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NEW SERIES INVOLVING BINOMIAL COEFFICIENTS (I)

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ABSTRACT. In this paper, we deduce several new identities on infinite series with denominators of summands containing both binomial coefficients and linear parts. For example, we evaluate the sums

$$\sum_{k=1}^{\infty}\frac{x_0^k}{(2k-1)\binom{3k}{k}}\ \text{and}\ \sum_{k=0}^{\infty}\frac{x_0^k}{(3k+2)\binom{3k}{k}}$$

for any $x_0\in(-27/4,27/4)$. For any $1<n\leqslant 85/4$, we obtain the following fast converging series for $\log n$:

$$\sum_{k=0}^{\infty}\frac{(2(n^2+6n+1)^2k+n^4+30n^2+1)(n-1)^{4k}}{(4k+1)(-n)^k(n+1)^{2k}\binom{4k}{2k}}$$
$$=8n(n+1)^2-2n(n^2-1)\log n.$$

In addition, we pose many new conjectural series identities involving binomial coefficients; for example, we conjecture that

$$\sum_{k=1}^{\infty}\frac{9(21k-8)H_{k-1}^{(4)}+25/k^3}{k^3\binom{2k}{k}^3}=\frac{13\pi^6}{3780},$$

where $H_{k-1}^{(4)}$ denotes the fourth harmonic number $\sum_{0<j\leqslant k-1}j^{-4}$.

1. INTRODUCTION

The curious identity

$$\sum_{k=0}^{\infty}\frac{25k-3}{2^k\binom{3k}{k}}=\frac{\pi}{2},\tag{1.1}$$

was first announced by R. W. Gosper in 1974, and later used by Bellard [6] to find an algorithm for computing the nth decimal of π without computing the earlier ones. Almkvist, Krattenthaler and Petersson [2] gave a proof of the Gosper identity (1.1) by using the beta function

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx\quad(a>0\text{ and }b>0).$$

A celebrated result of Euler states that

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

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where the Gamma function is given by

$$\Gamma(x) = \int_0^{+\infty} \frac{t^{x-1}}{e^t} dt \quad \text{for } x > 0.$$

Motivated by Gosper's identity (1.1), the author [19] evaluated

$$\sum_{k=0}^{\infty} \frac{(ak+b)x_0^k}{\binom{3k}{k}}$$

for $x_0 \in (-27/4, 27/4)$, where a and b are real numbers. As observed in [19], $x^3 = x_0(x-1)$ for some $x \in (-3, c)$, where

$$c = \frac{3}{2}\left(\sqrt[3]{1+\sqrt{2}} - \frac{1}{\sqrt[3]{1+\sqrt{2}}}\right) = 0.8941\ldots. \tag{1.2}$$

By [19, (1.10)],

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{x^{3k}}{(x-1)^k\binom{3k}{k}} = &\frac{27(1-x)}{(x+3)(2x-3)^2} + \frac{3x(x-1)}{(2x-3)^3}\log(1-x) \\ &+ \frac{2x(x-1)(x^2-12x+9)q(x)}{(x+3)(2x-3)^3\sqrt{(1-x)(x+3)}},\end{aligned} \tag{1.3}$$

where

$$q(x) = \begin{cases} \arctan \frac{x}{x+2}\sqrt{\frac{3+x}{1-x}} & \text{if } -2 < x < 1, \\ -\frac{\pi}{2} & \text{if } x = -2, \\ \arctan \frac{x}{x+2}\sqrt{\frac{3+x}{1-x}} - \pi & \text{if } -3 < x < -2. \end{cases} \tag{1.4}$$

Note that the identity (1.3) is clearer than Batir's complicated formula for $\sum_{k=0}^{\infty} x_0^k/\binom{3k}{k}$ (cf. [4, (3.3)]).

In view of the above, it is natural to ask whether for some $a, b \in \mathbb{Z} \setminus \{0\}$ we can evaluate

$$\sum_{k=0}^{\infty} \frac{x^{3k}}{(ak+b)(x-1)^k\binom{3k}{k}},$$

where $x \in (-3, c)$. In this direction, we obtain the following new result.

Theorem 1.1. *Let $-3 < x < c$ with $x \neq 0$, where the constant c is given by (1.2). Then*

$$\sum_{k=1}^{\infty} \frac{x^{3k}}{(2k-1)(x-1)^k\binom{3k}{k}} = \frac{x(1-x)\log(1-x)}{2x-3} - \frac{2x(x^2-3)q(x)}{3(2x-3)\sqrt{(1-x)(x+3)}}. \tag{1.5}$$

Also,

$$\sum_{k=0}^{\infty} \frac{x^{3k}}{(3k+1)(x-1)^k\binom{3k}{k}} = \frac{3(1-x)\log(1-x)}{2x(2x-3)} + \frac{(x-3)q(x)}{x(2x-3)}\sqrt{\frac{1-x}{x+3}} \tag{1.6}$$

and

$$\sum_{k=0}^{\infty}\frac{x^{3k}}{(3k+2)(x-1)^k\binom{3k}{k}}=\frac{3(3-x)(1-x)\log(1-x)}{2x^2(2x-3)} -\frac{(x^2+6x-9)q(x)}{x^2(2x-3)}\sqrt{\frac{1-x}{x+3}}. \tag{1.7}$$

Remark 1.1. In 2013, Batir [5, Theorem 1] gave a very complicated formula for the series $\sum_{k=1}^{\infty}x_0^k/((3k+1)\binom{3k}{k})$ with $x_0\in(-27/4,27/4)$.

Taking $x=-\sqrt{3}$ in (1.5), we immediately obtain the following corollary.

Corollary 1.1. *We have*

$$\sum_{k=1}^{\infty}\frac{(\frac{9-3\sqrt{3}}{2})^k}{(2k-1)\binom{3k}{k}}=(\sqrt{3}-1)\log(\sqrt{3}+1). \tag{1.8}$$

Theorem 1.1 with $x=-1$ yields the following corollary obtained by Campbell and Levrie [7].

Corollary 1.2. *We have the following identities:*

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)2^k\binom{3k}{k}}=\frac{2}{5}\log 2-\frac{\pi}{30}, \tag{1.9}$$

$$\sum_{k=0}^{\infty}\frac{1}{(3k+1)2^k\binom{3k}{k}}=\frac{\pi+3\log 2}{5}, \tag{1.10}$$

$$\sum_{k=0}^{\infty}\frac{1}{(3k+2)2^k\binom{3k}{k}}=\frac{7\pi-24\log 2}{10}. \tag{1.11}$$

Theorem 1.1 with $x=-2$ yields the following corollary.

Corollary 1.3. *We have the following identities:*

$$\sum_{k=1}^{\infty}\frac{8^k}{(2k-1)3^k\binom{3k}{k}}=\frac{2\sqrt{3}}{63}\pi+\frac{6}{7}\log 3, \tag{1.12}$$

$$\sum_{k=0}^{\infty}\frac{8^k}{(3k+1)3^k\binom{3k}{k}}=\frac{5\sqrt{3}\,\pi+9\log 3}{28}, \tag{1.13}$$

$$\sum_{k=0}^{\infty}\frac{8^k}{(3k+2)3^k\binom{3k}{k}}=\frac{17\sqrt{3}\,\pi-45\log 3}{56}. \tag{1.14}$$

Recall that the harmonic numbers are given by

$$H_n=\sum_{0<k\leqslant n}\frac{1}{k}\quad (n=0,1,2,\ldots).$$

We also establish the following theorem.

Theorem 1.2. *We have*

$$\sum_{k=1}^\infty\frac{H_{3k}-H_k}{(2k-1)2^k\binom{3k}{k}}=\frac{\pi^2}{36}-\frac{\pi}{18}-\frac{2}{15}(G+\log^2 2)+\frac{\log 2}{3},\tag{1.15}$$

$$\sum_{k=1}^\infty\frac{H_{2k}-H_k}{(3k+1)2^k\binom{3k}{k}}=-\frac{(\pi-2\log 2)(\pi-6\log 2)}{80},\tag{1.16}$$

$$\sum_{k=0}^\infty\frac{H_{3k+1}-H_k}{(3k+1)2^k\binom{3k}{k}}=\frac{\pi^2}{24}+\frac{4}{5}G-\frac{\log^2 2}{5},\tag{1.17}$$

and

$$\sum_{k=1}^\infty\frac{H_{2k}-H_k}{(3k+2)2^k\binom{3k}{k}}=\frac{3}{5}\log^2 2-9\log 2+\frac{\pi}{20}(\pi+7\log 2+30),\tag{1.18}$$

where G is the Catalan constant given by

$$G:=\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^2}.$$

By [19, Remarks 1.2 and 1.3], there are simple closed formulas for the series

$$\sum_{k=0}^\infty\frac{x_0^k}{\binom{4k}{2k}}\quad\text{and}\quad\sum_{k=0}^\infty\frac{kx_0^k}{\binom{4k}{2k}}$$

with $x_0\in(-16,16)$. Clearly, if $0<x_0<16$ then $x_0=1/x^2$ for some $x>1/4$, if $-16<x_0<0$ then $x_0=4/(x(1-x))$ for some $x>(1+\sqrt{2})/2$ or $x<(1-\sqrt{2})/2$.

For $x>1$ or $x<0$, as in [19] we define

$$R(x):=\sqrt{x}\,\mathrm{arctanh}\frac{1}{\sqrt{x}}=\begin{cases}\frac{\sqrt{x}}{2}\log\frac{\sqrt{x}+1}{\sqrt{x}-1} & \text{if } x>1,\\ \sqrt{|x|}\arctan\frac{1}{\sqrt{|x|}} & \text{if } x<0,\end{cases}\tag{1.19}$$

where $\mathrm{arctanh}\,t$ is the inverse hyperbolic tangent function.

Now we state our third theorem.

