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# ON SOME DETERMINANTS INVOLVING THE TANGENT FUNCTION

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ABSTRACT. Let $p$ be an odd prime and let $a,b\in\mathbb{Z}$ with $p\nmid ab$. In this paper we mainly evaluate

$$T_p^{(\delta)}(a,b,x):=\det\left[x+\tan\pi\frac{aj^2+bk^2}{p}\right]_{\delta\leqslant j,k\leqslant(p-1)/2}\quad(\delta=0,1).$$

For example, in the case $p\equiv 3\pmod 4$, we show that $T_p^{(1)}(a,b,0)=0$ and

$$T_p^{(0)}(a,b,x)=\begin{cases}2^{(p-1)/2}p^{(p+1)/4} & \text{if } (\frac{ab}{p})=1,\\ p^{(p+1)/4} & \text{if } (\frac{ab}{p})=-1,\end{cases}$$

where $(\frac{\cdot}{p})$ is the Legendre symbol. When $(\frac{-ab}{p})=-1$, we also evaluate the determinant $\det[x+\cot\pi\frac{aj^2+bk^2}{p}]_{1\leqslant j,k\leqslant(p-1)/2}$. In addition, we pose several conjectures one of which states that for any prime $p\equiv 3\pmod 4$ there is an integer $x_p\equiv 1\pmod p$ such that

$$\det\left[\sec 2\pi\frac{(j-k)^2}{p}\right]_{0\leqslant j,k\leqslant p-1}=-p^{(p+3)/2}x_p^2.$$

## 1. INTRODUCTION

Let $p$ be an odd prime. It is well known that the numbers

$$0^2,\ 1^2,\ \ldots,\ \left(\frac{p-1}{2}\right)^2$$

are pairwise incongruent modulo $p$. In [10], the author investigated the determinants

$$S(d,p)=\det\left[\left(\frac{j^2+dk^2}{p}\right)\right]_{1\leqslant j,k\leqslant(p-1)/2}$$

and

$$T(d,p)=\det\left[\left(\frac{j^2+dk^2}{p}\right)\right]_{0\leqslant j,k\leqslant(p-1)/2},$$

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where $d$ is an integer not divisible by $p$, and $(\frac{\cdot}{p})$ is the Legendre symbol. In particular, Sun [10] showed that if $(\frac{d}{p})=1$ then

$$\left(\frac{-S(d,p)}{p}\right)=1 \ \text{ and } \ T(d,p)=\frac{p-1}{2}S(d,p).$$

Inspired by the determinants $S(d,p)$ and $T(d,p)$ with $d\in\mathbb{Z}$ and $p\nmid d$, and noting that the tangent function $\tan x$ has period $\pi$, for $a,b\in\mathbb{Z}$ we introduce

$$T_p^{(0)}(a,b,x):=\det\left[x+\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2} \tag{1.1}$$

and

$$T_p^{(1)}(a,b,x):=\det\left[x+\tan\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}, \tag{1.2}$$

and denote $T_p^{(0)}(a,b,0)$ and $T_p^{(1)}(a,b,0)$ by $T_p^{(0)}(a,b)$ and $T_p^{(1)}(a,b)$, respectively. To study the novel determinants $T_p^{(0)}(a,b,x)$ and $T_p^{(1)}(a,b,x)$, we first find their values by numerical experiments via `Mathematica`, and then seek for detailed proofs via related known results involving roots of unity.

Now we present our main results.

**Theorem 1.1.** *Let $p>3$ be a prime, and let $a,b\in\mathbb{Z}$ with $p\nmid ab$.*

(i) *Assume that $p\equiv 1 \pmod 4$. Then*

$$T_p^{(0)}(a,b,-x)=-T_p^{(0)}(a,b,x), \tag{1.3}$$

*and in particular $T_p^{(0)}(a,b)=0$. If $(\frac{ab}{p})=1$ and $b\equiv ac^2 \pmod p$ with $c\in\mathbb{Z}$, then*

$$T_p^{(1)}(a,b,x)=\left(\frac{2c}{p}\right)p^{(p-3)/4}\varepsilon_p^{(\frac{a}{p})(2-(\frac{2}{p}))h(p)}, \tag{1.4}$$

*where $\varepsilon_p$ and $h(p)$ are the fundamental unit and the class number of the real quadratic field $\mathbb{Q}(\sqrt{p})$ respectively. When $(\frac{ab}{p})=-1$, we have*

$$T_p^{(1)}(a,b,x)=T_p^{(1)}(a,b)=\pm 2^{(p-1)/2}p^{(p-3)/4}. \tag{1.5}$$

(ii) *Suppose that $p\equiv 3 \pmod 4$. Then*

$$T_p^{(1)}(a,b,-x)=-T_p^{(1)}(a,b,x), \tag{1.6}$$

*and in particular $T_p^{(1)}(a,b)=0$. Also,*

$$T_p^{(0)}(a,b,x)=\begin{cases}2^{(p-1)/2}p^{(p+1)/4} & \text{if } (\frac{ab}{p})=1,\\ p^{(p+1)/4} & \text{if } (\frac{ab}{p})=-1.\end{cases} \tag{1.7}$$

*Remark* 1.1. When $p$ is a prime with $p\equiv 1\pmod 4$, and $a$ and $b$ are integers with $(\frac{ab}{p})=-1$, we are unable to determine the sign in (1.5). For any prime $p\equiv 3\pmod 4$ and integers $a$ and $b$ with $p\nmid ab$, the identity (1.7) looks surprising and interesting. We believe that Theorem 1.1 has certain potential applications.

**Theorem 1.2.** *Let $n>1$ be an odd integer, and let $a$ and $b$ be integers with $\gcd(ab,n)=1$. Then*

$$\det\left[x+\tan\pi\frac{aj+bk}{n}\right]_{0\leqslant j,k\leqslant n-1}+\det\left[-x+\tan\pi\frac{aj+bk}{n}\right]_{0\leqslant j,k\leqslant n-1}=0 \tag{1.8}$$

*and*

$$\det\left[x+\tan\pi\frac{aj+bk}{n}\right]_{1\leqslant j,k\leqslant n-1}=\left(\frac{-ab}{n}\right)n^{n-2}, \tag{1.9}$$

*where $(\frac{\cdot}{n})$ is the Jacobi symbol.*

For the cotangent function, we establish the following two theorems.

**Theorem 1.3.** *Let $p>3$ be a prime, and let $a,b\in\mathbb{Z}$ with $(\frac{-ab}{p})=-1$. Then*

$$\begin{aligned}&\det\left[x+\cot\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}\\ &=\begin{cases}T_p^{(1)}(a,b)/(-p)^{(p-1)/4}=\pm 2^{(p-1)/2}/\sqrt{p} & \text{if } p\equiv 1\pmod 4,\\ (-1)^{(h(-p)+1)/2}(\frac{a}{p})2^{(p-1)/2}/\sqrt{p} & \text{if } p\equiv 3\pmod 4,\end{cases}\end{aligned} \tag{1.10}$$

*where $h(-p)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{p}\,\mathrm{i})$ with $\mathrm{i}=\sqrt{-1}$.*

*Remark* 1.2. It is known that $2\nmid h(-p)$ for each prime $p\equiv 3\pmod 4$. In 1961 Mordell [8] even proved that for any prime $p>3$ with $p\equiv 3\pmod{4}$ we have

$$\frac{p-1}{2}!\equiv(-1)^{(h(-p)+1)/2}\pmod p.$$

**Theorem 1.4.** *For any odd prime $p$, we have*

$$D_p:=\det\left[\cot\pi\frac{jk}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}\in\begin{cases}\mathbb{Q} & \text{if } p\equiv 1\pmod 4,\\ \sqrt{p}\ \mathbb{Q} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

We are going to provide several lemmas in the next section and then prove Theorem 1.1 in Section 3. Theorems 1.2–1.4 will be shown in Section 4. In Section 5, we pose some conjectures on determinants involving trigonometric tangent function.

