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ON CERTAIN DETERMINANTS AND RELATED LEGENDRE SYMBOLS

HAN WANG AND ZHI-WEI SUN

Abstract. Let p be an odd prime. For $b,c\in\mathbb{Z}$, we study the Legendre symbol $(\frac{D_p^*(b,c)}{p})$, where $D_p^*(b,c)$ denotes the determinant of the matrix $[(i^2+bij+cj^2)^{p-3}]_{1\le i,j\le p-1}$. For example, we prove that if $p\equiv 2 \pmod 3$ then

$$D_p^*(1,1)\equiv \det\left[\frac{1}{(i^2+ij+j^2)^2}\right]_{1\le i,j\le p-1}\equiv -x^2 \pmod p$$

for some integer $x\not\equiv 0 \pmod p$. We also show that

$$\left(\frac{D_p^*(2,2)}{p}\right)=\left(\frac{p}{3}\right)(-1)^{(p+1)/8}$$

if $p\equiv 7 \pmod 8$.

1. Introduction

For an $n\times n$ matrix $[a_{ij}]_{1\le i,j\le n}$ over a commutative ring with identity, we use $\det[a_{ij}]_{1\le i,j\le n}$ or $|a_{ij}|_{1\le i,j\le n}$ to denote its determinant.

Let p be an odd prime, and let $b,c\in\mathbb{Z}$. Sun [4] introduced

$$(b,c)_p=\det\left[\left(\frac{i^2+bij+cj^2}{p}\right)\right]_{1\le i,j\le p-1}\quad\text{and}\quad [b,c]_p=\det\left[\left(\frac{i^2+bij+cj^2}{p}\right)\right]_{0\le i,j\le p-1},$$

and proved the following results:

$$\left(\frac{c}{p}\right)=-1\Longrightarrow (b,c)_p=0,\tag{1.1}$$

and

$$\left(\frac{c}{p}\right)=1\Longrightarrow [b,c]_p=\begin{cases}\frac{p-1}{2}(b,c)_p & \text{if } p\nmid b^2-4c,\\ \frac{1-p}{p-2}(b,c)_p & \text{if } p\mid b^2-4c,\end{cases}$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol. Grinberg, Sun and Zhao [1, Theorem 1.3] determined $(\frac{S_c(b,p)}{p})$ in the case $p\nmid bc$, where

$$S_c(b,p)=\det\left[\left(\frac{i^2+bj^2+c}{p}\right)\right]_{1\le i,j\le (p-1)/2}.$$

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For each prime $p \equiv 5 \pmod 6$, Sun [4] conjectured that

$$2\det\left[\frac{1}{i^2-ij+j^2}\right]_{1\le i,j\le p-1}$$

is a quadratic residue modulo p. This was confirmed by Wu, She, and Ni [8].

For any odd prime p and a p-adic integer $x \not\equiv 0 \pmod p$, clearly

$$\frac{1}{x} \equiv x^{p-2} \pmod p \quad \text{and} \quad \frac{1}{x^2} \equiv x^{p-3} \pmod p.$$

Let p be an odd prime, and let $b, c \in \mathbb{Z}$. Sun [5] showed that for any integer n with $(p-1)/2 < n < p-1$ we have

$$\det[(i^2+bij+cj^2)^n]_{0\le i,j\le p-1} \equiv 0 \pmod p.$$

Sun [5] also introduced

$$D_p(b,c) = \det[(i^2+bij+cj^2)^{p-2}]_{1\le i,j\le p-1},$$

and proved that for any prime $p > 3$ with $p \equiv 3 \pmod 4$ we have

$$D_p(b,-1) \equiv D_p(2,2) \equiv 0 \pmod p.$$

By Wu, She, and Ni [8], we actually have $(\frac{D_p(1,1)}{p}) = (\frac{-2}{p})$ if $p \equiv 2 \pmod 3$. Recently, Luo and Sun [3] proved that

$$\left(\frac{D_p(1,1)}{p}\right) = \begin{cases} 0 & \text{if } p \equiv 7 \pmod 9, \\ 1 & \text{if } p \equiv 1, 4 \pmod 9, \end{cases} \tag{1.2}$$

and that

$$\left(\frac{D_p(2,2)}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8, \\ 0 & \text{if } p \equiv 5 \pmod 8. \end{cases} \tag{1.3}$$

Their tools include generalized trinomial coefficients and Lucas sequences. Similar to (1.1), Wu and She [7] extended a result of Sun [5] by proving that $D_p(b,c) \equiv 0 \pmod p$ if $(\frac{c}{p}) = -1$.

We first present a basic result which is similar to (1.1) and [5, Theorem 1.2].

Theorem 1.1. *Let p be an odd prime, and let $b, c \in \mathbb{Z}$ with $(\frac{c}{p}) = -1$. For any integer n in the interval $[1, p-1]$, we have*

$$\det[(i^2+bij+cj^2)^n]_{1\le i,j\le p-1} \equiv 0 \pmod p. \tag{1.4}$$

Proof. For $j = 1, \ldots, p-1$, let $\pi_c(j) = \{cj\}_p$, the least nonnegative residue of cj modulo p. By Zolotarev's Lemma (cf. [9]), the sign of $\pi_c \in S_{p-1}$ is exactly the Legendre symbol $(\frac{c}{p})$. Observe that

$$\begin{aligned} &c^{n(p-1)}\det[(i^2+bij+cj^2)^n]_{1\le i,j\le p-1} \\ &= \det\left[(ci^2+bi(cj)+(cj)^2)^n\right]_{1\le i,j\le p-1} \\ &= \det\left[(ci^2+bi\pi_c(j)+\pi_c(j)^2)^n\right]_{1\le i,j\le p-1} \end{aligned}$$

$$= \sum_{\sigma \in S_{p-1}} \mathrm{sign}(\sigma) \prod_{i=1}^{p-1} (ci^2 + bi\pi_c(\sigma(i)) + \pi_c(\sigma(i))^2)^n$$

$$= \mathrm{sign}(\pi_c) \sum_{\tau \in S_{p-1}} \mathrm{sign}(\tau) \prod_{i=1}^{p-1} (ci^2 + bi\tau(i) + \tau(i)^2)^n$$

$$= \left(\frac{c}{p}\right) \det[(i^2 + bij + cj^2)^n]_{1 \le i,j \le p-1} = -\det[(i^2 + bij + cj^2)^n]_{1 \le i,j \le p-1}$$

Thus, with the aid of Fermat's little theorem, we obtain (1.4). □

Let p be an odd prime, and let $b, c \in \mathbb{Z}$. In contrast to the notation $D_p(b,c)$, we introduce

$$D_p^*(b,c) = \det[(i^2 + bij + cj^2)^{p-3}]_{1 \le i,j \le p-1}. \tag{1.5}$$

If $(\frac{b^2-4c}{p}) = -1$, then $i^2 + bij + cj^2 \not\equiv 0 \pmod p$ for all $i, j = 1, \ldots, p-1$, and hence

$$D_p^*(b,c) \equiv \det\left[\frac{1}{(i^2 + bij + cj^2)^2}\right]_{1 \le i,j \le p-1} \pmod p.$$

The notations $D_p(b,c)$ and $D_p^*(b,c)$ are motivated by Wolstenholme's congruences (cf. [6])

$$\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod p$$

provided $p > 3$.