Theorem 1.3. (i) *Let $x>1/4$. Then*

$$\sum_{k=1}^\infty\frac{2(8x^2+6x+1)k-4x^2-6x-1}{(2k-1)x^{2k}\binom{4k}{2k}}=\frac{24x^2\mathrm{arccot}\sqrt{4x-1}}{(4x-1)^{3/2}}+\frac{2x+1}{4x-1}\tag{1.20}$$

and

$$\sum_{k=1}^\infty\frac{2(8x^2-6x+1)k-4x^2+6x-1}{(2k-1)x^{2k}\binom{4k}{2k}}=\frac{12x^2}{(4x+1)^{3/2}}\log\frac{\sqrt{4x+1}+1}{\sqrt{4x+1}-1}+\frac{2x-1}{4x+1}.\tag{1.21}$$

Also,

$$\sum_{k=0}^{\infty}\frac{2(4x+1)k+2x+1}{(4k+1)x^{2k}\binom{4k}{2k}}=\frac{8x^2}{4x-1}\left(\frac{\operatorname{arccot}\sqrt{4x-1}}{\sqrt{4x-1}}+1\right), \tag{1.22}$$

$$\sum_{k=0}^{\infty}\frac{2(4x-1)k+2x-1}{(4k+1)x^{2k}\binom{4k}{2k}}=\frac{8x^2}{4x+1}-\frac{4x^2}{(4x+1)^{3/2}}\log\frac{\sqrt{4x+1}+1}{\sqrt{4x+1}-1}, \tag{1.23}$$

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{2(4x+1)(8x+1)k+48x^2+18x+1}{(4k+3)x^{2k}\binom{4k}{2k}}\\&\quad=8x^2\frac{8x+1}{4x-1}+\frac{24x^2}{(4x-1)^{3/2}}\operatorname{arccot}\sqrt{4x-1},\end{aligned} \tag{1.24}$$

and

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{2(4x-1)(8x-1)k+48x^2-18x+1}{(4k+3)x^{2k}\binom{4k}{2k}}\\&\quad=8x^2\frac{8x-1}{4x+1}+\frac{12x^2}{(4x+1)^{3/2}}\log\frac{\sqrt{4x+1}+1}{\sqrt{4x+1}-1}.\end{aligned} \tag{1.25}$$

(ii) *If $x>(1+\sqrt{2})/2$ or $x<(1-\sqrt{2})/2$, then*

$$\sum_{k=1}^{\infty}\frac{(2(2x-1)^2k-2x^2-x+2)4^k}{(2k-1)x^k(1-x)^k\binom{4k}{2k}}=3(1-x)\frac{R(x)}{x}-1, \tag{1.26}$$

$$\sum_{k=0}^{\infty}\frac{(2(2x-1)^2k+2x^2-3x+2)4^k}{(4k+1)(x(1-x))^k\binom{4k}{2k}}=(1-x)(R(x)-2x) \tag{1.27}$$

and

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(2(2x-1)^2(4x-3)k+24x^3-42x^2+25x-6)4^k}{(4k+3)(x(1-x))^k\binom{4k}{2k}}\\&\quad=(1-x)(3R(x)-2x(4x-3)).\end{aligned} \tag{1.28}$$

Remark 1.2. When $x>(1+\sqrt{2})/2$ or $x<(1-\sqrt{2})/2$, we have $1-x<(1-\sqrt{2})/2$ or $1-x>(1+\sqrt{2})/2$, thus all the three identities in Theorem 1.3(ii) with x replaced by $1-x$ also hold. In view of this, Theorem 1.3 yields the values of the series

$$\sum_{k=1}^{\infty}\frac{k^r x_0^k}{(2k-1)\binom{4k}{2k}},\ \sum_{k=1}^{\infty}\frac{k^r x_0^k}{(4k+1)\binom{4k}{2k}}\ \text{and}\ \sum_{k=1}^{\infty}\frac{k^r x_0^k}{(4k+3)\binom{4k}{2k}}$$

for any $r\in\{0,1\}$ and $x_0\in(-16,16)$.

Corollary 1.4. *We have*

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)\binom{4k}{2k}}=\frac{3\sqrt5}{10}\log\frac{1+\sqrt5}{2}-\frac{\sqrt3}{36}\pi,\tag{1.29}$$

$$\sum_{k=0}^{\infty}\frac{1}{(4k+1)\binom{4k}{2k}}=\frac{2\sqrt5}{5}\log\frac{1+\sqrt5}{2}+\frac{\sqrt3}{9}\pi,\tag{1.30}$$

$$\sum_{k=0}^{\infty}\frac{1}{(4k+3)\binom{4k}{2k}}=\frac{7\sqrt3}{9}\pi-\frac{18\sqrt5}{5}\log\frac{1+\sqrt5}{2}.\tag{1.31}$$

Proof. Putting $x=1$ in (1.20) and (1.21), we get the following two identities:

$$\sum_{k=1}^{\infty}\frac{30k-11}{(2k-1)\binom{4k}{2k}}=\frac{4\pi}{3\sqrt3}+1,\tag{1.32}$$

$$\sum_{k=1}^{\infty}\frac{6k+1}{(2k-1)\binom{4k}{2k}}=\frac{24}{5\sqrt5}\log\frac{1+\sqrt5}{2}+\frac15.\tag{1.33}$$

Note that $5\times(1.33)-(1.20)$ yields the first identity in Corollary 1.4. Similarly, the second and the third identities in Corollary 1.4 follow from the last four identities in Theorem 1.3(i). □

The following corollary provides fast converging series for $\log n$ with $1<n\leqslant 85/4$.

Corollary 1.5. *For*

$$1<n<\frac{\sqrt{(1+\sqrt2)/2}+1}{\sqrt{(1+\sqrt2)/2}-1}=21.2666866\ldots,$$

we have the following formulas for $\log n$*:*

$$\begin{aligned}\sum_{k=1}^{\infty}&\frac{(2(n^2+6n+1)^2k-n^4-16n^3+2n^2-16n-1)(n-1)^{4k}}{(2k-1)(-n)^k(n+1)^{2k}\binom{4k}{2k}}\\&=-(n-1)^4-\frac{6n(n-1)^3}{n+1}\log n.\end{aligned}\tag{1.34}$$

$$\begin{aligned}\sum_{k=0}^{\infty}&\frac{(2(n^2+6n+1)^2k+n^4+30n^2+1)(n-1)^{4k}}{(4k+1)(-n)^k(n+1)^{2k}\binom{4k}{2k}}\\&=8n(n+1)^2-2n(n^2-1)\log n,\end{aligned}\tag{1.35}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty}&\frac{(2(n^2+6n+1)^2(n^2+14n+1)k+c_n)(n-1)^{4k}}{(4k+3)(-n)^k(n+1)^{2k}\binom{4k}{2k}}\\&=8n(n+1)^2(n^2+14n+1)-6n(n+1)(n-1)^3\log n,\end{aligned}\tag{1.36}$$

where

$$c_n=n^6+46n^5+287n^4+868n^3+287n^2+46n+1.$$

Proof. Let $x = (n+1)^2/(n-1)^2$. Then $x > (1+\sqrt{2})/2 > 1$,

$$\frac{x(1-x)}{4} = -\frac{n(n+1)^2}{(n-1)^4} \quad \text{and} \quad R(x) = \frac{\sqrt{x}}{2}\log\frac{\sqrt{x}+1}{\sqrt{x}-1} = \frac{n+1}{2(n-1)}\log n.$$

Applying the identities (1.26), (1.27) and (1.28), we get the desired identities (1.34), (1.35) and (1.36) respectively. □

We are going to prove Theorems 1.1-1.2 and Theorem 1.3 in Sections 2 and 3 respectively.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. The generalized harmonic numbers are given by

$$H_n^{(m)} = \sum_{0<k\leqslant n} \frac{1}{k^m} \quad (m \in \mathbb{Z}^+ \text{ and } n \in \mathbb{N}).$$

Inspired by Ramanujan-type series (cf. [9, Chapter 14] and [14]), the work of Chu and Zhang [8], Guo and Lian [12], Wei [22], and Wei and Ruan [24], the author [18, 20, 21] posed many conjectures on series with summands involving generalized harmonic numbers.

In Sections 4 and 5, we pose various new conjectures on series whose summands involve two or more binomial coefficients, many of which also involve generalized harmonic numbers. Sections 4 and 5 also contain related p-adic congruences, where p is an odd prime. By Wolstenholme's theorem, $H_{p-1} \equiv 0 \pmod{p^2}$ and $H_{p-1}^{(2)} \equiv 0 \pmod{p}$ if $p > 3$. As usual, we let $(\frac{\cdot}{p})$ be the Legendre symbol. For any $a \in \mathbb{Z}$ not divisible by p, we define the Fermat quotient $q_p(a)$ as the integer $(a^{p-1}-1)/p$. Some p-adic congruences in Sections 4 and 5 involve Bernoulli numbers $B_0, B_1, \ldots$ or Euler numbers $E_0, E_1, \ldots$. For each $n \in \mathbb{N}$, the Bernoulli polynomial $B_n(x)$ and the Euler polynomial $E_n(x)$ are given by

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k} \quad \text{and} \quad E_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k},$$

respectively.

2. Proofs of Theorems 1.1-1.2

Lemma 2.1. *If $-3 < x < 1$, then*

$$\arctan\frac{x+1}{\sqrt{(1-x)(3+x)}} + \arctan\frac{x-1}{\sqrt{(1-x)(3+x)}} = q(x), \tag{2.1}$$

where $q(x)$ is defined by (1.4).

This lemma is known, see [19, Lemma 3.1].

Proof of (1.6). For each $k \in \mathbb{N}$, we have

$$\begin{aligned}\frac{1}{(3k+1)\binom{3k}{k}} &= \frac{k!(2k)!}{(3k+1)!} = \frac{\Gamma(k+1)\Gamma(2k+1)}{\Gamma(3k+2)} \\ &= B(k+1, 2k+1) = \int_0^1 x^{2k}(1-x)^k dx.\end{aligned} \tag{2.2}$$

Thus

$$\begin{aligned}&\sum_{k=0}^\infty\frac{x^{3k}}{(3k+1)(x-1)^k\binom{3k}{k}}\\&=\sum_{k=0}^\infty\frac{x^{3k}}{(x-1)^k}\int_0^1 t^{2k}(1-t)^k dt=\int_0^1\sum_{k=0}^\infty\left(\frac{x^3t^2(1-t)}{x-1}\right)^k dt\end{aligned}$$

and hence

$$\sum_{k=0}^\infty\frac{x^{3k}}{(3k+1)(x-1)^k\binom{3k}{k}}=\int_0^1\frac{dt}{1-x^3t^2(1-t)/(x-1)}. \tag{2.3}$$

Since

$$\begin{aligned}&\frac{2x-3}{x^3t^2(t-1)+x-1}+\frac{1}{tx-1}\\&=\frac{(t-1)x+2}{t^2x^2+tx(1-x)-x+1}\\&=\frac{1}{2x}\cdot\frac{2tx^2+x(1-x)}{t^2x^2+tx(1-x)-x+1}+\frac{2(3-x)}{(2tx-x+1)^2+(1-x)(x+3)},\end{aligned}$$

we have

$$\begin{aligned}&(2x-3)\int_0^1\frac{dt}{x^3t^2(t-1)+x-1}+\int_0^1\frac{dt}{tx-1}\\&=\frac{1}{2x}\int_0^1\frac{2tx^2+x(1-x)}{t^2x^2+tx(1-x)-x+1}dt\\&\quad+\frac{2(3-x)}{(1-x)(x+3)}\int_0^1\frac{dt}{((2tx-x+1)/\sqrt{(1-x)(x+3)})^2+1}\end{aligned}$$

and hence

$$\begin{aligned}&(2x-3)\int_0^1\frac{dt}{x^3t^2(t-1)+x-1}+\frac{\log|xt-1|}{x}\bigg|_{t=0}^1\\&=\frac{1}{2x}\log(t^2x^2+tx(1-x)-x+1)\bigg|_{t=0}^1\\&\quad+\frac{2(3-x)}{(1-x)(x+3)}\cdot\frac{\sqrt{(1-x)(x+3)}}{2x}\arctan\frac{2tx-x+1}{\sqrt{(1-x)(x+3)}}\bigg|_{t=0}^1\\&=\frac{1}{2x}(\log 1-\log(1-x))+\frac{(3-x)Q(x)}{x\sqrt{(1-x)(x+3)}},\end{aligned}$$

where

$$Q(x):=\arctan\frac{x+1}{\sqrt{(1-x)(x+3)}}-\arctan\frac{1-x}{\sqrt{(1-x)(x+3)}}=q(x)$$

by Lemma 2.1. Therefore

$$(2x-3)\int_0^1\frac{dt}{x^3t^2(t-1)+x-1}=-\frac{3}{2x}\log(1-x)+\frac{(3-x)q(x)}{x\sqrt{(1-x)(x+3)}}. \tag{2.4}$$