## 2. Some Lemmas

**Lemma 2.1.** *Let $A=[a_{jk}]_{0\leqslant j,k\leqslant n}$ be a matrix over a field. Then*

$$\det[x+a_{jk}]_{0\leqslant j,k\leqslant n}=\det A+x\det B, \tag{2.1}$$

*where $B=[b_{jk}]_{1\leqslant j,k\leqslant n}$ with $b_{jk}=a_{jk}-a_{j0}-a_{0k}+a_{00}$.*

*Proof.* As $(x+a_{jk})-(x+a_{0k})=a_{jk}-a_{0k}$ for all $0<j\leqslant n$ and $0\leqslant k\leqslant n$, we have

$$\det[x+a_{jk}]_{0\leqslant j,k\leqslant n}=\begin{vmatrix} x+a_{00} & x+a_{01} & x+a_{02} & \ldots & x+a_{0n}\\ a_{10}-a_{00} & a_{11}-a_{01} & a_{12}-a_{02} & \ldots & a_{1n}-a_{0n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n0}-a_{00} & a_{n1}-a_{01} & a_{n2}-a_{02} & \ldots & a_{nn}-a_{0n}\end{vmatrix}$$
$$=\begin{vmatrix} x & x & x & \ldots & x\\ a_{10}-a_{00} & a_{11}-a_{01} & a_{12}-a_{02} & \ldots & a_{1n}-a_{0n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n0}-a_{00} & a_{n1}-a_{01} & a_{n2}-a_{02} & \ldots & a_{nn}-a_{0n}\end{vmatrix}$$
$$+\begin{vmatrix} a_{00} & a_{01} & a_{02} & \ldots & a_{0n}\\ a_{10}-a_{00} & a_{11}-a_{01} & a_{12}-a_{02} & \ldots & a_{1n}-a_{0n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n0}-a_{00} & a_{n1}-a_{01} & a_{n2}-a_{02} & \ldots & a_{nn}-a_{0n}\end{vmatrix},$$

and hence $\det[x+a_{jk}]_{0\leqslant j,k\leqslant n}-\det A$ coincides with

$$\begin{vmatrix} x & 0 & \ldots & 0\\ a_{10}-a_{00} & a_{11}-a_{01}-(a_{10}-a_{00}) & \ldots & a_{1n}-a_{0n}-(a_{10}-a_{00})\\ \vdots & \vdots & \ddots & \vdots\\ a_{n0}-a_{00} & a_{n1}-a_{01}-(a_{n0}-a_{00}) & \ldots & a_{nn}-a_{0n}-(a_{n0}-a_{00})\end{vmatrix}=x\det B.$$

This concludes the proof of (2.1). □

**Corollary 2.1.** *Let $m$ and $n$ be positive integers with $2\nmid n$. Let $f:\mathbb{Z}\to\mathbb{R}$ be an odd function, where $\mathbb{R}$ is the field of real numbers. Then, for any integer $d$, the determinant*

$$\det\left[x+f((j+d)^m-(k+d)^m)\right]_{0\leqslant j,k\leqslant n}$$

*does not depend on $x$.*

*Proof.* Let

$$a_{jk}=f((j+d)^m-(k+d)^m)\quad \text{for } j,k=0,\ldots,n.$$

For $1\leqslant j,k\leqslant n$ set $b_{jk}=a_{jk}-a_{j0}-a_{0k}+a_{00}$. As $f$ is an odd function, we have

$$\begin{aligned}b_{jk}&=f((j+d)^m-(k+d)^m)-f((j+d)^m-d^m)-f(d^m-(k+d)^m)\\&=-f((k+d)^m-(j+d)^m)+f((k+d)^m-d^m)+f(d^m-(j+d)^m)=-b_{kj}.\end{aligned}$$

Thus

$$\det[b_{jk}]_{1\leqslant j,k\leqslant n}=(-1)^n\det[b_{kj}]_{1\leqslant j,k\leqslant n}=-\det[b_{jk}]_{1\leqslant j,k\leqslant n}$$

and hence $\det[b_{jk}]_{1\leqslant j,k\leqslant n}=0$. Applying Lemma 2.1, we immediately get the desired result. □

The following lemma is Frobenius' extension (cf. [3] and [9, (8)]) of Cauchy's determinant identity (cf. [7, (5.5)]).

**Lemma 2.2.** *We have*

$$\det\left[z+\frac{1}{x_j+y_k}\right]_{0\leqslant j,k\leqslant n}=\frac{\prod_{0\leqslant j<k\leqslant n}(x_k-x_j)(y_k-y_j)}{\prod_{j=0}^n\prod_{k=0}^n(x_j+y_k)}\left(1+z\sum_{k=0}^n(x_k+y_k)\right). \tag{2.2}$$

*Proof.* We present here an induction proof of (2.2) by using Cauchy's determinant identity which is the special case $z=0$ of (2.2).

In the case $n=0$, both sides of (2.2) coincide with $z+1/(x_0+y_0)$.

Now, let $n$ be a positive integer, and suppose that

$$\det\left[z+\frac{1}{x_j+y_k}\right]_{1\leqslant j,k\leqslant n}=\frac{\prod_{1\leqslant j<k\leqslant n}(x_k-x_j)(y_k-y_j)}{\prod_{j=1}^n\prod_{k=1}^n(x_j+y_k)}\left(1+z\sum_{k=1}^n(x_k+y_k)\right). \tag{2.3}$$

By Lemma 2.1 and (2.2) in the case $z=0$,

$$\det\left[z+\frac{1}{x_j+y_k}\right]_{0\leqslant j,k\leqslant n}=\frac{\prod_{0\leqslant j<k\leqslant n}(x_k-x_j)(y_k-y_j)}{\prod_{j=0}^n\prod_{k=0}^n(x_j+y_k)}+z\det[b_{jk}]_{1\leqslant j,k\leqslant n}, \tag{2.4}$$

where

$$b_{jk}=\frac{1}{x_j+y_k}-\frac{1}{x_j+y_0}-\frac{1}{x_0+y_k}+\frac{1}{x_0+y_0}=\frac{(x_j-x_0)(y_k-y_0)(x_j+y_k+x_0+y_0)}{(x_0+y_0)(x_j+y_0)(x_0+y_k)(x_j+y_k)}.$$

With the aid of (2.3), we have

$$\begin{aligned}\det[b_{jk}]_{1\leqslant j,k\leqslant n}&=\prod_{j=1}^n\frac{x_j-x_0}{x_j+y_0}\times\prod_{k=1}^n\frac{y_k-y_0}{y_k+x_0}\times\det\left[\frac{1}{x_0+y_0}+\frac{1}{x_j+y_k}\right]_{1\leqslant j,k\leqslant n}\\&=\prod_{k=1}^n\frac{(x_k-x_0)(y_k-y_0)}{(x_k+y_0)(y_k+x_0)}\times\frac{\prod_{1\leqslant j<k\leqslant n}(x_k-x_j)(y_k-y_j)}{\prod_{j=1}^n\prod_{k=1}^n(x_j+y_k)}\left(1+\frac{\sum_{k=1}^n(x_k+y_k)}{x_0+y_0}\right)\\&=\frac{\prod_{0\leqslant j<k\leqslant n}(x_k-x_j)(y_k-y_j)}{\prod_{j=0}^n\prod_{k=0}^n(x_j+y_k)}\sum_{k=0}^n(x_k+y_k).\end{aligned}$$

Combining this with (2.4), we obtain the desired (2.2). This concludes the proof. □

An analogue of Lemma 2.2 for Pfaffians can be found in Okada's paper [9].