Now we state our main results.

Theorem 1.2. *Let p be an odd prime. Then*

$$\left(\frac{D_p^*(1,1)}{p}\right) = \begin{cases} \left(\frac{-1}{p}\right) & \text{if } p \equiv 2 \pmod 3, \\ \left(\frac{p}{5}\right) \text{ or } 0 & \text{if } p \equiv 1 \pmod 3. \end{cases} \tag{1.6}$$

Consequently, when $p \equiv 2 \pmod 3$ the p-adic integer

$$-\det\left[\frac{1}{(i^2 + ij + j^2)^2}\right]_{1 \le i,j \le p-1}$$

is a quadratic residue modulo p.

Theorem 1.3. *Let p be an odd prime. Then*

$$\left(\frac{D_p^*(2,2)}{p}\right) = \begin{cases} 0 \text{ or } 1 & \text{if } p \equiv 1 \pmod 8 \\ \left(\frac{p}{3}\right)(-1)^{\frac{p+1}{8}} & \text{if } p \equiv 7 \pmod 8, \\ 0 & \text{if } p \equiv \pm 3 \pmod 8. \end{cases} \tag{1.7}$$

Remark 1.1. Note that for any prime $p\equiv 3 \pmod 4$ we have

$$D_p^*(2,2)\equiv \det\left[\frac{1}{((i+j)^2+j^2)^2}\right]_{1\le i,j\le p-1} \pmod p.$$

Let $n\in\mathbb{N}=\{0,1,2,\ldots\}$ and $b,c\in\mathbb{Z}$. The generalized trinomial coefficients

$$\binom{n}{k}_{b,c}\quad (k\in\mathbb{Z})$$

are given by

$$\left(x+b+\frac{c}{x}\right)^n=\sum_{k\in\mathbb{Z}}\binom{n}{k}_{b,c}x^k. \tag{1.8}$$

We will make use of generalized trinomial coefficients to prove Theorems 1.2 and 1.3 in Sections 2 and 3, respectively. Note that Theorems 1.2 and 1.3 cannot be deduced from Luo and Sun's results (1.2) and (1.3), and their proofs are somewhat sophisticated.

2. Proof of Theorem 1.2

Lemma 2.1. ([2, Lemma 10]) *Let R be a commutative ring with identity, and let $P(x)=\sum_{i=0}^{n-1}a_ix^i\in R[x]$. Then*

$$\det[P(X_iY_j)]_{1\le i,j\le n}=a_0a_1\cdots a_{n-1}\prod_{1\le i,j\le n}(X_i-X_j)(Y_i-Y_j).$$

Lemma 2.2. (Luo and Sun [3, (3.2)]) *For any odd prime p, we have*

$$\prod_{1\le i,j\le p-1}(i-j)\left(\frac1i-\frac1j\right)=(-1)^{(p+1)/2}\prod_{j=1}^{p-2}(j!)^2. \tag{2.1}$$

Lemma 2.3. ([3, Lemma 2.1]) *Let p be an odd prime, and let $b,c\in\mathbb{Z}$. For $k\in\{-p+2,\ldots,p-2\}$, we have*

$$(4c^2-b)\binom{p-2}{k}_{b,c}\equiv\begin{cases}\binom{p-1}{-1}_{b,c}+c\binom{p-1}{1}_{b,c}-b \pmod p & \text{if } k=0,\\ (k+1)\binom{p-1}{k-1}_{b,c}-(k-1)c\binom{p-1}{k+1}_{b,c} \pmod p & \text{otherwise.}\end{cases} \tag{2.2}$$

Lemma 2.4. *Let p be an odd prime, and let $b,c\in\mathbb{Z}$. For $k\in\{-p+3,\ldots,p-3\}$, we have*

$$(k+3)\binom{p-2}{k-1}_{b,c}-(k-3)c\binom{p-2}{k+1}_{b,c}-2\binom{p-1}{k}_{b,c}\equiv 2(4c-b^2)\binom{p-3}{k}_{b,c} \pmod p. \tag{2.3}$$

Proof. For $n\in\mathbb{N}$ and $k\in\mathbb{Z}$, we simply write $\left[{n\atop k}\right]$ for $\binom{n}{k}_{b,c}$.

Taking derivatives of both sides of the following identity

$$\sum_{k=-p+1}^{p-1}\begin{bmatrix}p-1\\k\end{bmatrix}x^k=\left(x+b+\frac{c}{x}\right)^{p-1},$$

we get

$$\sum_{k=-p+1}^{p-1} k\begin{bmatrix}p-1\\k\end{bmatrix}x^{k-1} = (p-1)\left(x+b+\frac{c}{x}\right)^{p-2}\left(1-\frac{c}{x^2}\right). \tag{2.4}$$

Comparing the coefficients of x^{k-1} on both sides of (2.4), we obtain

$$k\begin{bmatrix}p-1\\k\end{bmatrix} = (p-1)\left(\begin{bmatrix}p-2\\k-1\end{bmatrix} - c\begin{bmatrix}p-2\\k+1\end{bmatrix}\right). \tag{2.5}$$

Taking derivatives of both sides of (2.4), we get

$$\sum_{k=-p+1}^{p-1} k(k-1)\begin{bmatrix}p-1\\k\end{bmatrix}x^{k-2} = (p-1)(p-2)\left(x+b+\frac{c}{x}\right)^{p-3}\left(1-\frac{c}{x^2}\right)^2 + \frac{2c}{x^3}(p-1)\left(x+b+\frac{c}{x}\right)^{p-2}. \tag{2.6}$$

Comparing the coefficients of x^{k-2} on both sides of (2.6), we deduce that

$$k(k-1)\begin{bmatrix}p-1\\k\end{bmatrix} = (p-1)(p-2)\left(\begin{bmatrix}p-3\\k-2\end{bmatrix} - 2c\begin{bmatrix}p-3\\k\end{bmatrix} + c^2\begin{bmatrix}p-3\\k+2\end{bmatrix}\right) + 2c(p-1)\begin{bmatrix}p-2\\k+1\end{bmatrix}. \tag{2.7}$$

