}^{n}\frac{\zeta^{k(km+l)/2}}{1-\zeta^{km}} = \sum_{k=1}^{n}\frac{(-1)^k q^{-k(n-k)/2}}{1-q^k}$$

is $-(n+1)/4$. This concludes the proof of Theorem 1.3. □

References

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