**Lemma 2.3** (Huang and Pan [4])**.** *Let $n>1$ be an odd integer, and let $c$ be any integer relatively prime to $n$. For each $j=1,\ldots,(n-1)/2$, let $\rho_c(j)$ be the unique $r\in\{1,\ldots,(n-1)/2\}$ with $cj$ congruent to $r$ or $-r$ modulo $n$. For the permutation $\rho_c$ on $\{1,\ldots,(n-1)/2\}$, its sign is given by*

$$\operatorname{sign}(\rho_c)=\left(\frac{c}{n}\right)^{(n+1)/2}. \tag{2.5}$$

**Lemma 2.4** (Sun [11])**.** *Let $p>3$ be a prime. Let $\zeta=e^{2\pi i/p}$ and $a\in\mathbb{Z}$ with $p\nmid a$.*

(i) *If $p\equiv1\pmod 4$, then*

$$\prod_{1\leqslant j<k\leqslant(p-1)/2}(\zeta^{aj^2}+\zeta^{ak^2})=\pm\varepsilon_p^{(\frac{a}{p})h(p)((\frac{2}{p})-1)/2} \tag{2.6}$$

*and*

$$\prod_{1\leqslant j<k\leqslant (p-1)/2}(\zeta^{aj^2}-\zeta^{ak^2})^2=(-1)^{(p-1)/4}p^{(p-3)/4}\varepsilon_p^{(\frac{a}{p})h(p)}. \tag{2.7}$$

(ii) *Suppose that* $p\equiv 3 \pmod 4$. *Then*

$$\prod_{1\leqslant j<k\leqslant (p-1)/2}(\zeta^{aj^2}+\zeta^{ak^2})=1, \tag{2.8}$$

*and*

$$\prod_{1\leqslant j<k\leqslant (p-1)/2}(\zeta^{aj^2}-\zeta^{ak^2})=\begin{cases}(-p)^{(p-3)/8} & \text{if } p\equiv 3 \pmod 8,\\ (-1)^{(p+1)/8+(h(-p)-1)/2}(\frac{a}{p})p^{(p-3)/8}\mathrm{i} & \text{if } p\equiv 7 \pmod 8.\end{cases} \tag{2.9}$$

*Also,*

$$\prod_{k=1}^{(p-1)/2}(1-\zeta^{ak^2})=(-1)^{(h(-p)+1)/2}\left(\frac{a}{p}\right)\sqrt{p}\,\mathrm{i}. \tag{2.10}$$

**Lemma 2.5.** *Let* $p>3$ *be a prime, and let* $a,b\in\mathbb{Z}$ *with* $(\frac{-ab}{p})=-1$. *Then*

$$\prod_{j=1}^{(p-1)/2}\prod_{k=1}^{(p-1)/2}\left(1-e^{2\pi\mathrm{i}(aj^2+bk^2)/p}\right)=p^{(p-1)/4}\times\begin{cases}1 & \text{if } p\equiv 1 \pmod 4,\\ (-1)^{(h(-p)-1)/2}(\frac{a}{p})\mathrm{i} & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{2.11}$$

*Proof.* For $m\in\mathbb{Z}$ set

$$\begin{aligned}r(m):&=\left|\left\{(j,k):\ 1\leqslant j,k\leqslant\frac{p-1}{2}\ \text{and}\ aj^2+bk^2\equiv m \pmod p\right\}\right|\\ &=\left|\left\{1\leqslant x\leqslant p-1:\ \left(\frac{x}{p}\right)=1\ \text{and}\ \left(\frac{m-ax}{p}\right)=\left(\frac{b}{p}\right)\right\}\right|.\end{aligned}$$

Note that $r(0)=0$ since $(\frac{-ab}{p})\neq 1$.

Let $m\in\{1,\ldots,p-1\}$. Then

$$\begin{aligned}r(m)=&\sum_{\substack{0<x<p\\ p\nmid ax-m}}\frac{(\frac{x}{p})+1}{2}\cdot\frac{(\frac{b(m-ax)}{p})+1}{2}\\ =&\frac{1}{4}\sum_{x=1}^{p-1}\left(\left(\frac{bx(m-ax)}{p}\right)+\left(\frac{x}{p}\right)+\left(\frac{b(m-ax)}{p}\right)+1\right)-\frac{(\frac{am}{p})+1}{4}\\ =&\frac{1}{4}\sum_{x=0}^{p-1}\left(\frac{-abx^2+bmx}{p}\right)+\frac{1}{4}\sum_{x=0}^{p-1}\left(\frac{x}{p}\right)+\frac{1}{4}\sum_{x=0}^{p-1}\left(\frac{-abx+bm}{p}\right)\\ &-\frac{1}{4}\left(\frac{bm}{p}\right)+\frac{p-1}{4}-\frac{(\frac{am}{p})+1}{4}.\end{aligned}$$

It is well known that for any $a_0,a_1,a_2\in\mathbb{Z}$ with $p\nmid a_0$ or $p\nmid a_1$ we have

$$\sum_{x=0}^{p-1}\left(\frac{a_0x^2+a_1x+a_2}{p}\right)=\begin{cases}-(\frac{a_0}{p}) & \text{if } p\nmid a_1^2-4a_0a_2,\\ (p-1)(\frac{a_0}{p}) & \text{if } p\mid a_1^2-4a_0a_2.\end{cases}\tag{2.12}$$

(See, e.g., [2, p. 58].) Therefore

$$r(m)=-\frac14\left(\frac{-ab}{p}\right)+\frac{p-1}{4}-\frac{(\frac{am}{p})+(\frac{bm}{p})+1}{4}=\frac{p-1}{4}-\frac{1-(\frac{-1}{p})}{4}\left(\frac{am}{p}\right).$$

In view of the above,

$$\begin{aligned}&\prod_{j=1}^{(p-1)/2}\prod_{k=1}^{(p-1)/2}\left(1-e^{2\pi\mathrm{i}(aj^2+bk^2)/p}\right)\\ =&\prod_{m=1}^{p-1}(1-e^{2\pi\mathrm{i}m/p})^{r(m)}=\frac{\prod_{m=1}^{p-1}(1-e^{2\pi\mathrm{i}m/p})^{(p-1+(\frac{a}{p})(1-(\frac{-1}{p})))/4}}{\prod_{\substack{0<m<p\\(\frac{m}{p})=1}}(1-e^{2\pi\mathrm{i}m/p})^{(\frac{a}{p})(1-(\frac{-1}{p}))/2}}.\end{aligned}$$

Clearly,

$$\prod_{m=1}^{p-1}(1-e^{2\pi\mathrm{i}m/p})=\lim_{x\to1}\frac{x^p-1}{x-1}=p.$$

As (2.10) holds for $p\equiv3\pmod 4$, we have

$$\prod_{\substack{0<m<p\\(\frac{m}{p})=1}}(1-e^{2\pi\mathrm{i}m/p})^{(1-(\frac{-1}{p}))/2}=\begin{cases}1 & \text{if } p\equiv1\pmod 4,\\ (-1)^{(h(-p)+1)/2}\sqrt{p}\,\mathrm{i} & \text{if } p\equiv3\pmod 4.\end{cases}$$

Thus the desired (2.11) follows. □

## 3. Proof of Theorem 1.1

For convenience, we set $n=(p-1)/2$ and $\zeta=e^{2\pi\mathrm{i}/p}$. Since $p>3$ and

$$\sum_{k=0}^{n}k^2=\frac{n(n+1)(2n+1)}{6}=\frac{p^2-1}{24}p\equiv0\pmod p,$$

we have

$$\prod_{k=0}^{n}\zeta^{k^2}=1.\tag{3.1}$$

As

$$\tan x=\frac{2\sin x}{2\cos x}=\frac{(e^{\mathrm{i}x}-e^{-\mathrm{i}x})/\mathrm{i}}{e^{\mathrm{i}x}+e^{-\mathrm{i}x}}=\frac{-\mathrm{i}(e^{2\mathrm{i}x}-1)}{e^{2\mathrm{i}x}+1}=-\mathrm{i}+\frac{2\mathrm{i}}{e^{2\mathrm{i}x}+1},$$

we also have

$$\mathrm{i}+\tan\pi\frac{aj^2+bk^2}{p}=\frac{2\mathrm{i}}{\zeta^{aj^2+bk^2}+1}\quad\text{for all } j,k=0,\dots,n.\tag{3.2}$$

For each $\delta\in\{0,1\}$ and integer $d\not\equiv 0 \pmod p$, we claim that

$$T_p^{(\delta)}(a,\pm ad^2,x)=\left(\frac{d}{p}\right)^{n+1}T_p^{(\delta)}(a,\pm a,x). \tag{3.3}$$

We now explain this. For $k=1,\ldots,n$ let $\rho_d(k)$ be the unique $r\in\{1,\ldots,n\}$ with $dk$ congruent to $r$ or $-r$ modulo $p$. In view of Lemma 2.3,

$$\begin{aligned}T_p^{(1)}(a,\pm ad^2,x)&=\sum_{\tau\in S_n}\operatorname{sign}(\tau)\prod_{j=1}^n\left(x+\tan\pi\frac{aj^2\pm a(d\tau(j))^2}{p}\right)\\&=\operatorname{sign}(\rho_d)\sum_{\tau\in S_n}\operatorname{sign}(\rho_d\tau)\prod_{j=1}^n\left(x+\tan\pi\frac{aj^2\pm a\rho_d(\tau(j))^2}{p}\right)\\&=\left(\frac{d}{p}\right)^{n+1}T_p^{(1)}(a,\pm a,x).\end{aligned}$$

If we extend the function $\rho_d$ by defining $\rho_d(0)=0$, then the new $\rho_d$ is a permutation of $\{0,1,\ldots,n\}$ and its sign is the same as the old one. So, (3.3) also holds for $\delta=0$.