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $k \in \mathbb{Z}$, we have the recurrence

$$\begin{bmatrix}n\\k\end{bmatrix} = \begin{bmatrix}n-1\\k-1\end{bmatrix} + b\begin{bmatrix}n-1\\k\end{bmatrix} + c\begin{bmatrix}n-1\\k+1\end{bmatrix}$$

by Luo and Sun [3, (2.3)]. With the aid of this, we have

$$\begin{aligned}
&\begin{bmatrix}p-3\\k-2\end{bmatrix} - 2c\begin{bmatrix}p-3\\k\end{bmatrix} + c^2\begin{bmatrix}p-3\\k+2\end{bmatrix}\\
&= \begin{bmatrix}p-2\\k-1\end{bmatrix} - b\begin{bmatrix}p-3\\k-1\end{bmatrix} - c\begin{bmatrix}p-3\\k\end{bmatrix} - 2c\begin{bmatrix}p-3\\k\end{bmatrix} + c\left(\begin{bmatrix}p-2\\k+1\end{bmatrix} - b\begin{bmatrix}p-3\\k+1\end{bmatrix} - \begin{bmatrix}p-3\\k\end{bmatrix}\right)\\
&= \begin{bmatrix}p-2\\k-1\end{bmatrix} + c\begin{bmatrix}p-2\\k+1\end{bmatrix} - 4c\begin{bmatrix}p-3\\k\end{bmatrix} - b\left(\begin{bmatrix}p-3\\k-1\end{bmatrix} + c\begin{bmatrix}p-3\\k+1\end{bmatrix}\right)\\
&= \begin{bmatrix}p-2\\k-1\end{bmatrix} + c\begin{bmatrix}p-2\\k+1\end{bmatrix} - 4c\begin{bmatrix}p-3\\k\end{bmatrix} - b\left(\begin{bmatrix}p-2\\k\end{bmatrix} - b\begin{bmatrix}p-3\\k\end{bmatrix}\right)\\
&= \begin{bmatrix}p-2\\k-1\end{bmatrix} - b\begin{bmatrix}p-2\\k\end{bmatrix} + c\begin{bmatrix}p-2\\k+1\end{bmatrix} + (b^2-4c)\begin{bmatrix}p-3\\k\end{bmatrix}\\
&= \begin{bmatrix}p-2\\k-1\end{bmatrix} - \left(\begin{bmatrix}p-1\\k\end{bmatrix} - \begin{bmatrix}p-2\\k-1\end{bmatrix} - c\begin{bmatrix}p-2\\k+1\end{bmatrix}\right) + c\begin{bmatrix}p-2\\k+1\end{bmatrix} + (b^2-4c)\begin{bmatrix}p-3\\k\end{bmatrix}
\end{aligned}$$

and hence

$$\begin{aligned}
&\begin{bmatrix}p-3\\k-2\end{bmatrix} - 2c\begin{bmatrix}p-3\\k\end{bmatrix} + c^2\begin{bmatrix}p-3\\k+2\end{bmatrix}\\
&= -\begin{bmatrix}p-1\\k\end{bmatrix} + 2\begin{bmatrix}p-2\\k-1\end{bmatrix} + 2c\begin{bmatrix}p-2\\k+1\end{bmatrix} + (b^2-4c)\begin{bmatrix}p-3\\k\end{bmatrix}.
\end{aligned} \tag{2.8}$$

Combining (2.5), (2.7) and (2.8), we get

$$\begin{aligned}&(k-1)\left(\begin{bmatrix}p-2\\k-1\end{bmatrix}-c\begin{bmatrix}p-2\\k+1\end{bmatrix}\right)-2c(p-1)\begin{bmatrix}p-2\\k+1\end{bmatrix}\\=&\frac{k(k-1)}{p-1}\begin{bmatrix}p-1\\k\end{bmatrix}-2c(p-1)\begin{bmatrix}p-2\\k+1\end{bmatrix}\\=&(p-2)\left(\begin{bmatrix}p-3\\k-2\end{bmatrix}-2c\begin{bmatrix}p-3\\k\end{bmatrix}+c^2\begin{bmatrix}p-3\\k+2\end{bmatrix}\right)-2c(p-2)\begin{bmatrix}p-2\\k+1\end{bmatrix}\\=&(p-2)\left(-\begin{bmatrix}p-1\\k\end{bmatrix}+2\begin{bmatrix}p-2\\k-1\end{bmatrix}+(b^2-4c)\begin{bmatrix}p-3\\k\end{bmatrix}\right).\end{aligned}$$

Hence we have

$$\begin{aligned}&(k-1)\left(\begin{bmatrix}p-2\\k-1\end{bmatrix}-c\begin{bmatrix}p-2\\k+1\end{bmatrix}\right)+2c\begin{bmatrix}p-2\\k+1\end{bmatrix}\\\equiv&2\left(\begin{bmatrix}p-1\\k\end{bmatrix}-2\begin{bmatrix}p-2\\k-1\end{bmatrix}-(b^2-4c)\begin{bmatrix}p-3\\k\end{bmatrix}\right)\pmod p,\end{aligned}$$

that is,

$$(k+3)\begin{bmatrix}p-2\\k-1\end{bmatrix}-(k-3)c\begin{bmatrix}p-2\\k+1\end{bmatrix}-2\begin{bmatrix}p-1\\k\end{bmatrix}\equiv 2(4c-b^2)\begin{bmatrix}p-3\\k\end{bmatrix}\pmod p. \tag{2.9}$$

This concludes the proof. □

Proof of Theorem 1.2. Let $b, c\in\mathbb{Z}$. By [3, (2.2)], we have

$$\binom{n}{-k}_{b,c}=c^k\binom{n}{k}_{b,c}\quad\text{for all } n\in\mathbb{N}\text{ and } k\in\mathbb{Z}. \tag{2.10}$$

Thus

$$\begin{aligned}&(x^2+bx+c)^{p-3}-\binom{p-3}{0}_{b,c}x^{p-3}\\=&\sum_{\substack{k=-p-3\\k\neq0}}^{p-3}\binom{p-3}{k}_{b,c}x^{p-3+k}=\sum_{k=1}^{p-3}\left(\binom{p-3}{k}_{b,c}x^{p-3+k}+\binom{p-3}{k}_{b,c}c^kx^{p-3-k}\right)\\=&\sum_{k=2}^{p-3}\left(\binom{p-3}{k}_{b,c}x^{p-1}+\binom{p-3}{p-1-k}_{b,c}c^{p-1-k}\right)x^{k-2}+\binom{p-3}{1}_{b,c}x^{p-2}+\binom{p-3}{1}_{b,c}cx^{p-4}.\end{aligned} \tag{2.11}$$

Let $k\in\{-p+3,\ldots,p-3\}$. Taking $b=c=1$ in Lemma 2.4, we get

$$6\binom{p-3}{k}_{1,1}\equiv(k+3)\binom{p-2}{k-1}_{1,1}-(k-3)\binom{p-2}{k+1}_{1,1}-2\binom{p-1}{k}_{1,1}\pmod p. \tag{2.12}$$