Combining (2.3) with (2.4), we obtain

$$\sum_{k=0}^{\infty} \frac{x^{3k}}{(3k+1)(x-1)^k \binom{3k}{k}} = \frac{x-1}{2x-3} \left(-\frac{3\log(1-x)}{2x} + \frac{(3-x)q(x)}{x\sqrt{(1-x)(x+3)}} \right),$$

which is equivalent to the desired (1.6). □

Proof of (1.9). As (2.2) holds for any positive integer k, we have

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{x^{3k}}{(2k-1)(3k+1)(x-1)^k \binom{3k}{k}} \\ &= \sum_{k=1}^{\infty} \frac{x^{3k}}{(2k-1)(x-1)^k} \int_0^1 t^{2k}(1-t)^k dt = \int_0^1 S(x_0, t) dt, \end{aligned}$$

where $x_0 = x^3/(x-1)$ and

$$S(x_0, t) = \sum_{k=1}^{\infty} \frac{(t^2(1-t)x_0)^k}{2k-1} = t\sqrt{(1-t)x_0} \sum_{k=1}^{\infty} \frac{(t\sqrt{(1-t)x_0})^{2k-1}}{2k-1}.$$

If $x_0 > 0$, then

$$S(x_0, t) = t\sqrt{(1-t)x_0}\,\operatorname{arctanh}(t\sqrt{(1-t)x_0}),$$

where $\operatorname{arctanh} t$ is the inverse hyperbolic tangent function. If $x_0 < 0$, then

$$\begin{aligned} S(x_0, t) &= -t\sqrt{(t-1)x_0} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(t\sqrt{(t-1)x_0})^{2k-1}}{2k-1} \\ &= -t\sqrt{(t-1)x_0}\, \arctan(t\sqrt{(t-1)x_0}). \end{aligned}$$

For the function

$$f(t) = -\frac{2}{15}(3t+2)(1-t)^{3/2}$$

with $0 < x < 1$, it is easy to verify that

$$f'(t) = t\sqrt{1-t}.$$

If $x_0 > 0$, then

$$\begin{aligned} \int_0^1 S(x_0, t) dt &= \sqrt{x_0} \int_0^1 f'(t) \operatorname{arctanh}(t\sqrt{(1-t)x_0}) dt \\ &= \sqrt{x_0} f(t) \operatorname{arctanh}(t\sqrt{(1-t)x_0}) \Big|_{t=0}^{1} \\ &\quad - \sqrt{x_0} \int_0^1 f(t) \frac{(t\sqrt{1-t})'\sqrt{x_0}}{1-t^2(1-t)x_0} dt \\ &= -x_0 \int_0^1 \frac{f(t)(t\sqrt{1-t})'}{1-t^2(1-t)x_0} dt. \end{aligned}$$

Similarly, if $x_0<0$ then

$$\begin{aligned}\int_0^1 S(x_0,t)dt&=-\sqrt{-x_0}\int_0^1 f'(t)\ \arctan(t\sqrt{(t-1)x_0})dt\\&=-\sqrt{-x_0}f(t)\ \arctan(t\sqrt{(t-1)x_0})\Big|_{t=0}^1\\&\quad+\sqrt{-x_0}\int_0^1 f(t)\frac{(t\sqrt{1-t})'\sqrt{-x_0}}{1+t^2(t-1)x_0}dt\\&=-x_0\int_0^1\frac{f(t)(t\sqrt{1-t})'}{1-t^2(1-t)x_0}dt.\end{aligned}$$

Note that

$$f(t)(t\sqrt{1-t})'=-\frac{2}{15}(3t+2)(1-t)^{3/2}\left(\sqrt{1-t}-\frac{t}{2\sqrt{1-t}}\right)=\frac{(1-t)(9t^2-4)}{15}.$$

Therefore

$$\begin{aligned}\int_0^1 S(x_0,t)&=\frac{x_0}{15}\int_0^1\frac{(1-t)(4-9t^2)}{1-t^2(1-t)x_0}dt\\&=\frac{x^3}{15}\int_0^1\frac{(1-t)(4-9t^2)}{(tx-1)(t^2x^2+tx(1-x)-x+1)}dt.\end{aligned}$$

Note that

$$\begin{aligned}&\frac{x^3(1-t)(4-9t^2)}{(tx-1)(t^2x^2+tx(1-x)-x+1)}\\&=9+\frac{3-x-2x^2}{tx-1}+\frac{2x^2+x-3}{2x}\cdot\frac{2tx^2+x(1-x)}{t^2x^2+tx(1-x)-x+1}\\&\quad-\frac{2(2x^3+3x^2-6x+9)}{(2tx+1-x)^2+(1-x)(x+3)}.\end{aligned}$$

Thus

$$\begin{aligned}&15\int_0^1 S(x_0,t)dt-\int_0^1 9dt\\&=(2x^2+x-3)\left(-\frac{\log|tx-1|}{x}\Big|_{t=0}^1+\frac{1}{2x}\log(t^2x^2+tx(1-x)-x+1)\Big|_{t=0}^1\right)\\&\quad-\frac{2(2x^3+3x^2-6x+9)}{(1-x)(x+3)}\int_0^1\frac{dt}{((2tx+1-x)/\sqrt{(1-x)(x+3)})^2+1}\\&=(2x^2+x-3)\left(-\frac{\log(1-x)}{x}-\frac{\log(1-x)}{2x}\right)\\&\quad-\frac{2(2x^3+3x^2-6x+9)}{2x\sqrt{(1-x)(x+3)}}\arctan\frac{2tx-x+1}{\sqrt{(1-x)(x+3)}}\Big|_{t=0}^1\\&=\frac{3(1-x)(2x+3)}{2x}\log(1-x)-\frac{2x^3+3x^2-6x+9}{x\sqrt{(1-x)(x+3)}}q(x)\end{aligned}$$

with the aid of Lemma 2.1.

Note that

$$\frac{5}{(2k-1)(3k+1)}=\frac{2}{2k-1}-\frac{3}{3k+1}.$$

By the above two paragraphs,

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{2}{2k-1}-\frac{3}{3k+1}\right)\frac{x^{3k}}{(x-1)^k\binom{3k}{k}}\\&=5\sum_{k=1}^{\infty}\frac{x^{3k}}{(2k-1)(3k+1)(x-1)^k\binom{3k}{k}}=5\int_0^1 S(x_0,t)dt\\&=3+\frac{(1-x)(2x+3)}{2x}\log(1-x)-\frac{2x^3+3x^2-6x+9}{3x\sqrt{(1-x)(x+3)}}q(x)\end{aligned}$$

Combining this with (1.6), we obtain

$$\begin{aligned}&2\sum_{k=1}^{\infty}\frac{x^{3k}}{(2k-1)(x-1)^k\binom{3k}{k}}\\&=\frac{(1-x)\log(1-x)}{2x}\left(\frac{9}{2x-3}+2x+3\right)\\&\quad+\frac{q(x)}{3x\sqrt{(1-x)(x+3)}}\left(\frac{9(x-3)(1-x)}{2x+3}-2x^3-3x^2+6x-9\right)\\&=\frac{2x(1-x)\log(1-x)}{2x-3}-\frac{4x(x^2-3)}{3(2x-3)\sqrt{(1-x)(x+3)}}\end{aligned}$$

and hence (1.5) follows. □

Proof of (1.7). Set $x_0=x^3/(x-1)$.

Observe that

$$\begin{aligned}3\sum_{k=1}^{\infty}\frac{x_0^k}{\binom{3k}{k}}&=\sum_{k=0}^{\infty}\frac{3x_0^{k+1}}{\binom{3k+3}{k+1}}=x_0\sum_{k=0}^{\infty}\frac{x_0^k}{\binom{3k+2}{k}}\\&=x_0\sum_{k=0}^{\infty}\frac{(2k+1)(2k+2)x_0^k}{(3k+1)(3k+2)\binom{3k}{k}}\\&=\frac{2x_0}{9}\sum_{k=0}^{\infty}\left(2+\frac{2}{3k+1}+\frac{1}{3k+2}\right)\frac{x_0^k}{\binom{3k}{k}}.\end{aligned}$$

Therefore

$$\sum_{k=0}^{\infty}\frac{x_0^k}{(3k+2)\binom{3k}{k}}+2\sum_{k=0}^{\infty}\frac{x_0^k}{(3k+1)\binom{3k}{k}}=\left(\frac{27}{2x_0}-2\right)\sum_{k=0}^{\infty}\frac{x_0^k}{\binom{3k}{k}}-\frac{27}{2x_0}.$$

Combining this with (1.3) and (1.6), we obtain the desired (1.7). □

For $a>0$ and $b>0$, we have

$$\frac{\partial}{\partial a}B(a,b)=\int_0^1\frac{\partial}{\partial a}x^{a-1}(1-x)^{b-1}dx=\int_0^1 x^{a-1}(1-x)^{b-1}\log x\,dx;$$

on the other hand,

$$\frac{\partial}{\partial a}B(a,b) = \frac{\partial}{\partial a}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \Gamma(b)\frac{\partial}{\partial a}e^{\log\frac{\Gamma(a)}{\Gamma(a+b)}} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}(\psi(a)-\psi(a+b)),$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the well known polygamma function. Therefore

$$\int_0^1 x^{a-1}(1-x)^{b-1}\log x dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}(\psi(a)-\psi(a+b)), \tag{2.5}$$

for any $a > 0$ and $b > 0$.

It is well known that $\psi(n+1) = H_n - \gamma$ for all $n \in \mathbb{N}$, where $\gamma = 0.577\ldots$ is the Euler constant. In view of (2.5), for any $k \in \mathbb{N}$ we have

$$\begin{aligned}\int_0^1 x^k(1-x)^{2k}\log x dx &= \frac{\Gamma(k+1)\Gamma(2k+1)}{\Gamma(3k+2)}(\psi(k+1)-\psi(3k+2))\\ &= \frac{k!(2k)!}{(3k+1)!}(\psi(k+1)+\gamma-(\psi(3k+2)+\gamma))\end{aligned}$$

and hence

$$\int_0^1 x^k(1-x)^{2k}\log x dx = \frac{H_k - H_{3k+1}}{(3k+1)\binom{3k}{k}}. \tag{2.6}$$

Similarly,

$$\int_0^1 x^{2k}(1-x)^k\log x dx = \frac{H_{2k} - H_{3k+1}}{(3k+1)\binom{3k}{k}}. \tag{2.7}$$

Note that (2.7) minus (2.6) yields the identity

$$\frac{H_{2k}-H_k}{(3k+1)\binom{3k}{k}} = \int_0^1 x^{2k}(1-x)^k\log\frac{x}{1-x}dx, \tag{2.8}$$

which is similar to the identity

$$2n\int_0^1 x^{2n-1}(1-x)^n\log\frac{x}{1-x}dx = \frac{H_{2n-1}-H_n}{\binom{3n}{n}} \tag{2.9}$$

(with $n \in \mathbb{Z}^+$) observed by Au [3].