*Proof of the First Part of Theorem 1.1.* As $p\equiv 1 \pmod 4$, we have $n=(p-1)/2\equiv 0 \pmod 2$. For $q=n!$ we have $q^2\equiv -1 \pmod p$ by Wilson's theorem, hence

$$\begin{aligned}-T_p^{(0)}(a,b,x)&=\det\left[-x-\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant n}\\&=\det\left[-x+\tan\pi\frac{a(qj)^2+b(qk)^2}{p}\right]_{0\leqslant j,k\leqslant n}=T_p^{(0)}(a,b,-x)\end{aligned}$$

and thus $\det T_p^{(0)}(a,b)=0$.

*Case* 1. $(\frac{ab}{p})=1$.

In this case, $b\equiv ac^2 \pmod p$ for some integer $c\not\equiv 0 \pmod p$. Note that $b\equiv -a(qc)^2 \pmod p$ and hence

$$T_p^{(1)}(a,b,x)=\left(\frac{2c}{p}\right)T_p^{(1)}(a,-a,x)$$

by (3.3) and the equality $(\frac{q}{p})=(\frac{2}{p})$ (cf. [10, Lemma 2.3]).

By Corollary 2.1,

$$\det\left[x+\tan\pi\frac{aj^2-ak^2}{p}\right]_{1\leqslant j,k\leqslant n}=\det\left[x+\tan\pi\frac{a(j+1)^2-a(k+1)^2}{p}\right]_{0\leqslant j,k\leqslant n-1}$$

does not depend on $x$. So, with the aid of (3.2), we get

$$T_p^{(1)}(a,-a,x)=\det\left[\mathrm{i}+\tan\pi\frac{aj^2-ak^2}{p}\right]_{1\leqslant j,k\leqslant n}$$

$$= \det\left[\frac{2\mathrm{i}}{e^{2\pi \mathrm{i}a(j^2-k^2)/p}+1}\right]_{1\leqslant j,k\leqslant n}$$

$$= \prod_{k=1}^{n}(2\mathrm{i}\zeta^{ak^2}) \times \det\left[\frac{1}{\zeta^{aj^2}+\zeta^{ak^2}}\right]_{1\leqslant j,k\leqslant n}.$$

(Recall that $\zeta = e^{2\pi \mathrm{i}/p}$.) In light of Lemma 2.2,

$$\det\left[\frac{1}{\zeta^{aj^2}+\zeta^{ak^2}}\right]_{1\leqslant j,k\leqslant n} = \frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2}{\prod_{k=1}^{n}(\zeta^{ak^2}+\zeta^{ak^2}) \times \prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}+\zeta^{ak^2})^2}.$$

Therefore,

$$T_p^{(1)}(a,-a,x) = \mathrm{i}^n \prod_{1\leqslant j<k\leqslant n}\left(\frac{\zeta^{ak^2}-\zeta^{aj^2}}{\zeta^{ak^2}+\zeta^{aj^2}}\right)^2$$

$$= (-1)^{(p-1)/4}\frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2}{\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}+\zeta^{ak^2})^2} = p^{(p-3)/4}\varepsilon_p^{(\frac{a}{p})(2-(\frac{2}{p}))h(p)}$$

with the aid of Lemma 2.4(i).

*Case* 2. $(\frac{ab}{p}) = -1$.

Recall that $T_p^{(1)}(a,b) = \det[c_{jk}]_{1\leqslant j,k\leqslant n}$ with $c_{jk} = \tan\pi(aj^2+bk^2)/p$. By Lemma 2.1,

$$T_p^{(1)}(a,b,x) = \det[x+c_{jk}]_{1\leqslant j,k\leqslant n} = T_p^{(1)}(a,b) + x\det[d_{jk}]_{1<j,k\leqslant n}, \tag{3.4}$$

where $d_{jk} = c_{jk} - c_{j1} - c_{1k} + c_{11}$. In light of (3.2) and (3.4),

$$\det\left[\frac{2\mathrm{i}}{\zeta^{aj^2+bk^2}+1}\right]_{1\leqslant j,k\leqslant n} = \det[\mathrm{i}+c_{jk}]_{1\leqslant j,k\leqslant n} = T_p^{(1)}(a,b) + \mathrm{i}\det[d_{jk}]_{1<j,k\leqslant n},$$

and hence (1.5) is implied by

$$D_p(a,b) := \det\left[\frac{2\mathrm{i}}{\zeta^{aj^2+bk^2}+1}\right]_{1\leqslant j,k\leqslant n} = \pm 2^{(p-1)/2}p^{(p-3)/4}. \tag{3.5}$$

(Note that both $T_p^{(1)}(a,b)$ and $\det[d_{jk}]_{1<j,k\leqslant n}$ are real numbers.)

In view of Lemma 2.2 and (3.1),

$$D_p(a,b) = \prod_{k=1}^{n}\left(\frac{2\mathrm{i}}{\zeta^{bk^2}}\right) \times \det\left[\frac{1}{\zeta^{aj^2}+\zeta^{-bk^2}}\right]_{1\leqslant j,k\leqslant n}$$

$$= \frac{(2\mathrm{i})^n}{\prod_{k=1}^{n}\zeta^{bk^2}} \times \frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2})}{\prod_{j=1}^{n}\prod_{k=1}^{n}(\zeta^{aj^2}+\zeta^{-bk^2})}$$

$$= (-1)^{(p-1)/4}2^{(p-1)/2}\frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2})}{\prod_{j=1}^{n}\prod_{k=1}^{n}(\zeta^{aj^2+bk^2}+1)}.$$

Note that

$$\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2+bk^2}+1)=\prod_{j=1}^n\prod_{k=1}^n\frac{1-\zeta^{2aj^2+2bk^2}}{1-\zeta^{aj^2+bk^2}}=1$$

by Lemma 2.5. So

$$D_p(a,b)=(-1)^{(p-1)/4}2^{(p-1)/2}\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2}). \tag{3.6}$$

Observe that

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2(\zeta^{-bk^2}-\zeta^{-bj^2})^2=p^{(p-3)/2}\varepsilon_p^{((\frac{a}{p})+(\frac{-b}{p}))h(p)}=p^{(p-3)/2}$$

by (2.7). Therefore (3.5) holds and hence so does (1.5).

In view of the above, we have completed the proof of part (i) of Theorem 1.1. □

*Proof of the Second Part of Theorem 1.1.* As $p\equiv 3 \pmod 4$, we have $n=(p-1)/2\equiv 1 \pmod 2$.

*Case* 1. $(\frac{ab}{p})=-1$.

In this case, $b\equiv -ad^2 \pmod p$ for some integer $d\not\equiv 0 \pmod p$, and hence by (3.3) we have

$$T_p^{(0)}(a,b,x)=T_p^{(0)}(a,-a,x)\ \text{ and }\ T_p^{(1)}(a,b,x)=T_p^{(1)}(a,-a,x).$$

As

$$T_p^{(1)}(a,-a,-x)=\det\left[-x+\tan\pi\frac{ak^2-aj^2}{p}\right]_{1\leqslant j,k\leqslant n}=(-1)^nT_p^{(1)}(a,-a,x)=-T_p^{(1)}(a,-a,x),$$

we get $T_p^{(1)}(a,b,-x)=-T_p^{(1)}(a,b,x)$, and in particular $T_p^{(1)}(a,b)=0$.