Putting $b=c=1$ in (2.2) and noting (2.10), we obtain

$$3\binom{p-2}{k}_{1,1}\equiv\begin{cases}2\binom{p-1}{1}_{1,1}-1\pmod p & \text{if } k=0,\\ (k+1)\binom{p-1}{k-1}_{1,1}-(k-1)\binom{p-1}{k+1}_{1,1}\pmod p & \text{if } k\neq 0.\end{cases}\tag{2.13}$$

Combining (2.12) with (2.13), we see that

$$\begin{aligned}&18\binom{p-3}{k}_{1,1}\\ &\equiv 3(k+3)\binom{p-2}{k-1}_{1,1}-3(k-3)\binom{p-2}{k+1}_{1,1}-6\binom{p-1}{k}_{1,1}\\ &\equiv\begin{cases}-2\binom{p-1}{-3}_{1,1}+8\binom{p-1}{1}_{1,1}-4\pmod p & \text{if } k=-1,\\ -2\binom{p-1}{3}_{1,1}+8\binom{p-1}{1}_{1,1}-4\pmod p & \text{if } k=1,\\ k(k+3)\binom{p-1}{k-2}_{1,1}-2(k^2-3)\binom{p-1}{k}_{1,1}+k(k-3)\binom{p-1}{k+2}_{1,1}\pmod p & \text{if } k\neq\pm1.\end{cases}\end{aligned}$$

For each $k\in\{0,\ldots,p-3\}$, we have

$$\binom{p-1}{p-k}_{1,1}\equiv\left(\frac{k}{3}\right)\pmod p$$

by Luo and Sun [3, (2.14)], and hence

$$\begin{aligned}18\binom{p-3}{k}_{1,1}&\equiv 3(k+3)\binom{p-2}{k-1}_{1,1}-3(k-3)\binom{p-2}{k+1}_{1,1}-6\binom{p-1}{k}_{1,1}\\ &\equiv\begin{cases}-2\left(\frac{p}{3}\right)+8\left(\frac{p-1}{3}\right)-4\pmod p & \text{if } k=1,\\ k(k+3)\left(\frac{p-k+2}{3}\right)-2(k^2-3)\left(\frac{p-k}{3}\right)+k(k-3)\left(\frac{p-k-2}{3}\right)\pmod p & \text{if } 2\le k\le p-3.\end{cases}\end{aligned}\tag{2.14}$$

When $2\le k\le p-3$, we have

$$\begin{aligned}&18\binom{p-3}{k}_{1,1}+18\binom{p-3}{p-1-k}_{1,1}\\ &\equiv k(k+3)\left(\frac{p-k+2}{3}\right)+(p-1-k)(p+2-k)\left(\frac{k+3}{3}\right)\\ &\quad-2(k^2-3)\left(\frac{p-k}{3}\right)-2((k+1)^2-3)\left(\frac{k+1}{3}\right)\\ &\quad+k(k-3)\left(\frac{p-k-2}{3}\right)+(p-1-k)(p-4-k)\left(\frac{k-1}{3}\right)\pmod p\end{aligned}$$

and hence

$$\begin{aligned}&18\binom{p-3}{k}_{1,1}+18\binom{p-3}{p-1-k}_{1,1}\\&\equiv\begin{cases}(-3k^2+3k+6)\binom{k+1}{3}+(3k^2+9k)\binom{k+2}{3}\pmod p & \text{if } p\equiv 1\pmod 3,\\ -6k\binom{k+1}{3}+6\binom{k+2}{3}\pmod p & \text{if } p\equiv 2\pmod 3.\end{cases}\end{aligned}\tag{2.15}$$

In view of (2.11) and (2.14), we obtain

$$\begin{aligned}&18(x^2+x+1)^{p-3}\\&\equiv 18\binom{p-3}{0}_{1,1}x^{p-3}+\sum_{k=2}^{p-3}\left(18\binom{p-3}{k}_{1,1}+18\binom{p-3}{p-1-k}_{1,1}\right)x^{k-2}\\&\quad+18\binom{p-3}{1}_{1,1}(x^{p-2}+x^{p-4})\\&\equiv 6\left(\frac{p}{3}\right)x^{p-3}+\sum_{k=2}^{p-3}\left(18\binom{p-3}{k}_{1,1}+18\binom{p-3}{p-1-k}_{1,1}\right)x^{k-2}\\&\quad+\left(-2\left(\frac{p}{3}\right)+8\left(\frac{p+2}{3}\right)-4\right)(x^{p-2}+x^{p-4})\pmod p.\end{aligned}$$

Thus, with the aid of (2.15), we have

$$\begin{aligned}&18(x^2+x+1)^{p-3}\\&\equiv 6\left(\frac{p}{3}\right)(x^{p-3}-x^{p-2}-x^{p-4})\\&\quad+\begin{cases}18\sum_{k=2}^{p-3}\left(-\frac{(k+1)(k-2)}{6}\binom{k+1}{3}+\frac{k(k+3)}{6}\binom{k+2}{3}\right)x^{k-2}\pmod p & \text{if } p\equiv 1\pmod 3,\\ 18\sum_{k=2}^{p-3}\left(-\frac{k}{3}\binom{k+1}{3}+\frac{1}{3}\binom{k+2}{3}\right)x^{k-2}\pmod p & \text{if } p\equiv 2\pmod 3.\end{cases}\end{aligned}\tag{2.16}$$

Let $F(x)=(x^2+x+1)^{p-3}$. For $1\le i,j\le p-1$, we have

$$\frac{(i^2+ij+j^2)^{p-3}}{j^{2(p-3)}}=\left(\frac{i^2}{j^2}+\frac{i}{j}+1\right)^{p-3}=F\left(\frac{i}{j}\right),$$

and hence

$$\left(\frac{D_p^*(1,1)}{p}\right)=\left(\frac{|F(i/j)|_{1\le i,j\le p-1}}{p}\right)\tag{2.17}$$

by Fermat's little theorem.

Case 1. $p\equiv 1\pmod 3$.