Proof of Theorem 1.2. In view of (2.8),

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{H_{2k}-H_k}{(3k+1)2^k\binom{3k}{k}}\\ &= \sum_{k=0}^{\infty}\frac{1}{2^k}\int_0^1 (x^2(1-x))^k\log\frac{x}{1-x}dx\\ &= \int_0^1\sum_{k=0}^{\infty}\left(\frac{x^2(1-x)}{2}\right)^k\log\frac{x}{1-x}dx = \int_0^1\frac{\log(x/(1-x))}{1-x^2(1-x)/2}dx\\ &= 2\int_0^1\frac{\log(x/(1-x))}{x^3-x^2+2}dx = -\frac{(\pi-2\log 2)(\pi-6\log 2)}{80},\end{aligned}$$

where the last equality can be yielded by `Mathematica`. Similarly, with the aid of (2.6), we have

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{H_k-H_{3k+1}}{(3k+1)2^k\binom{3k}{k}}\\&=\sum_{k=0}^{\infty}\frac{1}{2^k}\int_0^1(x(1-x)^2)^k\log x dx=\int_0^1\sum_{k=0}^{\infty}\left(\frac{t^2(1-t)}{2}\right)^k\log(1-t)dt\\&=\int_0^1\frac{\log(1-t)}{1-t^2(1-t)/2}dt=2\int_0^1\frac{\log(1-t)}{t^3-t^2+2}dt=\frac{24\log^2 2-5\pi^2-96G}{120},\end{aligned}$$

where the last equality can be yielded by `Mathematica`. So we have proved (1.16) and (1.17).

In light of (1.9), (1.10) and (1.16), to prove (1.15) we only need to note that

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{H_k-H_{3k+1}}{(3k+1)(2k-1)2^k\binom{3k}{k}}\\&=\int_0^1\sum_{k=1}^{\infty}\frac{(x(1-x)^2/2)^k}{2k-1}\log x dx\\&=\int_0^1(1-x)\sqrt{\frac{x}{2}}\operatorname{arctanh}\left((1-x)\sqrt{\frac{x}{2}}\right)\log x dx\\&=\frac{960G+\pi(25\pi+136)-24(5\log^2 2+4\log 2+63)}{1800}\end{aligned}$$

where the last equality can be yielded by `Mathematica`. □

3. Proof of Theorem 1.3

Lemma 3.1 ([19]). (i) *For any $x>1/4$, we have*

$$\sum_{k=0}^{\infty}\frac{2(4x+1)k-2x+1}{x^{2k}\binom{4k}{2k}}=\frac{8x^2}{(4x-1)^2}\left(\frac{3}{\sqrt{4x-1}}\operatorname{arccot}\sqrt{4x-1}-4x+4\right)\tag{3.1}$$

and

$$\sum_{k=0}^{\infty}\frac{2(4x-1)k-2x-1}{x^{2k}\binom{4k}{2k}}=\frac{8x^2}{(4x+1)^2}\left(\frac{3R(4x+1)}{4x+1}-4x-4\right).\tag{3.2}$$

(ii) *If $x>(1+\sqrt{2})/2$ or $x<(1-\sqrt{2})/2$, then*

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(2(2x-1)^2(2x-3)k-(4x^3-16x^2+7x+6))4^k}{(x(1-x))^k\binom{4k}{2k}}\\&=(1-x)\left(3R(x)+4x(x-3)\right).\end{aligned}\tag{3.3}$$

Lemma 3.2. *For any integer $n\geqslant 0$ and complex number $z\neq 0$, we have the following identities:*

$$\sum_{k=0}^{n}\frac{2(1-16z)k^2+(32z+1)k-6z-1}{(2k-1)z^k\binom{4k}{2k}}=6z+\frac{n+1}{z^n\binom{4n}{2n}},\tag{3.4}$$

$$\sum_{k=0}^{n}\frac{2(1-16z)k^2+3k+2z+1}{(4k+1)z^k\binom{4k}{2k}}=2z+\frac{(n+1)(2n+1)}{(4n+1)z^n\binom{4n}{2n}},\tag{3.5}$$

$$\sum_{k=0}^{n}\frac{2(1-16z)k+3k+18z+1}{(4k+3)z^k\binom{4k}{2k}}=6z+\frac{(n+1)(2n+1)}{(4n+3)z^n\binom{4n}{2n}}.\tag{3.6}$$

Proof. The three identities can be easily proved by induction on n. □

Proof of Theorem 1.3(i). We just prove (1.21), (1.22) and (1.25) as the other three identities in Theorem 1.3(i) can be proved similarly.

Since $x>1/4$, by (3.4) with $z=x^2$ we have

$$\sum_{k=0}^{n}\frac{2(1-16x^2)k^2+(32x^2+1)k-6x^2-1}{(2k-1)x^{2k}\binom{4k}{2k}}=6x^2+\frac{n+1}{x^{2n}\binom{4n}{2n}}$$

for any integer $n\geqslant 0$. Letting $n\to+\infty$, we then obtain

$$\sum_{k=0}^{\infty}\frac{2(1-16x^2)k^2+(32x^2+1)k-6x^2-1}{(2k-1)x^{2k}\binom{4k}{2k}}=6x^2.$$

It follows that

$$\begin{aligned}12x^2=&-(4x+1)\sum_{k=0}^{\infty}\frac{(2(4x-1)k-2x-1)}{x^{2k}\binom{4k}{2k}}\\&+\sum_{k=0}^{\infty}\frac{2(2x-1)(4x-1)k-4x^2+6x-1}{(2k-1)x^{2k}\binom{4k}{2k}}.\end{aligned}$$

Combining this with (3.2) and noting that

$$R(4x+1)=\frac{\sqrt{4x+1}}{2}\log\frac{\sqrt{4x+1}+1}{\sqrt{4x+1}-1},\tag{3.7}$$

we get (1.21).

By (3.5) with $z=x^2$, we have

$$\sum_{k=0}^{n}\frac{2(1-16x^2)k^2+3k+2x^2+1}{(4k+1)x^{2k}\binom{4k}{2k}}=2x^2+\frac{(n+1)(2n+1)}{(4n+1)x^{2n}\binom{4n}{2n}}$$

for any integer $n\geqslant 0$, and hence

$$\sum_{k=0}^{\infty}\frac{2(1-16x^2)k^2+3k+2x^2+1}{(4k+1)x^{2k}\binom{4k}{2k}}=2x^2.$$

It follows that

$$8x^2 = (1-4x)\sum_{k=0}^{\infty}\frac{(2(4x+1)k-2x+1)}{x^{2k}\binom{4k}{2k}} + 3\sum_{k=0}^{\infty}\frac{2(4x+1)k+2x+1}{(4k+1)x^{2k}\binom{4k}{2k}}.$$

Combining this with (3.1), we immediately get (1.22).

By (3.6) with $z = x^2$, we have

$$\sum_{k=0}^{n}\frac{2(1-16x^2)k^2+3k+18x^2+1}{(4k+3)x^{2k}\binom{4k}{2k}} = 6x^2 + \frac{(n+1)(2n+1)}{(4n+3)x^{2n}\binom{4n}{2n}}$$

for any integer $n \geqslant 0$, and hence

$$\sum_{k=0}^{\infty}\frac{2(1-16x^2)k^2+3k+18x^2+1}{(4k+3)x^{2k}\binom{4k}{2k}} = 6x^2.$$

It follows that

$$24x^2 = -(4x+1)\sum_{k=0}^{\infty}\frac{(2(4x-1)k-2x-1)}{x^{2k}\binom{4k}{2k}} + \sum_{k=0}^{\infty}\frac{2(4x-1)(8x-1)k+48x^2-18x+1}{(4k+3)x^{2k}\binom{4k}{2k}}.$$

Combining this with (3.2) and noting (3.7), we immediately get (1.25). □

Proof of Theorem 1.3(ii). Assume that $x > (1+\sqrt{2})/2$ or $x < (1-\sqrt{2})/2$. By (3.4) with $z = x(1-x)/4$, we have

$$\sum_{k=0}^{n}\frac{2(1-4x(1-x))k+(8x(1-x)+1)k-3x(1-x)/2-1}{(2k-1)(x(1-x)/4)^k\binom{4k}{2k}} = \frac{3}{2}x(1-x) + \frac{n+1}{(x(1-x)/4)^n\binom{4n}{2n}}$$

for any integer $n \geqslant 0$, and hence

$$\sum_{k=0}^{\infty}\frac{2(1-4x(1-x))k+(8x(1-x)+1)k-3x(1-x)/2-1}{(2k-1)(x(1-x)/4)^k\binom{4k}{2k}} = \frac{3}{2}x(1-x).$$

It follows that

$$3x(1-x)(2x-3) = \sum_{k=0}^{\infty}\frac{2(2x-1)^2(2x-3)k-(4x^3-16x^2+7x+6)}{(x(1-x)/4)^k\binom{4k}{2k}} - x\sum_{k=0}^{\infty}\frac{2(2x-1)^2k-2x^2-x+2}{(2k-1)(x(1-x)/4)^k\binom{4k}{2k}}.$$

Combining this with (3.3), we obtain

$$\begin{aligned}&x\sum_{k=1}^\infty\frac{2(2x-1)^2k-2x^2-x+2}{(2k-1)(x(1-x)/4)^k\binom{4k}{2k}}\\&=(1-x)3R(x)-x(1-x)(2x+3)-x(2x^2+x-2)\\&=3(1-x)R(x)-x,\end{aligned}$$

which is equivalent to (1.26).

Now we prove (1.28). By (3.6) with $z=x(1-x)/4$, we have

$$\begin{aligned}&\sum_{k=0}^n\frac{2(1-4x(1-x))k+3k+9x(1-x)/2+1}{(4k+3)(x(1-x)/4)^k\binom{4k}{2k}}\\&\qquad=\frac32x(1-x)+\frac{(n+1)(2n+1)}{(4n+3)(x(1-x)/4)^n\binom{4n}{2n}}\end{aligned}$$

for any integer $n\geqslant0$, and hence

$$\sum_{k=0}^\infty\frac{4(2x-1)^2k^2+6k+9x(1-x)+2}{(4k+3)(x(1-x)/4)^k\binom{4k}{2k}}=3x(1-x).$$

It follows that

$$\begin{aligned}6x(1-x)(2x-3)=&\sum_{k=0}^\infty\frac{2(2x-1)^2(2x-3)k-(4x^3-16x^2+7x+6)}{(x(1-x)/4)^k\binom{4k}{2k}}\\&-\sum_{k=0}^\infty\frac{2(2x-1)^2(4x-3)k+24x^3-42x^2+25x-6}{(4k+3)(x(1-x)/4)^k\binom{4k}{2k}}.\end{aligned}$$

Combining this with (3.3), we obtain

$$\begin{aligned}&\sum_{k=0}^\infty\frac{2(2x-1)^2(4x-3)k+24x^3-42x^2+25x-6}{(4k+3)(x(1-x)/4)^k\binom{4k}{2k}}\\&=(1-x)(3R(x)+4x(x-3))-6x(1-x)(2x-3)\\&=(1-x)(3R(x)-2x(4x-3)).\end{aligned}$$

This proves (1.28).