To obtain the equality $T_p^{(0)}(a,b,x)=p^{(p+1)/4}$, we now determine $T_p^{(0)}(a,-a,x)$ which equals $T_p^{(0)}(a,b,x)$. In view of Corollary 2.1 and (3.2), we have

$$\begin{aligned}T_p^{(0)}(a,-a,x)&=\det\left[\mathrm{i}+\tan\pi\frac{aj^2-ak^2}{p}\right]_{0\leqslant j,k\leqslant n}\\&=\det\left[\frac{2\mathrm{i}}{\zeta^{a(j^2-k^2)}+1}\right]_{0\leqslant j,k\leqslant n}\\&=\prod_{k=0}^n(2\mathrm{i}\zeta^{ak^2})\times\det\left[\frac{1}{\zeta^{aj^2}+\zeta^{ak^2}}\right]_{0\leqslant j,k\leqslant n}.\end{aligned}$$

By Lemma 2.2,

$$\det\left[\frac{1}{\zeta^{aj^2}+\zeta^{ak^2}}\right]_{0\leqslant j,k\leqslant n}=\frac{\prod_{0\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2}{\prod_{k=0}^n(\zeta^{ak^2}+\zeta^{ak^2})\times\prod_{0\leqslant j<k\leqslant n}(\zeta^{aj^2}+\zeta^{ak^2})^2}.$$

Therefore,

$$T_p^{(0)}(a,-a,x)=\mathrm{i}^{n+1}\prod_{0\leqslant j<k\leqslant n}\left(\frac{\zeta^{ak^2}-\zeta^{aj^2}}{\zeta^{ak^2}+\zeta^{aj^2}}\right)^2$$

$$= (-1)^{(p+1)/4}\prod_{k=1}^{n}\left(\frac{\zeta^{ak^2}-1}{\zeta^{ak^2}+1}\right)^2 \times \frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2}{\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}+\zeta^{ak^2})^2}.$$

By Lemma 2.4(ii),

$$\prod_{k=1}^{n}(\zeta^{ak^2}-1)^2 = -p \quad\text{and}\quad \prod_{k=1}^{n}(\zeta^{ak^2}+1)^2 = \prod_{k=1}^{n}\frac{(\zeta^{2ak^2}-1)^2}{(\zeta^{ak^2}-1)^2} = \frac{-p}{-p} = 1,$$

and

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})^2 = (-p)^{(p-3)/4} \text{ and } \prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}+\zeta^{aj^2})^2 = 1.$$

Therefore

$$T_p^{(0)}(a,-a,x) = (-1)^{(p+1)/4}(-p)(-p)^{(p-3)/4} = p^{(p+1)/4}$$

as desired.

*Case* 2. $(\frac{ab}{p}) = 1$.

In this case, $b \equiv ac^2 \pmod p$ for some $c \in \mathbb{Z}$ with $p \nmid c$, and hence by (3.3) we have $T_p^{(0)}(a,b,x) = T_p^{(0)}(a,a,x)$ and $T_p^{(1)}(a,b,x) = T_p^{(1)}(a,a,x)$ since $n+1$ is even.

Clearly $T_p^{(0)}(a,a) = \det[a_{jk}]_{0\leqslant j,k\leqslant n}$ with $a_{jk} = \tan\pi(aj^2+ak^2)/p$. By Lemma 2.1,

$$T_p^{(0)}(a,a,x) = \det[x+a_{jk}]_{0\leqslant j,k\leqslant n} = T_p^{(0)}(a,a) + x\det[b_{jk}]_{1\leqslant j,k\leqslant n} \tag{3.7}$$

where

$$b_{jk} := a_{jk} - a_{j0} - a_{0k} + a_{00} = \tan\pi\frac{aj^2+ak^2}{p} - \tan\pi\frac{aj^2}{p} - \tan\pi\frac{ak^2}{p}.$$

Using the well known identity

$$\tan(x_1+x_2) = \frac{\tan x_1 + \tan x_2}{1-\tan x_1\tan x_2},$$

we obtain

$$b_{jk} = \tan\pi\frac{aj^2}{p} \times \tan\pi\frac{ak^2}{p} \times \tan\pi\frac{aj^2+ak^2}{p}$$

and hence

$$\det[b_{jk}]_{1\leqslant j,k\leqslant n} = T_p^{(1)}(a,a)\prod_{j=1}^{n}\tan^2\pi\frac{aj^2}{p}. \tag{3.8}$$

In view of (3.2), (3.7) and (3.8),

$$\det\left[\frac{2\mathrm{i}}{\zeta^{a(j^2+k^2)}+1}\right]_{0\leqslant j,k\leqslant n} = \det[\mathrm{i}+a_{jk}]_{0\leqslant j,k\leqslant n} = T_p^{(0)}(a,a) + \mathrm{i}T_p^{(1)}(a,a)\prod_{j=1}^{n}\tan^2\pi\frac{aj^2}{p}.$$

Thus

$$T_p^{(0)}(a,a) = 2^{(p-1)/2}p^{(p+1)/4},\ T_p^{(1)}(a,a) = 0 \text{ and } \det[b_{jk}]_{1\leqslant j,k\leqslant n} = 0 \tag{3.9}$$

if and only if

$$\det\left[\frac{2\mathrm{i}}{\zeta^{a(j^2+k^2)}+1}\right]_{0\leqslant j,k\leqslant n}=2^{(p-1)/2}p^{(p+1)/4}. \tag{3.10}$$

With the aid of Lemma 2.2,

$$\begin{aligned}\det\left[\frac{2\mathrm{i}}{\zeta^{a(j^2+k^2)}+1}\right]_{0\leqslant j,k\leqslant n}&=\prod_{k=0}^{n}\frac{2\mathrm{i}}{\zeta^{ak^2}}\times\det\left[\frac{1}{\zeta^{aj^2}+\zeta^{-ak^2}}\right]_{0\leqslant j,k\leqslant n}\\&=\frac{(2\mathrm{i})^{n+1}}{\prod_{k=0}^{n}\zeta^{ak^2}}\times\frac{\prod_{0\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-ak^2}-\zeta^{-aj^2})}{\prod_{j=0}^{n}\prod_{k=0}^{n}(\zeta^{aj^2}+\zeta^{-ak^2})}.\end{aligned}$$

This, together with (3.1), yields

$$\det\left[\frac{2\mathrm{i}}{\zeta^{a(j^2+k^2)}+1}\right]_{0\leqslant j,k\leqslant n}(-1)^{(p+1)/4}2^{(p+1)/2}\frac{\prod_{0\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-ak^2}-\zeta^{-aj^2})}{\prod_{j=0}^{n}\prod_{k=0}^{n}(\zeta^{a(j^2+k^2)}+1)}. \tag{3.11}$$

By Lemma 2.4(ii),

$$\begin{aligned}&\prod_{0\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-ak^2}-\zeta^{-aj^2})\\&=\prod_{k=1}^{n}(\zeta^{ak^2}-1)(\zeta^{-ak^2}-1)\times\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}-\zeta^{ak^2})(\zeta^{-aj^2}-\zeta^{-ak^2})\\&=p\times p^{(p-3)/4}=p^{(p+1)/4}.\end{aligned}$$

In view of Lemma 2.4(ii) and Lemma 2.5,

$$\begin{aligned}\prod_{j=0}^{n}\prod_{k=0}^{n}(\zeta^{a(j^2+k^2)}+1)&=(\zeta^0+1)\prod_{j=1}^{n}\left(\frac{1-\zeta^{2aj^2}}{1-\zeta^{aj^2}}\right)^2\times\prod_{j=1}^{n}\prod_{k=1}^{n}\frac{1-\zeta^{2a(j^2+k^2)}}{1-\zeta^{a(j^2+k^2)}}\\&=2\left(\frac{2}{p}\right)^2\left(\frac{2}{p}\right)=2(-1)^{(p+1)/4}.\end{aligned}$$

Combining these with (3.11), we get (3.10) and hence (3.9) holds. In view of (3.7) and (3.9), we finally obtain that

$$T_p^{(0)}(a,b,x)=T_p^{(0)}(a,a,x)=2^{(p-1)/2}p^{(p+1)/4}.$$

By the above, we have finished the proof of part (ii) of Theorem 1.1. □

## 4. Proofs of Theorems 1.2-1.4

The following lemma is Frobenius' extension (cf. [1]) of the Zolotarev lemma [14].

**Lemma 4.1.** *Let $n$ be a positive odd integer, and let $a\in\mathbb{Z}$ be relatively prime to $n$. For $j=0,\ldots,n-1$, let $\lambda_a(j)$ be the least nonnegative residue of $aj$ modulo $n$. Then the permutation $\lambda_a$ of $\{0,\ldots,n-1\}$ has the sign* $\mathrm{sign}(\lambda_a)=(\frac{a}{n})$.

We also need another lemma.