Applying Lemma 2.1 with $P(x) = F(x)$, $X_i = i$ and $Y_j = 1/j$, and noting the identity (2.1), we get

$$\begin{aligned}
&\left|F\left(\frac{i}{j}\right)\right|_{1\le i,j\le p-1} \\
&\equiv \frac{1}{27}\prod_{k=2}^{p-3}\left(-\frac{(k+1)(k-2)}{6}\binom{k+1}{3}+\frac{k(k+3)}{6}\binom{k+2}{3}\right)\times\prod_{1\le i<j\le p-1}(i-j)\left(\frac{1}{i}-\frac{1}{j}\right) \\
&\equiv \frac{1}{27}(-1)^{\frac{p+1}{2}}(-1)^{\frac{p-4}{3}}\prod_{j=1}^{p-2}(j!)^2\prod_{k\in\{2+3a:\ 0\le a\le\frac{p-7}{3}\}}\frac{k(k+3)}{6} \\
&\quad\times\prod_{k\in\{4+3a:\ 0\le a\le\frac{p-7}{3}\}}\frac{(k+1)(k-2)}{6}\times\prod_{k\in\{3+3a:\ 0\le a\le\frac{p-7}{3}\}}\frac{k^2+k-1}{3} \\
&\equiv \frac{1}{27}(-1)^{\frac{p+1}{2}+\frac{p-4}{3}}\prod_{j=1}^{p-2}(j!)^2\times\prod_{k\in\{2+3a:\ 0\le a\le\frac{p-7}{3}\}}\left(\frac{k(k+3)}{3}\right)^2\times 3^{-\frac{p-4}{3}}\prod_{\substack{3\le k\le p-4\\ 3\mid k}}(k^2+k-1) \\
&\equiv 3^{-\frac{p+5}{3}}(-1)^{\frac{p+1}{2}+\frac{p-4}{3}}\prod_{j=1}^{p-2}(j!)^2\times\prod_{k\in\{2+3a:\ 0\le a\le\frac{p-7}{3}\}}\left(\frac{k(k+3)}{3}\right)^2 \\
&\quad\times\prod_{\substack{3\le k<\frac{p-1}{2}\\ 3\mid k}}(k^2+k-1)((p-1-k)^2+(p-1-k)-1)\times\left(\left(\frac{p-1}{2}\right)^2+\frac{p-1}{2}-1\right),
\end{aligned}$$

and hence

$$\begin{aligned}
\left|F\left(\frac{i}{j}\right)\right|_{1\le i,j\le p-1} &\equiv 3^{-\frac{p+5}{3}}(-1)^{\frac{p+1}{2}+\frac{p-4}{3}}\frac{p^2-5}{4}\prod_{j=1}^{p-2}(j!)^2\times\prod_{k\in\{2+3a:\ 0\le a\le\frac{p-7}{3}\}}\left(\frac{k(k+3)}{3}\right)^2 \\
&\quad\times\prod_{\substack{3\le k<\frac{p-1}{2}\\ 3\mid k}}(k^2+k-1)^2 \pmod p.
\end{aligned} \tag{2.18}$$

Observe that

$$\left(\frac{3}{p}\right)^{-\frac{p+5}{3}}\left(\frac{-1}{p}\right)^{\frac{p+1}{2}+\frac{p-4}{3}}\left(\frac{-5}{p}\right)=\left(\frac{-1}{p}\right)^{\frac{p+1}{2}+\frac{p-4}{3}+1}\left(\frac{5}{p}\right)=(-1)^{\frac{p-1}{2}\cdot\frac{p+1}{2}}\left(\frac{5}{p}\right)=\left(\frac{5}{p}\right). \tag{2.19}$$

Combining (2.17), (2.18) and (2.19), we obtain

$$\left(\frac{D_p^*(1,1)}{p}\right)=\left(\frac{|F(i/j)|_{1\le i,j\le p-1}}{p}\right)=\left(\frac{5}{p}\right)\text{ or }0.$$

Case 2. $p \equiv 2 \pmod 3$.

By Lemma 2.1 and the identity (2.1), we have

$$\begin{aligned}\left|F\left(\frac{i}{j}\right)\right|_{1\le i,j\le p-1} &\equiv -\frac{1}{27}\prod_{k=2}^{p-3}\left(-\frac{k}{3}\binom{\frac{k+1}{3}}{}+\frac{1}{3}\binom{\frac{k+2}{3}}{}\right)\prod_{1\le i<j\le p-1}(i-j)\left(\frac{1}{i}-\frac{1}{j}\right)\\ &\equiv -\frac{1}{27}(-1)^{\frac{p+1}{2}}(-1)^{\frac{p-5}{3}}\prod_{j=1}^{p-2}(j!)^2\times\prod_{k\in\{2+3a:\ 0\le a\le\frac{p-5}{3}\}}\frac{1}{3}\\ &\quad\times\prod_{k\in\{4+3a:\ 0\le a\le\frac{p-8}{3}\}}\frac{k}{3}\times\prod_{k\in\{3+3a:\ 0\le a\le\frac{p-8}{3}\}}\frac{k+1}{3}\\ &\equiv 3^{-(p-4)-3}(-1)^{\frac{p+1}{2}+\frac{p-5}{3}+1}\prod_{j=1}^{p-2}(j!)^2\times\prod_{k\in\{4+3a:\ 0\le a\le\frac{p-8}{3}\}}k^2.\end{aligned}$$

Combining this with (2.17), we finally obtain

$$\left(\frac{D_p^*(1,1)}{p}\right)=\left(\frac{|F(i/j)|_{1\le i,j\le p-1}}{p}\right)=\left(\frac{3}{p}\right)^{-p+1}\left(\frac{-1}{p}\right)^{\frac{p+1}{2}+\frac{p-5}{3}+1}=\left(\frac{-1}{p}\right)^{\frac{p+1}{2}+\frac{p-5}{3}+1}=\left(\frac{-1}{p}\right).$$

In view of the above, we have completed our proof of Theorem 1.2. □

3. Proof of Theorem 1.3

Lemma 3.1. *Let $p>5$ be a prime, and let $b,c\in\mathbb{Z}$. Then*

$$\left(\frac{D_p^*(b,c)}{p}\right)=\left(\frac{c}{p}\right)^{\frac{(p-1)(p-3)}{8}}\left(\frac{\binom{p-3}{0}_{b,c}\binom{p-3}{1}_{b,c}^2}{p}\right)\left(\frac{W(\frac{p-1}{2})\prod_{k=2}^{(p-3)/2}W(k)^2}{p}\right),\tag{3.1}$$

where

$$W(k)=\binom{p-3}{k}_{b,c}+\binom{p-3}{p-1-k}_{b,c}c^{p-1-k}.$$

Proof. Let $G(x)=(x^2+bx+c)^{p-3}$. For $1\le i,j\le p-1$, we have

$$\frac{(i^2+bij+cj^2)^{p-3}}{j^{2(p-3)}}=\left(\frac{i^2}{j^2}+b\frac{i}{j}+c\right)^{p-3}=G\left(\frac{i}{j}\right),$$

and hence

$$\left(\frac{D_p^*(b,c)}{p}\right)=\left(\frac{|G(i/j)|_{1\le i,j\le p-1}}{p}\right)$$

by Fermat's little theorem. In view of (2.11), and Lemmas 2.1 and 2.2, we see that