The identity (1.27) can be proved similarly by using (1.27) and (3.3). □

4. Conjectural series with summands involving two binomial coefficients

In this section, we propose some new conjectures on series whose summands involve two binomial coefficients and generalized harmonic numbers. Recall that the Riemann zeta function is given by

$$\zeta(s)=\sum_{n=1}^\infty\frac1{n^s}\quad\text{for }\Re(s)>1.$$

Conjecture 4.1 (2023-08-21). (i) *We have*

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^2\binom{2k}{k}\binom{3k}{k}}\left(\frac{7k-2}{2k-1}H_{k-1}-\frac{1}{6k}\right)=\frac{\pi^2\log 2}{6}-\frac{11}{12}\zeta(3), \tag{4.1}$$

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^2\binom{2k}{k}\binom{3k}{k}}\left(\frac{7k-2}{2k-1}H_{2k-1}+\frac{5}{12k}\right)=\frac{\pi^2\log 2}{12}-\frac{29}{24}\zeta(3), \tag{4.2}$$

$$\sum_{k=1}^\infty \frac{(-1)^k((7k-2)H_{k-1}^{(2)}-(6k-2)/(2k-1)^2)}{(2k-1)k^2\binom{2k}{k}\binom{3k}{k}}=\frac{\pi^4}{144}, \tag{4.3}$$

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^2\binom{2k}{k}\binom{3k}{k}}\left(\frac{7k-2}{2k-1}H_{k-1}^{(3)}-\frac{17}{32k^3}\right)=\frac{\pi^2}{2}\zeta(3)-\frac{45}{8}\zeta(5). \tag{4.4}$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{(p-1)/2}(-1)^k\binom{2k}{k}\binom{3k}{k}\left(\frac{7k+2}{2k+1}H_k+\frac{1}{6k}\right)\equiv -4q_p(2)-4p\,q_p(2)^2 \pmod{p^2},$$

$$\sum_{p/2<k<p}(-1)^k\binom{2k}{k}\binom{3k}{k}\left(\frac{7k+2}{2k+1}H_k+\frac{1}{6k}\right)\equiv 6p\,q_p(2)^2 \pmod{p^2},$$

$$\sum_{k=1}^{(p-1)/2}(-1)^k\binom{2k}{k}\binom{3k}{k}\left(\frac{7k+2}{2k+1}H_{2k}-\frac{5}{12k}\right)\equiv q_p(2)-2p\,q_p(2)^2 \pmod{p^2},$$

$$\sum_{p/2<k<p}(-1)^k\binom{2k}{k}\binom{3k}{k}\left(\frac{7k+2}{2k+1}H_{2k}-\frac{5}{12k}\right)\equiv -3q_p(2)+3p\,q_p(2)^2 \pmod{p^2}.$$

Also,

$$\sum_{k=1}^{(p-3)/2}\frac{(-1)^k\binom{2k}{k}\binom{3k}{k}}{2k+1}\left((7k+2)H_k^{(2)}+\frac{6k+2}{(2k+1)^2}\right)\equiv -2-\frac{p}{12}B_{p-3} \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1}(-1)^k\binom{2k}{k}\binom{3k}{k}\left(\frac{7k+2}{2k+1}H_k^{(3)}+\frac{17}{32k^3}\right)\equiv \frac{45H_{p-1}}{4p^2}+9p^2B_{p-5} \pmod{p^3}.$$

(iii) *For any odd prime $p\neq 5$, we have*

$$\sum_{k=1}^{p-1}(-1)^k(7k+2)\frac{\binom{2k}{k}\binom{3k}{k}}{2k+1}\equiv -21pH_{p-1} \pmod{p^5}.$$

Remark 4.1. Chu and Zhang [8, Example 24] observed that

$$\sum_{k=1}^\infty \frac{(-1)^k(7k-2)}{(2k-1)k^2\binom{2k}{k}\binom{3k}{k}}=-\frac{\pi^2}{12}.$$

Conjecture 4.2 (2023-09-11). *We have*

$$\sum_{k=1}^\infty\frac{(-1)^k((28k^2-41k+9)(2H_{2k-1}+5H_{k-1})-21k+3)}{(2k-1)k\binom{2k}{k}\binom{3k}{k}}=\frac{21}{2}\zeta(3)-\frac32\pi^2\log2+\frac{\pi^2}{4}\tag{4.5}$$

and

$$\sum_{k=1}^\infty\frac{(-1)^k(6(28k^2-41k+9)(H_{3k-1}-6H_{k-1})-142k+125)}{(2k-1)k\binom{2k}{k}\binom{3k}{k}}=\frac{15}{2}\pi^2\log2-\frac{147}{4}\zeta(3).\tag{4.6}$$

Remark 4.2. In view of the identity in Remark 4.1, we have

$$\sum_{k=1}^\infty\frac{(-1)^k(28k^2-41k+9)}{(2k-1)k\binom{2k}{k}\binom{3k}{k}}=\frac{\pi^2}{8}-\frac12,$$

because $2k(28k^2-41k+9)+3(7k-2)=(2k-1)(28k^2-27k+6)$, and

$$\sum_{k=1}^n\frac{(-1)^k(28k^2-27k+6)}{k^2\binom{2k}{k}\binom{3k}{k}}=-1+\frac{(-1)^n}{\binom{2n}{n}\binom{3n}{n}}$$

tends to -1 as $n\to+\infty$.

Conjecture 4.3 (2023-09-11). *We have*

$$\sum_{k=1}^\infty\frac{(-1)^k(12(21k^2-23k+4)(2H_{2k-1}+5H_{k-1})+35k-121)}{(2k-1)\binom{2k}{k}\binom{3k}{k}}=\frac{63}{2}\zeta(3)-\frac92\pi^2\log2+\frac74\pi^2-4\tag{4.7}$$

and

$$\sum_{k=1}^\infty\frac{(-1)^k(108(21k^2-23k+4)(H_{3k-1}-6H_{k-1})-3247k+2537)}{(2k-1)\binom{2k}{k}\binom{3k}{k}}=-\frac{1323}{8}\zeta(3)+\frac{135}{4}\pi^2\log2-\frac{95}{16}\pi^2+\frac{95}{4}.\tag{4.8}$$

Remark 4.3. In view of the first identity in Remark 4.2, we have

$$\sum_{k=1}^\infty\frac{(-1)^k(21k^2-23k+4)}{(2k-1)\binom{2k}{k}\binom{3k}{k}}=\frac{\pi^2}{32}-\frac12,$$

because $4k(21k^2-23k+4)-(28k^2-41k+9)=3(2k-1)^2(7k-3)$, and

$$\sum_{k=1}^n\frac{(-1)^k(2k-1)(7k-3)}{k\binom{2k}{k}\binom{3k}{k}}=-\frac12+\frac{(-1)^n(n+1)}{2\binom{2n}{n}\binom{3n}{n}}$$

tends to $-1/2$ as $n\to+\infty$.

Conjecture 4.4 (2023-08-21)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}\binom{3k}{k}}\left(\frac{56k^2-32k+5}{(2k-1)^2}(2H_{2k-1}^{(2)}-H_{k-1}^{(2)})-\frac{7}{4k^2}\right)=9\zeta(5). \quad (4.9)$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\binom{2k}{k}\binom{3k}{k}\left(\frac{56k^2+32k+5}{(2k+1)^2}(2H_{2k}^{(2)}-H_k^{(2)})+\frac{7}{4k^2}\right)$$
$$\equiv 66\frac{H_{p-1}}{p^2}-\frac{1608}{5}p^2B_{p-5} \pmod{p^3}.$$

Provided $p>5$, we have

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\binom{2k}{k}\binom{3k}{k}\left(\frac{56k^2+32k+5}{(2k+1)^2}(2H_{2k}^{(2)}-H_k^{(2)})+\frac{7}{4k^2}\right)$$
$$\equiv 18\frac{H_{p-1}}{p^2}-28p \pmod{p^2}.$$

(iii) *For any prime $p>3$, we have*

$$\sum_{k=1}^{(p-3)/2}(-1)^{k-1}(56k^2+32k+5)\frac{\binom{2k}{k}\binom{3k}{k}}{k(2k+1)^2}\equiv 12+20p \pmod{p^2}.$$

Remark 4.4. Chu and Zhang [8, Example 21] proved that

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(56k^2-32k+5)}{(2k-1)^2k^3\binom{2k}{k}\binom{3k}{k}}=4\zeta(3).$$

Conjecture 4.5 (2023-08-21)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}\binom{3k}{k}}\left(\frac{56k^2-32k+5}{(2k-1)^2}H_{2k-1}^{(3)}+\frac{1}{k^3}\right)=\frac{\pi^6}{1512}+3\zeta(3)^2 \quad (4.10)$$

and

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}\binom{3k}{k}}\left(\frac{56k^2-32k+5}{(2k-1)^2}\left(4H_{2k-1}^{(4)}-H_{k-1}^{(4)}\right)-\frac{15}{4k^4}\right)$$
$$=205\zeta(7)-\frac{62}{3}\pi^2\zeta(5)+\frac{\pi^4}{5}\zeta(3). \quad (4.11)$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{(p-3)/2}\frac{(-1)^k}{k}\binom{2k}{k}\binom{3k}{k}\left(\frac{56k^2+32k+5}{(2k+1)^2}H_{2k}^{(3)}-\frac{1}{k^3}\right)\equiv 0 \pmod{p}$$

and

$$p\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\binom{2k}{k}\binom{3k}{k}\left(\frac{56k^2+32k+5}{(2k+1)^2}H_{2k}^{(3)}-\frac{1}{k^3}\right)$$
$$\equiv 6\frac{H_{p-1}}{p^2}-\frac{858}{5}p^2B_{p-5} \pmod{p^3}.$$

Also,

$$\sum_{k=1}^{(p-3)/2}\frac{(-1)^k}{k}\binom{2k}{k}\binom{3k}{k}\left(\frac{56k^2+32k+5}{(2k+1)^2}(4H_{2k}^{(4)}-H_k^{(4)})+\frac{15}{4k^4}\right)$$
$$\equiv 36B_{p-5} \pmod{p}$$

and

$$p^2\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\binom{2k}{k}\binom{3k}{k}\left(\frac{56k^2+32k+5}{(2k+1)^2}(4H_{2k}^{(4)}-H_k^{(4)})+\frac{15}{4k^4}\right)$$
$$\equiv 24\frac{H_{p-1}}{p^2}-\frac{3996}{5}p^2B_{p-5} \pmod{p^3}.$$

Remark 4.5. This conjecture looks quite challenging.