**Lemma 4.2.** *Let $n>1$ be an odd number and let $a\in\mathbb{Z}$ with $\gcd(a,n)=1$. Then*

$$\prod_{1\leqslant j<k\leqslant n-1}\left(e^{2\pi\mathrm{i}ak/n}-e^{2\pi\mathrm{i}aj/n}\right)^2=(-1)^{(n-1)/2}n^{n-2}. \tag{4.1}$$

*Proof.* Let $\zeta=e^{2\pi\mathrm{i}a/n}$. Clearly,

$$\prod_{r=1}^{n-1}(1-\zeta^r)=\lim_{x\to1}\frac{x^n-1}{x-1}=n \tag{4.2}$$

and hence

$$\begin{aligned}(-1)^{\binom{n-1}{2}}\prod_{1\leqslant j<k\leqslant n-1}(\zeta^k-\zeta^j)^2&=\prod_{j=1}^{n-1}\prod_{\substack{k=1\\k\neq j}}^{n-1}(\zeta^j-\zeta^k)=\prod_{j=1}^{n-1}\prod_{\substack{k=1\\k\neq j}}^{n-1}\zeta^j(1-\zeta^{k-j})\\&=\prod_{j=1}^{n-1}\left(\frac{(\zeta^j)^{n-2}}{1-\zeta^{-j}}\prod_{\substack{k=0\\k\neq j}}^{n-1}(1-\zeta^{k-j})\right)\\&=\frac{\zeta^{(n-1)\sum_{j=0}^{n-1}j}}{\prod_{j=1}^{n-1}(\zeta^j-1)}\prod_{j=1}^{n-1}\prod_{r=1}^{n-1}(1-\zeta^r)=n^{n-2}.\end{aligned}$$

So (4.1) holds. □

*Proof of Theorem 1.2.* In view of Lemma 4.1, for each $\delta=0,1$ we have

$$\det\left[x+\tan\pi\frac{aj+bk}{n}\right]_{\delta\leqslant j,k\leqslant n-1}=\left(\frac{a}{n}\right)\det\left[x+\tan\pi\frac{j+bk}{n}\right]_{\delta\leqslant j,k\leqslant n-1}=\left(\frac{-ab}{n}\right)D_n^{(\delta)}(x),$$

where

$$D_n^{(\delta)}(x):=\det\left[x+\tan\pi\frac{j-k}{n}\right]_{\delta\leqslant j,k\leqslant n-1}.$$

Since

$$\begin{aligned}D_n^{(0)}(-x)&=\det\left[-x+\tan\pi\frac{k-j}{n}\right]_{0\leqslant j,k\leqslant n-1}=\det\left[-x-\tan\pi\frac{j-k}{n}\right]_{0\leqslant j,k\leqslant n-1}\\&=(-1)^n\det\left[x+\tan\pi\frac{j-k}{n}\right]_{0\leqslant j,k\leqslant n-1}=-D_n^{(0)}(x),\end{aligned}$$

we have

$$\det\left[-x+\tan\pi\frac{aj+bk}{n}\right]_{0\leqslant j,k\leqslant n-1}=-\left(\frac{-ab}{n}\right)D_n^{(0)}(x)=-\det\left[x+\tan\pi\frac{aj+bk}{n}\right]_{0\leqslant j,k\leqslant n-1}$$

and hence (1.8) holds.

Now it remains to show that $D_n^{(1)}(x)=n^{n-2}$. Write $\zeta=e^{2\pi\mathrm{i}/n}$. Similar to (3.2), we have

$$\mathrm{i}+\tan\pi\frac{j-k}{n}=\frac{2\mathrm{i}}{\zeta^{j-k}+1}\quad\text{for all } j,k=1,\ldots,n-1.$$

Thus

$$D_n^{(1)}(\mathrm{i})=\det\left[\frac{2\mathrm{i}}{\zeta^{j-k}+1}\right]_{1\leqslant j,k\leqslant n-1}=\prod_{k=1}^{n-1}(2\mathrm{i}\zeta^k)\times\det\left[\frac{1}{\zeta^j+\zeta^k}\right]_{1\leqslant j,k\leqslant n-1}.$$

By Lemma 2.2,

$$\begin{aligned}\det\left[\frac{1}{\zeta^j+\zeta^k}\right]_{1\leqslant j,k\leqslant n-1}&=\frac{\prod_{1\leqslant j<k\leqslant n-1}(\zeta^k-\zeta^j)^2}{\prod_{j=1}^{n-1}\prod_{k=1}^{n-1}(\zeta^j+\zeta^k)}\\&=\frac{\prod_{1\leqslant j<k\leqslant n-1}(\zeta^k-\zeta^j)^2}{\prod_{k=1}^{n-1}(2\zeta^k)\times\prod_{1\leqslant j<k\leqslant n-1}(\zeta^k+\zeta^j)^2}.\end{aligned}$$

Therefore

$$D_n^{(1)}(\mathrm{i})=\mathrm{i}^{n-1}\prod_{1\leqslant j<k\leqslant n-1}\frac{(\zeta^k-\zeta^j)^4}{(\zeta^{2k}-\zeta^{2j})^2}=(-1)^{(n-1)/2}\prod_{1\leqslant j<k\leqslant n-1}\frac{(\zeta^k-\zeta^j)^4}{(\zeta^{2k}-\zeta^{2j})^2}.$$

Combining this with Lemma 4.2, we immediately get $D_n^{(1)}(\mathrm{i})=(n^{n-2})^2/n^{n-2}=n^{n-2}$. By Lemma 2.1,

$$D_n^{(1)}(x)=D_n^{(1)}(0)+rx$$

for certain real number $r$. As $D_n^{(1)}(\mathrm{i})=n^{n-2}$, we have $D_n^{(1)}(0)=n^{n-2}$ and $r=0$. Thus $D_n^{(1)}(x)=D_n^{(1)}(0)=n^{n-2}$ as desired.

The proof of Theorem 1.2 is now complete. □

*Proof of Theorem 1.3.* For any nonzero real number $x\notin\pi\mathbb{Z}$, we obviously have

$$\cot x=\frac{\cos x}{\sin x}=\frac{(e^{\mathrm{i}x}+e^{-\mathrm{i}x})/2}{(e^{\mathrm{i}x}-e^{-\mathrm{i}x})/(2\mathrm{i})}=\mathrm{i}+\frac{2\mathrm{i}}{e^{2\mathrm{i}x}-1}.$$

Thus

$$-\mathrm{i}+\cot\pi\frac{aj^2+bk^2}{p}=\frac{2\mathrm{i}}{\zeta^{aj^2+bk^2}-1}\quad\text{for all } j,k=1,\ldots,n,$$

where $n=(p-1)/2$ and $\zeta=e^{2\pi\mathrm{i}/p}$. Let

$$C(x)=\det\left[x+\cot\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant n}.$$

Then

$$C(-\mathrm{i})=\det\left[\frac{2\mathrm{i}}{\zeta^{aj^2+bk^2}-1}\right]_{1\leqslant j,k\leqslant n}=\prod_{k=1}^{n}(2\mathrm{i}\zeta^{-bk^2})\times\det\left[\frac{1}{\zeta^{aj^2}-\zeta^{-bk^2}}\right]_{1\leqslant j,k\leqslant n}.$$

By Lemma 2.2,

$$\det\left[\frac{1}{\zeta^{aj^2}-\zeta^{-bk^2}}\right]_{1\leqslant j,k\leqslant n} = \frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(-\zeta^{-bk^2}-(-\zeta^{-bj^2}))}{\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2}-\zeta^{-bk^2})}$$
$$= (-1)^{\binom{n}{2}}\frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2})}{(\prod_{k=1}^n\zeta^{-bk^2})^n\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2+bk^2}-1)}.$$

Note that $\prod_{k=1}^n\zeta^{k^2}=1$ by (3.1). So

$$C(-\mathrm{i}) = (2\mathrm{i})^n\frac{(-1)^{\binom{n}{2}}\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2})}{(-1)^n\prod_{j=1}^n\prod_{k=1}^n(1-\zeta^{aj^2+bk^2})}. \tag{4.3}$$

*Case* 1. $p\equiv 1 \pmod 4$, i.e., $2\mid n$.