$$\begin{aligned}\left|G\left(\frac{i}{j}\right)\right|_{1\le i,j\le p-1} &= \binom{p-3}{0}_{b,c}\binom{p-3}{1}_{b,c}^2 c\prod_{k=2}^{p-3}\left(\binom{p-3}{k}_{b,c}+\binom{p-3}{p-1-k}_{b,c}c^{p-1-k}\right)\\ &\quad\times\prod_{1\le i<j\le p-1}(i-j)\left(\frac{1}{i}-\frac{1}{j}\right)\\ &= \binom{p-3}{0}_{b,c}\binom{p-3}{1}_{b,c}^2 c\prod_{k=2}^{p-3}W(k)\times(-1)^{\frac{p+1}{2}}\prod_{j=1}^{p-2}(j!)^2\\ &= \binom{p-3}{0}_{b,c}\binom{p-3}{1}_{b,c}^2 cW\left(\frac{p-1}{2}\right)\prod_{k=2}^{\frac{p-3}{2}}(W(k)W(p-1-k))\\ &\quad\times(-1)^{\frac{p+1}{2}}\prod_{j=1}^{p-2}(j!)^2.\end{aligned}$$

Since

$$W(p-1-k)=\binom{p-3}{p-1-k}_{b,c}+c^k\binom{p-3}{k}_{b,c}\equiv c^kW(k)\pmod p$$

for all $k=2,\ldots,(p-3)/2$, we have

$$\begin{aligned}\prod_{k=2}^{\frac{p-3}{2}}W(k)W(p-1-k) &= \prod_{k=2}^{\frac{p-3}{2}}\left(\frac{c^kW(k)^2}{p}\right)=\left(\frac{c}{p}\right)^{\sum_{k=2}^{\frac{p-3}{2}}k}\left(\frac{\prod_{k=2}^{(p-3)/2}W(k)^2}{p}\right)\\ &= \left(\frac{c}{p}\right)^{\frac{p^2-4p-5}{8}}\left(\frac{\prod_{k=2}^{(p-3)/2}W(k)^2}{p}\right).\end{aligned}$$

Combining the above, we immediately obtain the desired identity (3.1). □

Proof of Theorem 1.3. Applying Theorem 1.1 with $c=2$, we see that $(\frac{D_p^*(2,2)}{p})=0$ if $p\equiv\pm3\pmod 8$. Below we assume that $p\equiv\pm1\pmod 8$.

Let $k\in\{0,\ldots,p-3\}$. Taking $b=c=2$ in Lemma 2.4 and (2.2), we get

$$8\binom{p-3}{k}_{2,2}\equiv(k+3)\binom{p-2}{k-1}_{2,2}-2(k-3)\binom{p-2}{k+1}_{2,2}-2\binom{p-1}{k}_{2,2}\pmod p \tag{3.2}$$

and

$$4\binom{p-2}{k}_{2,2}\equiv\begin{cases}\binom{p-1}{-1}_{2,2}+2\binom{p-1}{1}_{2,2}-2\pmod p & \text{if } k=0,\\ (k+1)\binom{p-1}{k-1}_{2,2}-2(k-1)\binom{p-1}{k+1}_{2,2}\pmod p & \text{otherwise.}\end{cases} \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\begin{aligned} 32\binom{p-3}{k}_{2,2} &\equiv 4(k+3)\binom{p-2}{k-1}_{2,2} - 8(k-3)\binom{p-2}{k+1}_{2,2} - 8\binom{p-1}{k}_{2,2} \\ &\equiv \begin{cases} 20\binom{p-1}{1}_{2,2} - 8\binom{p-1}{3} - 8 \pmod p & \text{if } k=1, \\ k(k+3)\binom{p-1}{k-2}_{2,2} - 4(k+2)(k-2)\binom{p-1}{k}_{2,2} + 4k(k-3)\binom{p-1}{k+2}_{2,2} \pmod p & \text{otherwise}, \end{cases} \end{aligned} \tag{3.4}$$

and also

$$\begin{aligned} 32\binom{p-3}{p-1-k}_{2,2} \equiv\ & (k^2-k-2)\binom{p-1}{p-3-k}_{2,2} - 4(k^2+2k-3)\binom{p-1}{p-1-k}_{2,2} \\ & + 4(k^2+5k+4)\binom{p-1}{p+1-k}_{2,2} \pmod p \end{aligned} \tag{3.5}$$

if $2 \le k \le p-3$.

For any $2 \le k \le p-3$, define

$$W(k) = \binom{p-3}{k}_{2,2} + \binom{p-3}{p-1-k}_{2,2} 2^{p-1-k}.$$

Then

$$\begin{aligned} 32W(k) &= 32\binom{p-3}{k}_{2,2} + 2^{p-1-k} \times 32\binom{p-3}{p-1-k}_{2,2} \\ &= k(k+3)\binom{p-1}{k-2}_{2,2} - 4(k+2)(k-2)\binom{p-1}{k}_{2,2} + 4k(k-3)\binom{p-1}{k+2}_{2,2} \\ &\quad + 2^{p-1-k}\Bigg((k^2-k-2)\binom{p-1}{p-3-k}_{2,2} - 4(k^2+2k-3)\binom{p-1}{p-1-k}_{2,2} \\ &\quad + 4(k^2+5k+4)\binom{p-1}{p+1-k}_{2,2}\Bigg). \end{aligned}$$

Define the sequence $(u_n)_{n\ge0}$ by

$$u_0 = 0,\ u_1 = 1,\ \text{and } u_{n+1} = -2u_n - 2u_{n-1} \text{ for } n = 1,2,3,\ldots.$$

By Luo and Sun [3, (4.3)],

$$\binom{p-1}{p-k}_{2,2} \equiv u_k \pmod p \tag{3.6}$$

for all $k = 0, 1, \ldots, p-1$. Thus

$$\begin{aligned} 32W(k) = k(k+3)u_{p-k+2} - 4(k+2)(k-2)u_{p-k} + 4k(k-3)u_{p-k-2} \\ + 2^{-k}\left((k+1)(k-2)u_{k+3} - 4(k-1)(k+3)u_{k+1} + 4(k+1)(k+4)u_{k-1}\right) \end{aligned} \tag{3.7}$$

for all $k = 2, \ldots, p-3$.