Conjecture 4.6 (2023-10-01)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(28k^2-18k+3)}{(2k-1)^3k^4\binom{2k}{k}\binom{3k}{k}}=\frac{\pi^4}{45}. \tag{4.12}$$

Also,

$$\sum_{k=1}^{\infty}\frac{(-1)^k(5(28k^2-18k+3)H_{2k-1}+F_1(k))}{(2k-1)^3k^4\binom{2k}{k}\binom{3k}{k}}=65\zeta(5)-7\pi^2\zeta(3), \tag{4.13}$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}((28k^2-18k+3)(H_{3k-1}-H_{k-1})+2F_2(k))}{(2k-1)^3k^4\binom{2k}{k}\binom{3k}{k}}=\frac{20}{3}\zeta(5), \tag{4.14}$$

and

$$\sum_{k=1}^{\infty}\frac{(-1)^k((28k^2-18k+3)H_{k-1}^{(2)}-F_3(k))}{(2k-1)^3k^4\binom{2k}{k}\binom{3k}{k}}=\frac{\pi^6}{378}, \tag{4.15}$$

where

$$F_1(k):=\frac{80k^3-72k^2+24k-3}{k(2k-1)},\ F_2(k):=\frac{28k^3-24k^2+8k-1}{k(2k-1)}$$

and

$$F_3(k):=\frac{208k^4-240k^3+120k^2-30k+3}{4k^2(2k-1)^2}.$$

(ii) *For any prime $p>3$, we have*

$$p\sum_{k=1}^{(p-1)/2}\frac{(-1)^{k-1}(28k^2-18k+3)}{(2k-1)^3k^4\binom{2k}{k}\binom{3k}{k}}\equiv 12\frac{H_{p-1}}{p^2}+\frac{27}{5}p^2B_{p-5} \pmod{p^3}$$

and

$$p^2\sum_{k=1}^{p-1}(28k^2+18k+3)\frac{(-1)^k\binom{2k}{k}\binom{3k}{k}}{k^2(2k+1)^3}\equiv -8+8p^2-24p^5B_{p-5}\pmod{p^6}.$$

Remark 4.6. The series in (4.12) has converging rate $-1/27$.

Conjecture 4.7 (2023-09-11)**.** *We have*

$$\sum_{k=1}^\infty\frac{(-16)^k((40k^2-52k+7)(3H_{2k-1}+2H_{k-1})-20k-11)}{(2k-1)k\binom{2k}{k}\binom{4k}{2k}} = \frac92\pi^2-9\pi^2\log 2+84\zeta(3) \tag{4.16}$$

and

$$\sum_{k=1}^\infty\frac{(-16)^k((40k^2-52k+7)(9H_{4k-1}-3H_{2k-1}-17H_{k-1})-298k+233)}{(2k-1)k\binom{2k}{k}\binom{4k}{2k}} = \frac{99}{2}\pi^2\log 2-147\zeta(3). \tag{4.17}$$

Remark 4.7. Theorem 9 of [8] with $a=e=1$ and $b=c=d=1/2$ yields the identity

$$\sum_{k=1}^\infty\frac{(-16)^k(5k-1)}{(2k-1)k^2\binom{2k}{k}\binom{4k}{2k}}=-\frac{\pi^2}{2}.$$

In view of this, we have

$$\sum_{k=1}^\infty\frac{(40k^2-52k+7)(-16)^k}{(2k-1)k\binom{2k}{k}\binom{4k}{2k}}=\frac32\pi^2-4,$$

because $k(40k^2-52k+7)+3(5k-1)=(2k-1)^2(10k-3)$, and

$$\sum_{k=1}^n\frac{(-16)^k(2k-1)(10k-3)}{k^2\binom{2k}{k}\binom{4k}{2k}}=-4+\frac{(-1)^n4^{2n+1}}{\binom{2n}{n}\binom{4n}{2n}}$$

tends to -4 as $n\to+\infty$.

Conjecture 4.8 (2023-09-11)**.** *We have*

$$\sum_{k=1}^\infty\frac{(-16)^k(7(140k^2-94k-15)(3H_{2k-1}+2H_{k-1})+15(226k-249))}{(2k-1)\binom{2k}{k}\binom{4k}{2k}} = 882\zeta(3)-\frac{189}{2}\pi^2\log 2+\frac{771}{4}\pi^2-388 \tag{4.18}$$

and

$$\sum_{k=1}^{\infty}\frac{(-16)^k(7(140k^2-94k-15)(2H_{4k-1}+5H_{2k-1})+48(19k-45))}{(2k-1)\binom{2k}{k}\binom{4k}{2k}}$$
$$=1323\zeta(3)-63\pi^2\log 2+219\pi^2-346. \tag{4.19}$$

Remark 4.8. In view of the first identity in Remark 4.7, we have

$$\sum_{k=1}^{\infty}\frac{(140k^2-94k-15)(-16)^k}{(2k-1)\binom{2k}{k}\binom{4k}{2k}}=\frac{9}{4}\pi^2-20,$$

because

$$2k(140k^2-94k-15)-3(40k^2-52k+7)=7(2k-1)(20k^2-12k+3),$$

and

$$\sum_{k=1}^{n}\frac{(-16)^k(20k^2-12k+3)}{k\binom{2k}{k}\binom{4k}{2k}}=-4+\frac{(-1)^n(n+1)4^{2n+1}}{\binom{2n}{n}\binom{4n}{2n}}$$

tends to -4 as $n\to+\infty$.

Conjecture 4.9 (2023-08-22)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-16)^k((20k^2-8k+1)H_{k-1}^{(2)}-16k(4k-1)/(2k-1)^2)}{(2k-1)^2k^3\binom{2k}{k}\binom{4k}{2k}}=62\zeta(5), \tag{4.20}$$

$$\sum_{k=1}^{\infty}\frac{(-16)^k((20k^2-8k+1)H_{2k-1}^{(2)}-3k(4k-1)/(2k-1)^2)}{(2k-1)^2k^3\binom{2k}{k}\binom{4k}{2k}}$$
$$=\frac{31}{2}\zeta(5)-\frac{7}{4}\pi^2\zeta(3), \tag{4.21}$$

and

$$\sum_{k=1}^{\infty}\frac{(-16)^k(20k^2-8k+1)}{(2k-1)^2k^3\binom{2k}{k}\binom{4k}{2k}}\left(16H_{2k-1}^{(2)}-3H_{k-1}^{(2)}\right)=62\zeta(5)-28\pi^2\zeta(3). \tag{4.22}$$

(ii) *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}(20k^2+8k+1)}{(2k+1)^2k(-16)^k}\left(16H_{2k}^{(2)}-3H_k^{(2)}\right)\equiv-52B_{p-3}\pmod p.$$

When $p>3$, we have

$$\sum_{k=1}^{(p-3)/2}\frac{\binom{2k}{k}\binom{4k}{2k}(20k^2+8k+1)}{(2k+1)^2k(-16)^k}\left(16H_{2k}^{(2)}-3H_k^{(2)}\right)\equiv-24B_{p-3}\pmod p.$$

Remark 4.9. Chu and Zhang [8, Example 1] proved that

$$\sum_{k=1}^{\infty}\frac{(-16)^k(20k^2-8k+1)}{(2k-1)^2k^3\binom{2k}{k}\binom{4k}{2k}}=-14\zeta(3).$$

Conjecture 4.10 (2023-09-12). *We have*

$$\sum_{k=1}^{\infty}\frac{16^k((6k^2-11k+1)(4H_{2k-1}-H_{k-1})-18k-1)}{(2k-1)k\binom{2k}{k}\binom{4k}{2k}}=-24G-\frac{3}{4}\pi^3 \quad (4.23)$$

and

$$\sum_{k=1}^{\infty}\frac{16^k((24k^2-34k-7)(4H_{2k-1}-H_{k-1})-6k-115)}{(2k-1)\binom{2k}{k}\binom{4k}{2k}}=11-138G-\frac{9}{4}\pi^3, \quad (4.24)$$

where G denotes the Catalan constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$.

Remark 4.10. By Chu and Zhang [8, Exmaple 84],

$$\sum_{k=1}^{\infty}\frac{(6k-1)16^k}{(2k-1)k^2\binom{2k}{k}\binom{4k}{2k}}=8G.$$

This implies the identity

$$\sum_{k=1}^{\infty}\frac{(6k^2-11k+1)16^k}{(2k-1)k\binom{2k}{k}\binom{4k}{2k}}=1-6G,$$

because $4k(6k^2-11k+1)+3(6k-1)=(2k-1)(12k^2-16k+3)$, and

$$\sum_{k=1}^{n}\frac{16^k(12k^2-16k+3)}{k^2\binom{2k}{k}\binom{4k}{2k}}=4-\frac{4^{2n+1}}{\binom{2n}{n}\binom{4n}{2n}}$$

tends to 4 as $n\to+\infty$. Similarly, we have

$$\sum_{k=1}^{\infty}\frac{(24k^2-34k-7)16^k}{(2k-1)\binom{2k}{k}\binom{4k}{2k}}=7-18G,$$

because $k(24k^2-34k-7)-3(6k^2-11k+1)=(2k-1)(2k-3)(6k-1)$, and

$$\sum_{k=1}^{n}\frac{16^k(2k-3)(6k-1))}{k\binom{2k}{k}\binom{4k}{2k}}=4-\frac{4^{2n+1}(n+1)}{\binom{2n}{n}\binom{4n}{2n}}$$

tends to 4 as $n\to+\infty$.

Conjecture 4.11 (2023-10-03). *We have*

$$\sum_{k=1}^{\infty}\frac{(-16)^k((112k^2-32k+3)a_k-1536k-120(2k+1)/(2k-1))}{k^2(4k-1)(4k-3)\binom{3k}{k}\binom{6k}{3k}}=\pi^3, \quad (4.25)$$

where

$$a_k:=6H_{6k-1}-3H_{3k-1}+18H_{2k-1}-13H_{k-1}.$$

Remark 4.11. By [8, Example 29], we have the identity

$$\sum_{k=1}^{\infty}\frac{(-16)^k(112k^2-32k+3)}{k^2(4k-1)(4k-3)\binom{3k}{k}\binom{6k}{3k}}=-8G.$$

Conjecture 4.12 (2023-08-21). (i) *We have*

$$\sum_{k=1}^\infty\frac{256^{k-1}}{(2k-1)k^2\binom{3k}{k}\binom{6k}{3k}}\left(3(22k-1)H_{k-1}^{(2)}-\frac{328(6k-1)}{(2k-1)^2}\right)=-28\beta(4),\tag{4.26}$$

$$\sum_{k=1}^\infty\frac{256^{k-1}}{(2k-1)k^2\binom{3k}{k}\binom{6k}{3k}}\left((22k-1)H_{2k-1}^{(2)}-\frac{29(6k-1)}{(2k-1)^2}\right)=-2\beta(4)\tag{4.27}$$

and

$$\sum_{k=1}^\infty\frac{256^{k-1}(22k-1)}{(2k-1)k^2\binom{3k}{k}\binom{6k}{3k}}\left(328H_{2k-1}^{(2)}-87H_{k-1}^{(2)}\right)=156\beta(4),\tag{4.28}$$

where $\beta(4)=\sum_{k=0}^\infty(-1)^k/(2k+1)^4$.