In this case, $(\frac{ab}{p})=(\frac{-ab}{p})=-1$. By Lemma 2.5,

$$\prod_{j=1}^n\prod_{k=1}^n(1-\zeta^{aj^2+bk^2}) = p^{(p-1)/4}.$$

Combining this with (4.3) and (3.6), we get

$$C(-\mathrm{i}) = \frac{2^{(p-1)/2}}{p^{(p-1)/4}}\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2}) = \frac{D_p(a,b)}{(-p)^{(p-1)/4}},$$

where $D_p(a,b)$ is defined as in (3.5). Thus

$$C(-\mathrm{i}) = \frac{D_p(a,b)}{(-p)^{(p-1)/4}} = \frac{T_p^{(1)}(a,b)}{(-p)^{(p-1)/4}} = \pm\frac{2^{(p-1)/2}}{\sqrt{p}}$$

in view of (3.2) and (1.5). By Lemma 2.1, $C(x)=C(0)+rx$ for certain real number $r$. Since $C(-\mathrm{i})$ is real, we have $r=0$ and hence

$$C(x) = C(-\mathrm{i}) = \frac{T_p^{(1)}(a,b)}{(-p)^{(p-1)/4}} = \pm\frac{2^{(p-1)/2}}{\sqrt{p}}.$$

*Case* 2. $p\equiv 3 \pmod 4$, i.e., $2\nmid n$.

In light of (2.9),

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{ak^2}-\zeta^{aj^2})(\zeta^{-bk^2}-\zeta^{-bj^2}) = p^{(p-3)/4}.$$

Combining this with (2.11) and (4.3), we obtain

$$C(-\mathrm{i}) = (2\mathrm{i})^n(-1)^{\binom{n}{2}}\frac{p^{(p-3)/4}}{(-1)^n(-1)^{(h(-p)-1)/2}(\frac{a}{p})p^{(p-1)/4}\mathrm{i}} = \frac{2^n\mathrm{i}(\mathrm{i}^2)^{(n-1)/2}(-1)^{(n-1)/2}}{(-1)^{(h(-p)+1)/2}(\frac{a}{p})\sqrt{p}\,\mathrm{i}}$$

and hence

$$C(-\mathrm{i})=(-1)^{(h(-p)+1)/2}\left(\frac{a}{p}\right)\frac{2^{(p-1)/2}}{\sqrt{p}}$$

is a real number. Combining this with Lemma 2.1, we get that

$$C(x)=C(-\mathrm{i})=(-1)^{(h(-p)+1)/2}\left(\frac{a}{p}\right)\frac{2^{(p-1)/2}}{\sqrt{p}}.$$

In view of the above, we have completed the proof of Theorem 1.3. □

The following lemma is a well known result on quadratic Gauss sums (cf. [6, pp. 70-76]).

**Lemma 4.3.** *Let $p$ be an odd prime. Then, for any integer $a\not\equiv 0 \pmod p$, we have*

$$\sum_{x=0}^{p-1}e^{2\pi\mathrm{i}ax^2/p}=\left(\frac{a}{p}\right)\sum_{t=0}^{p-1}\left(\frac{t}{p}\right)e^{2\pi\mathrm{i}t/p}=\left(\frac{a}{p}\right)\sqrt{(-1)^{(p-1)/2}p}.$$

Let $p$ be an odd prime, and let $\zeta=e^{2\pi\mathrm{i}/p}$. For $a,b\in\mathbb{Z}$ with $p\nmid ab$, Lemmas 2.2 and 4.3 are helpful to evaluate $\det[z+1/(\zeta^{aj^2}+\zeta^{bk^2})]_{1\leqslant j,k\leqslant(p-1)/2}$. However, we actually only need the case $z=0$ in our previous proofs of Theorems 1.1–1.3.

*Proof of Theorem 1.4.* The Galois group $\mathrm{Gal}(\mathbb{Q}(e^{2\pi\mathrm{i}/p})/\mathbb{Q})$ consists of those $\mathbb{Q}$-automorphisms $\sigma_a$ $(1\le a\le p-1)$ with $\sigma_a(e^{2\pi\mathrm{i}/p})=e^{2\pi\mathrm{i}a/p}$. For any integer $x\not\equiv 0 \pmod p$, we have

$$\cot\pi\frac{x}{p}=\mathrm{i}\frac{e^{\pi\mathrm{i}x/p}+e^{-\pi\mathrm{i}x/p}}{e^{\pi\mathrm{i}x/p}-e^{-\pi\mathrm{i}x/p}}=\mathrm{i}\frac{e^{2\pi\mathrm{i}x/p}+1}{e^{2\pi\mathrm{i}x/p}-1}.$$

It follows that

$$\frac{D_p}{\mathrm{i}^{(p-1)/2}}=\det\left[\frac{e^{2\pi\mathrm{i}jk/p}+1}{e^{2\pi\mathrm{i}jk/p}-1}\right]_{1\leqslant j,k\leqslant(p-1)/2}.$$

Let $a\in\{1,\ldots,p-1\}$. By the last equality,

$$\begin{aligned}\sigma_a\left(\frac{D_p}{\mathrm{i}^{(p-1)/2}}\right)&=\sigma_a\Bigg(\sum_{\tau\in S_{(p-1)/2}}\mathrm{sign}(\tau)\prod_{j=1}^{(p-1)/2}\frac{e^{2\pi\mathrm{i}j\tau(j)/p}+1}{e^{2\pi\mathrm{i}j\tau(j)/p}-1}\Bigg)\\&=\sum_{\tau\in S_{(p-1)/2}}\mathrm{sign}(\tau)\prod_{j=1}^{(p-1)/2}\frac{e^{2\pi\mathrm{i}aj\tau(j)/p}+1}{e^{2\pi\mathrm{i}aj\tau(j)/p}-1}\\&=\det\left[\frac{e^{2\pi\mathrm{i}ajk/p}+1}{e^{2\pi\mathrm{i}ajk/p}-1}\right]_{1\leqslant j,k\leqslant(p-1)/2}=\frac{1}{\mathrm{i}^{(p-1)/2}}\det\left[\cot\pi\frac{ajk}{p}\right]_{1\le j,k\le(p-1)/2}.\end{aligned}$$

By Gauss' Lemma (see, e.g., [6, p. 52]),

$$\left(\frac{a}{p}\right)=(-1)^{|\{1\le j\le(p-1)/2:\ \{aj/p\}>1/2\}|},$$

where $\{x\}$ denotes the fractional part of a real number $x$. Therefore,

$$\sigma_a\left(\frac{D_p}{\mathrm{i}^{(p-1)/2}}\right)=\frac{(\frac{a}{p})}{\mathrm{i}^{(p-1)/2}}\det\left[\cot\pi\frac{\rho_a(j)k}{p}\right]_{1\le j,k\le (p-1)/2},$$

where $\rho_a(j)$ is the unique $r\in\{1,\ldots,(p-1)/2\}$ with $aj\equiv \pm r \pmod p$. Combining this with Lemma 2.3, we deduce that

$$\begin{aligned}\sigma_a\left(\frac{D_p}{\mathrm{i}^{(p-1)/2}}\right)&=\frac{(\frac{a}{p})}{\mathrm{i}^{(p-1)/2}}\left(\frac{a}{p}\right)^{(p+1)/2}\det\left[\cot\pi\frac{jk}{p}\right]_{1\le j,k\le (p-1)/2}\\&=\left(\frac{a}{p}\right)^{(p-1)/2}\frac{D_p}{\mathrm{i}^{(p-1)/2}}.\end{aligned}$$

If $p\equiv 1 \pmod 4$, then $\mathrm{i}^{(p-1)/2}=(-1)^{(p-1)/4}\in\mathbb{Q}$ and hence $\sigma_a(D_p)=D_p$. When $p\equiv 3 \pmod 4$, by Lemma 4.3 we have $\sqrt{-p}\in\mathbb{Q}(e^{2\pi\mathrm{i}/p})$ and

$$\sigma_a(\sqrt{-p})=\sigma_a\left(\sum_{x=0}^{p-1}e^{2\pi\mathrm{i}x^2/p}\right)=\sum_{x=0}^{p-1}e^{2\pi\mathrm{i}ax^2/p}=\left(\frac{a}{p}\right)\sqrt{-p},$$

therefore

$$\begin{aligned}\sigma_a\left((-1)^{(p+1)/4}\frac{D_p}{\sqrt{p}}\right)&=\sigma_a\left(\frac{D_p}{\mathrm{i}(-1)^{(p-3)/4}\sqrt{-p}}\right)=\frac{\sigma_a(D_p/\mathrm{i}^{(p-1)/2})}{\sigma_a(\sqrt{-p})}\\&=\frac{(\frac{a}{p})D_p/\mathrm{i}^{(p-1)/2}}{(\frac{a}{p})\sqrt{-p}}=(-1)^{(p+1)/4}\frac{D_p}{\sqrt{p}}\end{aligned}$$

and hence $\sigma_a(D_p/\sqrt{p})=D_p/\sqrt{p}$.