Clearly, (3.5) with $k=(p-1)/2$ yields that

$$32W\left(\frac{p-1}{2}\right)=\frac{p-1}{2}\cdot\frac{p+5}{2}u_{\frac{p+5}{2}}-4\frac{p+3}{2}\cdot\frac{p-5}{2}u_{\frac{p+1}{2}}+4\frac{p-1}{2}\cdot\frac{p-7}{2}u_{\frac{p-3}{2}}$$
$$+\left(\frac{2}{p}\right)\left(\frac{p+1}{2}\cdot\frac{p-5}{2}u_{\frac{p+5}{2}}-4\frac{p-3}{2}\cdot\frac{p+5}{2}u_{\frac{p+1}{2}}+4\frac{p+1}{2}\cdot\frac{p+7}{2}u_{\frac{p-3}{2}}\right). \tag{3.8}$$

By [3, (4.9)], for any $k\in\mathbb{N}$ we have

$$u_k=(-4)^{\lfloor\frac{k}{4}\rfloor}\times\begin{cases}0 & \text{if } k\equiv 0 \pmod 4,\\ 1 & \text{if } k\equiv 1 \pmod 4,\\ -2 & \text{if } k\equiv 2 \pmod 4,\\ 2 & \text{if } k\equiv 3 \pmod 4.\end{cases} \tag{3.9}$$

Case 1. $p\equiv 1 \pmod 8$.

By (3.4) and (3.6), we have

$$\binom{p-3}{0}_{2,2}=\frac{1}{2}\binom{p-1}{0}_{2,2}=\frac{u_p}{2}=\frac{1}{2}(-4)^{\lfloor\frac{p}{4}\rfloor}\equiv\frac{1}{2}\cdot 2^{\frac{p-1}{2}}=\frac{1}{2}\left(\frac{2}{p}\right)=\frac{1}{2} \pmod p. \tag{3.10}$$

Write $p=8q+1$ with $q\in\mathbb{N}$. In view of (3.8) and (3.9), we have

$$\begin{aligned}32W\left(\frac{p-1}{2}\right)&=2\left(4q(4q+3)u_{4q+3}-4(4q+2)(4q-2)u_{4q+1}+4(4q)(4q-3)u_{4q-1}\right)\\&=8q(4q+3)\times 2(-4)^q-8(4q+2)(4q-2)(-4)^q+32q(4q-3)\times 2(-4)^{q-1}\end{aligned}$$

and hence

$$\begin{aligned}W\left(\frac{p-1}{2}\right)&=(-1)^q\left(2q(4q+3)2^{2q-2}-(4q+2)(4q-2)2^{2q-2}-2q(4q-3)2^{2q-2}\right)\\&=(-1)^q2^{2q-2}\left(8q^2+6q-16q^2+4-8q^2+6q\right)\\&=(-1)^q2^{2q}(-q+1)(4q+1)=(-1)^{\frac{p+7}{8}}2^{2q}\cdot\frac{p-9}{8}\cdot\frac{p+1}{2}.\end{aligned}$$

Therefore,

$$\left(\frac{W(\frac{p-1}{2})}{p}\right)=\left(\frac{-1}{p}\right)^{\frac{p+7}{8}}\left(\frac{-1}{p}\right)=1. \tag{3.11}$$

Note that

$$\left(\frac{\binom{p-3}{0}_{2,2}}{p}\right)=\left(\frac{2}{p}\right)=1$$

by (3.10). Combining this with Lemma 3.1, (3.12) and (3.11), we obtain

$$\left(\frac{D_p^*(2,2)}{p}\right)\neq -1.$$

Case 2. $p \equiv 7 \pmod 8$.

In this case we write $p = 8q + 7$ with $q \in \mathbb{N}$. For $2 \le k \le p-3$, write $k = 4s + r$ with $s \in \mathbb{N}$ and $r \in \{0, 1, 2, 3\}$. We will first show that $W(k) \not\equiv 0 \pmod p$ for any $k \in \{2, 3, \ldots, p-3\}$.

Subcase 2.1. $r = 0$.

In this subcase, by (3.7), (3.9) and Fermat's little theorem, we have

$$\begin{aligned}
&32W(k)\\
\equiv\ & k(k+3)u_{p-k+2} - 4(k+2)(k-2)u_{p-k} + 4k(k-3)u_{p-k-2}\\
&+ 2^{-k}\left((k+1)(k-2)u_{k+3} - 4(k-1)(k+3)u_{k+1} + 4(k+1)(k+4)u_{k-1}\right)\\
\equiv\ & k(k+3)(-4)^{\lfloor\frac{p-k+2}{4}\rfloor} - 8(k+2)(k-2)(-4)^{\lfloor\frac{p-k}{4}\rfloor} + 4k(k-3)(-4)^{\lfloor\frac{p-k-2}{4}\rfloor}\\
&+ 2^{-k}\left(2(k+1)(k-2)(-4)^{\lfloor\frac{k+3}{4}\rfloor} - 4(k-1)(k+3)(-4)^{\lfloor\frac{k+1}{4}\rfloor} + 8(k+1)(k+4)(-4)^{\lfloor\frac{k-1}{4}\rfloor}\right)\\
\equiv\ & 4s(4s+3)(-4)^{2q-s+2} - 8(4s+2)(4s-2)(-4)^{2q-s+1} + 16s(4s-3)(-4)^{2q-s+1}\\
&+ 2^{-k}\left(2(4s+1)(4s-2)(-4)^{s} - 4(4s-1)(4s+3)(-4)^{s} + 8(k+1)(k+4)(-4)^{s-1}\right)\\
\equiv\ & 2k(k+3)(-4)^{-s} \times \left(\frac{2}{p}\right) + 4(k+2)(k-2)(-4)^{-s}\left(\frac{2}{p}\right) - 2k(k-3)\left(\frac{2}{p}\right)(-4)^{-s}\\
&+ 2^{-4s+1}(k+1)(k-2)(-4)^{s} - 2^{-4s+2}(k-1)(k+3)(-4)^{s} - 2^{-4s+1}(k+1)(k+4)(-4)^{s}\\
=\ & 2(-4)^{-s}(-4k-8) \not\equiv 0 \pmod p.
\end{aligned}$$

Subcase 2.2. $r = 1$.

In this subcase, by (3.7) and (3.9) we have

$$\begin{aligned}
32W(k) \equiv\ & k(k+3)u_{p-k+2} - 4(k+2)(k-2)u_{p-k} + 4k(k-3)u_{p-k-2}\\
&+ 2^{-k}\left((k+1)(k-2)u_{k+3} - 4(k-1)(k+3)u_{k+1} + 4(k+1)(k+4)u_{k-1}\right)\\
\equiv\ & 8(k+2)(k-2)(-4)^{\lfloor\frac{p-k}{4}\rfloor} + 2^{-k} \times 8(k-1)(k+3)(-4)^{\lfloor\frac{k+1}{4}\rfloor}\\
\equiv\ & 8(k+2)(k-2)(-4)^{2q-s+1} + 2^{-k} \times 8(k-1)(k+3)(-4)^{s}\\
\equiv\ & -4(k+2)(k-2)(-4)^{-s}\left(\frac{2}{p}\right) + 2^{-4s+2}(k-1)(k+3)(-4)^{s}\\
=\ & (-4)^{-s+1}(-1-2k) \not\equiv 0 \pmod p.
\end{aligned}$$

Subcase 2.3. $r = 2$.