(ii) *Let* $p>3$ *be a prime. Then*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{3k}{k}\binom{6k}{3k}}{(2k+1)256^k}\left(3(22k+1)H_k^{(2)}+\frac{328(6k+1)}{(2k+1)^2}\right)\equiv192E_{p-3}\pmod{p},$$

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{3k}{k}\binom{6k}{3k}}{(2k+1)256^k}\left((22k+1)H_{2k}^{(2)}+\frac{29(6k+1)}{(2k+1)^2}\right)\equiv16E_{p-3}\pmod{p},$$

and

$$\sum_{k=0}^{p-1}\frac{(22k+1)\binom{3k}{k}\binom{6k}{3k}}{(2k+1)256^k}\left(328H_{2k}^{(2)}-87H_k^{(2)}\right)\equiv4600E_{p-3}\pmod{p}.$$

Remark 4.12. Chu and Zhang [8, Example 50] proved that

$$\sum_{k=1}^\infty\frac{256^k(22k-1)}{(2k-1)k^2\binom{3k}{k}\binom{6k}{3k}}=128G.$$

Conjecture 4.13 (2023-09-12). *We have*

$$\sum_{k=1}^\infty\frac{256^k((22k^2-65k-48)\mathcal{H}(k)+258k+3970)}{(2k-1)k\binom{3k}{k}\binom{6k}{3k}}=20(3\pi^3+1216G)\tag{4.29}$$

and

$$\sum_{k=1}^\infty\frac{256^k(24(1408k^2-6318k+3557)\mathcal{H}(k)+145906k+533701)}{(2k-1)\binom{3k}{k}\binom{6k}{3k}}=16(159030G-450\pi^3-11381),\tag{4.30}$$

where

$$\mathcal{H}(k):=10H_{6k-1}-5H_{3k-1}-16H_{2k-1}+5H_{k-1}.$$

Remark 4.13. Based on the identity in Remark 4.12, we have

$$\sum_{k=1}^\infty\frac{256^k(22k^2-65k-48)}{(2k-1)k\binom{3k}{k}\binom{6k}{3k}}=16(1-30G)$$

and

$$\sum_{k=1}^{\infty}\frac{256^k(1408k^2-6318k+3557)}{(2k-1)\binom{3k}{k}\binom{6k}{3k}}=16(59+150G)$$

because of the following identities:

$$4k(22k^2-65k-48)+15(22k-1)=(2k-1)(44k^2-108k+15),$$

$$\sum_{k=1}^{n}\frac{256^k(44k^2-108k+15)}{k^2\binom{3k}{k}\binom{6k}{3k}}=64-\frac{4^{4n+3}}{\binom{3n}{n}\binom{6n}{3n}},$$

$$k(1408k^2-6318k+3557)+5(22k^2-65k-48)$$
$$=16(2k-1)(44k^2-172k+15),$$

$$\sum_{k=1}^{n}\frac{256^k(44k^2-172k+15)}{k\binom{3k}{k}\binom{6k}{3k}}=64-\frac{4^{4n+3}(n+1)}{\binom{3n}{n}\binom{6n}{3n}}.$$

5. Conjectural series with summands involving three or more binomial coefficients

Conjecture 5.1 (2023-08-17)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(21k-8)H_{k-1}^{(2)}+2/k}{k^3\binom{2k}{k}^3}=\frac{\pi^4}{360}. \tag{5.1}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=1}^{p-1}\binom{2k}{k}^3\left((21k+8)H_k^{(2)}-\frac{2}{k}\right)\equiv\frac{16}{5}p^4B_{p-5}\pmod{p^5}.$$

Remark 5.1. In 1993, Zeilberger [25] established the identity

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{6}.$$

The author [15] proved that

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3\equiv 8p+16p^4B_{p-3}\pmod{p^5}$$

for any prime p.

Conjecture 5.2 (2023-08-17)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(21k-8)H_{k-1}^{(3)}+1/k^2}{k^3\binom{2k}{k}^3}=\frac{62}{7}\zeta(5)-\frac{16}{21}\pi^2\zeta(3). \tag{5.2}$$

(ii) *For any prime $p>7$, we have*

$$\sum_{k=1}^{p-1}\binom{2k}{k}^3\left((21k+8)H_k^{(3)}-\frac{1}{k^2}\right)\equiv\frac{256}{7}\cdot\frac{H_{p-1}}{p}+\frac{752}{35}p^3B_{p-5}\pmod{p^4}.$$

Remark 5.2. The author [21, Conjecture 3.2] conjectured that

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(3)}+\frac{43}{8}H_{k-1}^{(3)}\right)=\frac{711}{28}\zeta(5)-\frac{29}{14}\pi^2\zeta(3),$$

which remains open.

Conjecture 5.3 (2023-08-17)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{9(21k-8)H_{k-1}^{(4)}+25/k^3}{k^3\binom{2k}{k}^3}=\frac{13\pi^6}{3780}. \tag{5.3}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=1}^{(p-1)/2}\binom{2k}{k}^3\left(9(21k+8)H_k^{(4)}-\frac{25}{k^3}\right)\equiv 96\frac{H_{p-1}}{p^2}+\frac{96}{5}p^2B_{p-5}\pmod{p^3}.$$

Remark 5.3. This conjecture looks quite challenging.

Conjecture 5.4 (2023-08-17)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{16^k}{k^3\binom{2k}{k}^3}\left((6k-2)H_{k-1}^{(2)}+\frac{1}{k}\right)=\frac{\pi^4}{24}. \tag{5.4}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\left((6k+2)H_k^{(2)}-\frac{1}{k}\right)$$
$$\equiv -4q_p(2)+2p\,q_p(2)^2-\frac{4}{3}p^2q_p(2)^3+p^3q_p(2)^4\pmod{p^4}.$$

Remark 5.4. In 2008, Guillera [11, Identity 1] used the WZ method to obtain that

$$\sum_{k=1}^{\infty}\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{2}.$$

Two q-analogues of this identity were given by Hou, Krattenthaler and Sun [13].

Conjecture 5.5 (2023-10-07)**.** *We have*

$$\sum_{k=1}^{\infty}\frac{(42k^2-23k+3)16^k}{k^3(2k-1)\binom{2k}{k}\binom{4k}{2k}^2}=\frac{\pi^2}{2}. \tag{5.5}$$

Also,

$$\sum_{k=1}^{\infty}\frac{16^k\left(P(k)(H_{2k-1}-H_{k-1})-\frac{196k^2-100k+13}{6(2k-1)}\right)}{k^3(2k-1)\binom{2k}{k}\binom{4k}{2k}^2}=\frac{2\pi^2\log 2-7\zeta(3)}{6},\quad (5.6)$$

$$\sum_{k=1}^{\infty}\frac{16^k\left(P(k)(H_{4k-1}-H_{2k-1})-\frac{28k^2-76k+19}{12(2k-1)}\right)}{k^3(2k-1)\binom{2k}{k}\binom{4k}{2k}^2}=\frac{2\pi^2\log 2+35\zeta(3)}{12},\quad (5.7)$$

$$\sum_{k=1}^{\infty}\frac{16^k\left(P(k)(4H^{(2)}_{4k-1}-H^{(2)}_{2k-1}-2H^{(2)}_{k-1})-\frac{6k+1}{k}\right)}{k^3(2k-1)\binom{2k}{k}\binom{4k}{2k}^2}=\frac{5\pi^4}{24},\quad (5.8)$$

$$\sum_{k=1}^{\infty}\frac{16^k\left(P(k)(4H^{(2)}_{4k-1}-5H^{(2)}_{2k-1}-3H^{(2)}_{k-1})+\frac{32k(3k-1)}{(2k-1)^2}\right)}{k^3(2k-1)\binom{2k}{k}\binom{4k}{2k}^2}=\frac{\pi^4}{6}\quad (5.9)$$

where $Q(k)=(3k-1)(14k-3)=42k^2-23k+3$.

Remark 5.5. The identity (5.5) and related p-adic congruences were announced in [18, Remark 4.14 and Conjecture 4.14].

Conjecture 5.6 (2023-10-13)**.** *We have*

$$\sum_{k=1}^{\infty}\frac{(92k^2-61k+9)64^k}{k^3(2k-1)\binom{2k}{k}\binom{3k}{k}\binom{4k}{2k}}=8\pi^2.\quad (5.10)$$

Also,

$$\sum_{k=1}^{\infty}\frac{64^k\left(Q(k)(3H_{3k-1}-5H_{k-1})-60k+21\right)}{k^3(2k-1)\binom{2k}{k}\binom{3k}{k}\binom{4k}{2k}}=48\pi^2\log 2-56\zeta(3),\quad (5.11)$$

$$\sum_{k=1}^{\infty}\frac{64^k\left(Q(k)(3H_{2k-1}-2H_{k-1})-\frac{124k^2-79k+13}{2k-1}\right)}{k^3(2k-1)\binom{2k}{k}\binom{3k}{k}\binom{4k}{2k}}=112\zeta(3),\quad (5.12)$$

$$\sum_{k=1}^{\infty}\frac{64^k\left(Q(k)(6H_{4k-1}-H_{k-1})-\frac{344k^2-281k+59}{2k-1}\right)}{k^3(2k-1)\binom{2k}{k}\binom{3k}{k}\binom{4k}{2k}}=560\zeta(3),\quad (5.13)$$

where $Q(k)=92k^2-61k+9$.

Remark 5.6. The identity (5.10) and related p-adic congruences were announced in [18, Remark 4.15 and Conjecture 4.15].

Conjecture 5.7 (2023-11-12)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(5535k^3-4689k^2+1110k-80)(-27)^{k-1}}{k^3(3k-1)(3k-2)\binom{3k}{k}\binom{6k}{3k}^2}=\sum_{k=1}^{\infty}\frac{\left(\frac{k}{3}\right)}{k^2}.\quad (5.14)$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{P(k)\binom{3k}{k}\binom{6k}{3k}^2}{(3k+1)(3k+2)(-27)^k}\equiv 4p\left(\frac{p}{3}\right)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{P(k)\binom{3k}{k}\binom{6k}{3k}^2}{(3k+1)(3k+2)(-27)^k}\equiv 40p\left(\frac{p}{3}\right)+\frac{400}{3}p^3B_{p-2}\left(\frac{1}{3}\right)\pmod{p^4},$$

where $P(k)=5535k^3+4689k^2+1110k+80$.

Remark 5.7. The series in (5.14) has converging rate $-1/1024$. In 2023, the author announced the identity (5.14) in an arXiv preprint. In 2024, J. Zuniga informed the author that Guillera had proved the identity via the WZ method.

Conjecture 5.8 (2023-08-17)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}\left((28k^2-18k+3)(4H_{2k-1}^{(3)}+3H_{k-1}^{(3)})+\frac{4}{k}\right)=-49\zeta(3)^2. \tag{5.15}$$

(ii) *For any odd prime* p*, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^4\binom{3k}{k}}{(-64)^k}\left((28k^2+18k+3)(4H_{2k}^{(3)}+3H_{k}^{(3)})-\frac{4}{k}\right)$$
$$\equiv -24q_p(2)+12p\,q_p(2)^2\pmod{p^2}.$$

Remark 5.8. This is motivated by the author's conjectural identity

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3)$$

published in [16].