By the above, if $p\equiv 1 \pmod 4$, then $\sigma(D_p)=D_p$ for all $\sigma\in\mathrm{Gal}(\mathbb{Q}(e^{2\pi\mathrm{i}/p})/\mathbb{Q})$, and hence $D_p\in\mathbb{Q}$ by Galois theory. Similarly, when $p\equiv 3 \pmod 4$ we have $D_p/\sqrt{p}\in\mathbb{Q}$.

The proof of Theorem 1.4 is now complete. □

## 5. Some open conjectures

**Conjecture 5.1.** *Let $p$ be any odd prime. Then*

$$\left(\frac{-2}{p}\right)\frac{\det\left[\cot\pi jk/p\right]_{1\leqslant j,k\leqslant (p-1)/2}}{2^{(p-3)/2}p^{(p-5)/4}}\in\{1,2,3,\ldots\},\tag{5.1}$$

*and this number is divisible by $h(-p)$ if $p\equiv 3 \pmod 4$.*

*Remark* 5.1. By Theorem 1.4, for any odd prime $p$ we have

$$\det\left[\cot\pi\frac{jk}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}\in p^{(p-1)/4}\mathbb{Q}.$$

**Conjecture 5.2.** *Let $n$ be a positive integer.*

(i) *The number*

$$s_n := (2n+1)^{-n/2}\det\left[\tan\pi\frac{jk}{2n+1}\right]_{1\leqslant j,k\leqslant n} \tag{5.2}$$

*is always an integer.*

(ii) *We have*

$$\det\left[\tan^2\pi\frac{jk}{2n+1}\right]_{1\leqslant j,k\leqslant n}\in(2n+1)^{(n+1)/2}4^{n-1}\mathbb{Z}. \tag{5.3}$$

*Remark* 5.2. Via `Mathematica` we find that

$$s_1=1,\ s_2=-2,\ s_3=s_4=4,\ s_5=48,\ s_6=-160,$$
$$s_7=32,\ s_8=2176,\ s_9=6912,\ s_{10}=0,\ s_{11}=273408.$$

Let $t_n$ denote the $n$th term of the sequence [5, A277445], which is the determinant of a matrix $T(n)=[t_{jk}]_{1\leqslant j,k\leqslant n}$ with entries among $0,\pm1$ such that

$$2\sum_{k=1}^n t_{jk}\sin\frac{\pi k}{2n+1}=\tan\frac{\pi j}{2n+1}\quad\text{ for all } j=1,\ldots,n.$$

We guess that $s_n=-t_n$ if $n\equiv3\pmod 4$, and $s_n=t_n$ otherwise.

**Conjecture 5.3.** *For any odd integer $n>1$, we have*

$$\det\left[\tan^2\pi\frac{j+k}{n}\right]_{1\leqslant j,k\leqslant n-1}\in n^{n-2}\mathbb{Z}. \tag{5.4}$$

*Remark* 5.3. We are able to prove that $\det[\tan^2\pi\frac{j-k}{n}]_{1\leqslant j,k\leqslant n-1}\in\mathbb{Z}$ for any odd integer $n>1$.

**Conjecture 5.4.** *Let $p\equiv3\pmod 4$ be a prime, and let $a,b\in\mathbb{Z}$ with $p\nmid ab$. Then*

$$\det\left[\tan^2\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant(p-1)/2}\in p^{(p-3)/4}\mathbb{Z} \tag{5.5}$$

*and*

$$\det\left[\tan^2\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant(p-1)/2}\in p^{(p+1)/4}\mathbb{Z}. \tag{5.6}$$

*If $(\frac{ab}{p})=1$, then*

$$\det\left[\cot^2\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant(p-1)/2}\in\frac{2^{p-3}}{p}\mathbb{Z}. \tag{5.7}$$

Let $p\equiv1\pmod 4$ be a prime, and let $a,b\in\mathbb{Z}$ with $p\nmid ab$. Choose $q\in\mathbb{Z}$ with $q^2\equiv-1\pmod p$. Then

$$\det\left[\left(\frac{a(qj)^2+b(qk)^2}{p}\right)\tan\pi\frac{a(qj)^2+b(qk)^2}{p}\right]_{0\leqslant j,k\leqslant(p-1)/2}$$

$$= (-1)^{(p+1)/2}\det\left[\left(\frac{aj^2+bk^2}{p}\right)\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}$$

and hence

$$\det\left[\left(\frac{aj^2+bk^2}{p}\right)\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}=0. \tag{5.8}$$

**Conjecture 5.5.** *Let $p\equiv 3 \pmod 4$ be a prime and let $a,b\in\mathbb{Z}$ with $p\nmid ab$. Then*

$$\det\left[\left(\frac{aj^2+bk^2}{p}\right)\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}\in p\mathbb{Z}. \tag{5.9}$$

*If $(\frac{ab}{p})=1$, then*

$$\sqrt{p}\det\left[\left(\frac{aj^2+bk^2}{p}\right)\cot\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}\in\mathbb{Z}. \tag{5.10}$$

*Remark* 5.4. For any prime $p\equiv 3 \pmod 4$, set

$$a_p^{\pm}:=\frac{1}{p}\det\left[\left(\frac{j^2\pm k^2}{p}\right)\tan\pi\frac{j^2\pm k^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}.$$

Via `Mathematica` we find that

$$a_3^+=a_3^-=-1,\ a_7^+=60,\ a_7^-=3,\ a_{11}^+=2^6\times 3^3,\ a_{11}^-=-373,$$
$$a_{19}^+=2^{12}\times 3\times 5^2\times 7\times 11\times 17 \text{ and } a_{19}^-=-5\times 7\times 89\times 3803.$$

**Conjecture 5.6.** *Let $p$ be an odd prime.*

(i) *Define*

$$S(p):=\det\left[\sec 2\pi\frac{jk}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2}.$$

*If $p\equiv 1 \pmod 4$, then $S(p)=0$. When $p\equiv 3 \pmod 4$, the number*

$$\frac{S(p)}{2^{(p-3)/2}(-p)^{(p+1)/4}}$$

*is a positive odd integer.*

(ii) *We have*

$$c_p:=\frac{1}{2^{(p-1)/2}p^{(p-5)/4}}\det\left[\csc 2\pi\frac{jk}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}\in\mathbb{Z}.$$

*Moreover, $c_p=0$ if $p\equiv 7 \pmod 8$.*

*Remark* 5.5. By the way we prove Theorem 1.4, we can show that $S(p)/p^{(p+1)/4}\in\mathbb{Q}$ and $c_p\in\mathbb{Q}$ for any odd prime $p$. In 2019 the author [12] conjectured that

$$\frac{1}{2n}\det\left[\cos\pi\frac{jk}{n}\right]_{0\leqslant j,k\leqslant n}=\det\left[\cos\pi\frac{jk}{n}\right]_{1\leqslant j,k\leqslant n}=(-1)^{\lfloor\frac{n+1}{2}\rfloor}\frac{n^{(n-1)/2}}{2^{(n-1)/2}} \tag{5.11}$$

for every positive integer $n$, this was later confirmed by Petrov (cf. the answer in [12]).

**Conjecture 5.7.** *For any prime* $p \equiv 3 \pmod 4$*, there is an integer* $x_p \equiv 1 \pmod p$ *such that*

$$\det\left[\sec 2\pi\frac{(j-k)^2}{p}\right]_{0\leqslant j,k\leqslant p-1} = -p^{(p+3)/2}x_p^2. \tag{5.12}$$

*Remark* 5.6. For $p = 3, 7, 11$, we may take $x_p = 1$ in (5.12). For each prime $p \equiv 3 \pmod 4$, the author [13] conjectured in 2021 that

$$\det\left[\sin 2\pi\frac{(j-k)^2}{p}\right]_{1\leqslant j,k\leqslant p-1} = -\frac{p^{(p-1)/2}}{2^{p-1}},$$

which was later confirmed by Kalmynin (cf. the answer in [13]).

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