In view of (3.7) and (3.9), we have

$$\begin{aligned}
&32W(k)\\
\equiv\ & k(k+3)u_{p-k+2} - 4(k+2)(k-2)u_{p-k} + 4k(k-3)u_{p-k-2}\\
&+ 2^{-k}\left((k+1)(k-2)u_{k+3} - 4(k-1)(k+3)u_{k+1} + 4(k+1)(k+4)u_{k-1}\right)\\
\equiv\ & 2k(k+3)(-4)^{\lfloor\frac{p-k+2}{4}\rfloor} - 4(k+2)(k-2)(-4)^{\lfloor\frac{p-k}{4}\rfloor} + 8k(k-3)(-4)^{\lfloor\frac{p-k-2}{4}\rfloor}
\end{aligned}$$

$$+\,2^{-k}\left((k+1)(k-2)(-4)^{\lfloor\frac{k+3}{4}\rfloor}-8(k-1)(k+3)(-4)^{\lfloor\frac{k+1}{4}\rfloor}+4(k+1)(k+4)(-4)^{\lfloor\frac{k-1}{4}\rfloor}\right)$$
$$\equiv 2k(k+3)(-4)^{2q-s+1}-4(k+2)(k-2)(-4)^{2q-s+1}+8k(k-3)(-4)^{2q-s}$$
$$+\,2^{-k}\left((k+1)(k-2)(-4)^{s+1}-8(k-1)(k+3)(-4)^{s}+4(k+1)(k+4)(-4)^{s}\right)$$
$$\equiv -(-4)^{-s}k(k+3)+2(-4)^{-s}(k+2)(k-2)+(-4)^{-s}k(k-3)$$
$$+\,2^{-4s-2}\left(-4(k+1)(k-2)(-4)^{s}-8(k-1)(k+3)(-4)^{s}+4(k+1)(k+4)(-4)^{s}\right)$$
$$=(-4)^{-s+1}(k-1)\not\equiv 0 \pmod p.$$

Subcase 2.4. $r=3$.

In this subcase, by (3.7) and (3.9) we have

$$\begin{aligned}
32W(k) &\equiv k(k+3)u_{p-k+2}-4(k+2)(k-2)u_{p-k}+4k(k-3)u_{p-k-2}\\
&\quad +2^{-k}\left((k+1)(k-2)u_{k+3}-4(k-1)(k+3)u_{k+1}+4(k+1)(k+4)u_{k-1}\right)\\
&\equiv -2k(k+3)(-4)^{\lfloor\frac{p-k+2}{4}\rfloor}-8k(k-3)(-4)^{\lfloor\frac{p-k-2}{4}\rfloor}\\
&\quad +2^{-k}(-2(k+1)(k-2)(-4)^{\lfloor\frac{k+3}{4}\rfloor}-8(k+1)(k+4)(-4)^{\lfloor\frac{k-1}{4}\rfloor})\\
&\equiv -2k(k+3)(-4)^{2q-s+1}-8k(k-3)(-4)^{2q-s}\\
&\quad +2^{-4s}\left((k+1)(k-2)(-4)^{s}-(k+1)(k+4)(-4)^{s}\right)\\
&\equiv k(k+3)(-4)^{-s}-k(k-3)(-4)^{-s}+(k+1)(k-2)(-4)^{-s}-(k+1)(k+4)(-4)^{-s}\\
&= -6(-4)^{-s}\not\equiv 0 \pmod p.
\end{aligned}$$

In view of the discussions for all the four subcases, we do have

$$W(k)\not\equiv 0 \pmod p \qquad \text{for any } 2\le k\le p-3. \tag{3.12}$$

By (3.4) and (3.6), we have

$$\begin{aligned}
32\binom{p-3}{1}_{2,2} &= 20\binom{p-1}{1}_{2,2}-8\binom{p-1}{3}_{2,2}-8\\
&\equiv 20u_{p-1}-8u_{p-3}-8=-40(-4)^{\lfloor\frac{p-1}{4}\rfloor}-8=10(-4)^{2q+2}-8\\
&=10\times 2^{4q+4}-8=20\times 2^{\frac{p-1}{2}}-8=20\left(\frac{2}{p}\right)-8=12 \pmod p
\end{aligned}$$

and

$$\binom{p-3}{0}_{2,2}=\frac12\binom{p-1}{0}_{2,2}\equiv\frac{u_p}{2}=(-4)^{\lfloor\frac{p}{4}\rfloor}=-\frac12 2^{\frac{p-1}{2}}\equiv -\frac12\left(\frac{2}{p}\right)=-\frac12 \pmod p.$$

Therefore,

$$\left(\frac{\binom{p-3}{1}_{2,2}}{p}\right)\neq 0 \ \text{ and } \ \left(\frac{\binom{p-3}{0}_{2,2}}{p}\right)=-1. \tag{3.13}$$

By (3.8) and (3.9), we have

$$32W\left(\frac{p-1}{2}\right) \equiv 2(4q+3)(4q+6)u_{4q+6} - 8(4q+5)(4q+1)u_{4q+4} + 8(4q+3)4qu_{4q+2}$$
$$\equiv 16(4q+3)(4q+6)(-4)^q - 16(4q+3)4q(-4)^q \pmod p$$

and hence

$$W\left(\frac{p-1}{2}\right) \equiv (4q+3)(2q+3)(-4)^q - (4q+3)2q(-4)^q$$
$$= (-1)^q((4q+3)(2q+3)2^{2q} - (4q+3)2q2^{2q}) = 3(-1)^q 2^{2q}\frac{p-1}{2} \pmod p.$$

Therefore

$$\left(\frac{W(\frac{p-1}{2})}{p}\right) = \left(\frac{3}{p}\right)(-1)^{\frac{p-7}{8}}\left(\frac{p-1}{p}\right)\left(\frac{2}{p}\right) = \left(\frac{3}{p}\right)(-1)^{\frac{p+1}{8}}. \tag{3.14}$$

Combining Lemma 3.1, (3.12), (3.13) and (3.14), we finally obtain

$$\left(\frac{D_p^*(2,2)}{p}\right) = \left(\frac{\binom{p-3}{0}_{2,2}}{p}\right)\left(\frac{W(\frac{p-1}{2})}{p}\right) = -1 \times \left(\frac{3}{p}\right)(-1)^{\frac{p+1}{8}} = \left(\frac{p}{3}\right)(-1)^{\frac{p+1}{8}}.$$

In view of the above, we have finished our proof of Theorem 1.3. □

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(Han Wang) Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China
E-mail address: hWang@smail.nju.edu.cn

(Zhi-Wei Sun, corresponding author) Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China
E-mail address: zwsun@nju.edu.cn