Conjecture 5.9 (2023-08-17)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^5\binom{2k}{k}^5}\left(8(205k^2-160k+32)(H_{2k-1}^{(3)}+H_{k-1}^{(3)})+\frac{125}{k}\right)=16\zeta(3)^2. \tag{5.16}$$

(ii) *For any prime* $p>3$*, we have*

$$\sum_{k=1}^{p-1}(-1)^k\binom{2k}{k}^5\left(8(205k^2+160k+32)(H_{2k}^{(3)}+H_{k}^{(3)})-\frac{125}{k}\right)$$
$$\equiv -3584p^4B_{p-5}\pmod{p^5}.$$

Remark 5.9. In 1997, Amdeberhan and Zeilbeger [1] obtained the identity

$$\sum_{k=1}^{\infty}\frac{(-1)^k(205k^2-160k+32)}{k^5\binom{2k}{k}^5}=-2\zeta(3).$$

Conjecture 5.10 (2023-08-17). (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^5\binom{2k}{k}^5}\left(16(205k^2-160k+32)(H_{2k-1}^{(4)}+3H_{k-1}^{(4)})-\frac{195}{k^2}\right)=32\zeta(7). \tag{5.17}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=1}^{p-1}(-1)^k\binom{2k}{k}^5\left(16(205k^2+160k+32)(H_{2k}^{(4)}+3H_k^{(4)})+\frac{195}{k^2}\right)$$
$$\equiv 1024\left(\frac{H_{p-1}}{p}-\frac{34}{5}p^3B_{p-5}\right)\pmod{p^4}.$$

Remark 5.10. Note that (5.17) gives a surprising series for $\zeta(7)$.

Conjecture 5.11 (2023-08-17). (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-256)^k}{k^5\binom{2k}{k}^5}\left((10k^2-6k+1)(8H_{2k-1}^{(3)}-H_{k-1}^{(3)})+\frac{2}{k}\right)=-196\zeta(3)^2. \tag{5.18}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^5}{(-256)^k}\left((10k^2+6k+1)(8H_{2k}^{(3)}-H_k^{(3)})-\frac{2}{k}\right)$$
$$\equiv -16q_p(2)+8p\,q_p(2)^2\pmod{p^2}.$$

Remark 5.11. It is known that

$$\sum_{k=1}^{\infty}\frac{(10k^2-6k+1)(-256)^k}{k^5\binom{2k}{k}^5}=-28\zeta(3)$$

(cf. [11, Identity 8]).

Conjecture 5.12 (2023-08-17). (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-256)^k}{k^5\binom{2k}{k}^5}(10k^2-6k+1)\left(16H_{2k-1}^{(4)}+3H_{k-1}^{(4)}\right)=-508\zeta(7). \tag{5.19}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^5}{(-256)^k}(10k^2+6k+1)\left(16H_{2k}^{(4)}+3H_k^{(4)}\right)\equiv-\frac{56}{3}pB_{p-3}\pmod{p^2}.$$

Remark 5.12. The identity (5.19) looks interesting and challenging.

Conjecture 5.13 (2023-08-17). (i) *We have*

$$\sum_{k=1}^{\infty}\frac{256^k}{k^7\binom{2k}{k}^7}\left((21k^3-22k^2+8k-1)\left(16H_{2k-1}^{(4)}+7H_{k-1}^{(4)}\right)+\frac{4}{k}\right)=\frac{31\pi^8}{1440}. \tag{5.20}$$

(ii) *For any prime $p>5$, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^7}{256^k}\left((21k^3+22k^2+8k+1)\left(16H_{2k}^{(4)}+7H_k^{(4)}\right)-\frac{4}{k}\right)$$
$$\equiv -32q_p(2)+16p\,q_p(2)^2-\frac{32}{3}p^2q_p(2)^3+8p^3q_p(2)^4 \pmod{p^4}.$$

Remark 5.13. This is inspired by the identity

$$\sum_{k=1}^{\infty}\frac{(21k^3-22k^2+8k-1)256^k}{k^7\binom{2k}{k}^7}=\frac{\pi^4}{8}$$

conjectured by Guillera [10].

Conjecture 5.14 (2023-10-13)**.** *We have*

$$\sum_{k=0}^{\infty}\frac{(66k^2+37k+4)\binom{2k}{k}\binom{3k}{k}\binom{4k}{2k}}{(2k+1)729^k}=\frac{27\sqrt{3}}{2\pi}. \tag{5.21}$$

Also,

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}\binom{3k}{k}\binom{4k}{2k}}{(2k+1)729^k}\left(P(k)(2H_{4k}-H_{2k}-3H_k)+60k+26\right)=\frac{81\sqrt{3}\,\log 3}{2\pi} \tag{5.22}$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}\binom{3k}{k}\binom{4k}{2k}}{(2k+1)729^k}\left(P(k)(3H_{3k}-2H_{2k}+3H_k)-\frac{108k^2+92k+23}{2k+1}\right)=0, \tag{5.23}$$

where $P(k)=66k^2+37k+4$.

Remark 5.14. Note that the series in (5.21) for $\sqrt{3}/\pi$ is not of the Ramanujan type.

Conjecture 5.15 (2023-10-13)**.** *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}\binom{4k}{2k}^2}{(2k+1)4096^k}\left((48k^2+32k+3)\left(H_{4k}^{(2)}-\frac{H_{2k}^{(2)}}{4}-\frac{H_k^{(2)}}{16}\right)-1\right)=-\frac{\pi}{6\sqrt{2}}. \tag{5.24}$$

Remark 5.15. This is motivated by the known identity

$$\sum_{k=0}^{\infty}\frac{(48k^2+32k+3)\binom{2k}{k}\binom{4k}{2k}^2}{(2k+1)4096^k}=\frac{8\sqrt{2}}{\pi}$$

(cf. [8, Example 82]).

Conjecture 5.16 (2023-06-19)**.** *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^5}{(-4096)^k}\left((20k^2+8k+1)H_k^{(3)}+\frac{8}{2k+1}\right)=\frac{64\zeta(3)}{\pi^2}. \tag{5.25}$$

Remark 5.16. This was motivated by the known identity

$$\sum_{k=0}^\infty(20k^2+8k+1)\frac{\binom{2k}{k}^5}{(-4096)^k}=\frac{8}{\pi^2}$$

(cf. [11, Identity 8]). The author [21, (145)] conjectured that

$$\sum_{k=0}^\infty\frac{\binom{2k}{k}^5}{(-4096)^k}\left((20k^2+8k+1)\left(8H_{2k}^{(2)}-3H_k^{(2)}\right)+4\right)=\frac{8}{3},$$

which was later confirmed by Wei [23].

Conjecture 5.17 (2023-08-19)**.** (i) *We have*

$$\sum_{k=0}^\infty\frac{\binom{2k}{k}^5}{(-4096)^k}(20k^2+8k+1)\left(64H_{2k}^{(4)}-3H_k^{(4)}\right)=-\frac{56}{45}\pi^2. \tag{5.26}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^5}{(-4096)^k}(20k^2+8k+1)\left(64H_{2k}^{(4)}-3H_k^{(4)}\right)\equiv-\frac{224}{3}pB_{p-3}\pmod{p^2}.$$

Remark 5.17. We haven't found similar results involving harmonic numbers of the fifth order.

Conjecture 5.18 (2023-06-19)**.** (i) *We have*

$$\sum_{k=0}^\infty\frac{\binom{2k}{k}^5}{(-2^{20})^k}\left((820k^2+180k+13)\left(9H_{2k}^{(3)}-H_k^{(3)}\right)+\frac{125}{2k+1}\right)=1024\frac{\zeta(3)}{\pi^2}. \tag{5.27}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^5}{(-2^{20})^k}\left((820k^2+180k+13)\left(9H_{2k}^{(3)}-H_k^{(3)}\right)+\frac{125}{2k+1}\right)\equiv-256B_{p-3}\pmod{p}.$$

Remark 5.18. This was motivated by the known identity

$$\sum_{k=0}^\infty(820k^2+180k+13)\frac{\binom{2k}{k}^5}{(-2^{20})^k}=\frac{128}{\pi^2}$$

(cf. [11, Identity 9]). The author [21, (150)] conjectured that

$$\sum_{k=0}^\infty\frac{\binom{2k}{k}^5}{(-2^{20})^k}\left((820k^2+180k+13)\left(11H_{2k}^{(2)}-3H_k^{(2)}\right)+43\right)=\frac{128}{3},$$

which was later confirmed by Wei [23].

Conjecture 5.19 (2023-08-19). (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^5}{(-2^{20})^k}\left((820k^2+180k+13)\left(49H_{2k}^{(4)}-3H_k^{(4)}\right)-\frac{195}{(2k+1)^2}\right)=-\frac{896}{45}\pi^2 \tag{5.28}$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^5}{(-2^{20})^k}\left((820k^2+180k+13)\left(351H_{2k}^{(5)}-11H_k^{(5)}\right)+\frac{275}{(2k+1)^3}\right)=512\left(85\frac{\zeta(5)}{\pi^2}-7\zeta(3)\right). \tag{5.29}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^5}{(-2^{20})^k}\left((820k^2+180k+13)\left(49H_{2k}^{(4)}-3H_k^{(4)}\right)-\frac{195}{(2k+1)^2}\right)\equiv 0 \pmod{p}.$$

Remark 5.19. It seems that (5.29) is the first nontrivial series for $\zeta(5)/\pi^2$.

Conjecture 5.20 (2023-11-15). (i) *We have*

$$\sum_{k=0}^{\infty}\frac{(92k^3+54k^2+12k+1)\binom{2k}{k}^7}{(6k+1)256^k\binom{3k}{k}\binom{6k}{3k}}=\frac{12}{\pi^2}, \tag{5.30}$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^7\left((92k^3+54k^2+12k+1)(2H_{6k}-H_{3k}+15H_k)-f(k)\right)}{(6k+1)256^k\binom{3k}{k}\binom{6k}{3k}}=-288\frac{\log 2}{\pi^2} \tag{5.31}$$

with $f(k)=2(5064k^3+2828k^2+542k+35)/(3(6k+1))$.

(ii) *Let p be an odd prime. If $p>3$, then*

$$\sum_{k=0}^{(p-1)/2}\frac{(92k^3+54k^2+12k+1)\binom{2k}{k}^7}{(6k+1)256^k\binom{3k}{k}\binom{6k}{3k}}\equiv p^2+\frac{93}{80}p^7B_{p-5}\pmod{p^8}$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{(92k^3-54k^2+12k-1)256^k\binom{3k}{k}\binom{6k}{k}}{(6k-1)k^5\binom{2k}{k}^7}\equiv 252\frac{H_{p-1}}{p^2}+\frac{783}{10}p^2B_{p-5}\pmod{p^3}.$$

When $p\neq 5$, we have

$$\sum_{k=0}^{p-1}\frac{(92k^3+54k^2+12k+1)\binom{2k}{k}^7}{(6k+1)256^k\binom{3k}{k}\binom{6k}{3k}}\equiv p^2+\frac{7}{10}p^5B_{p-3}\pmod{p^6}.$$

Remark 5.20. The hypergeometric series in (5.30) has converging rate 4/27.

Conjecture 5.21 (2023-11-15)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}^7\left((92k^3+54k^2+12k+1)(20H_{2k}^{(2)}-7H_k^{(2)})+44k+10\right)}{(6k+1)256^k\binom{3k}{k}\binom{6k}{3k}}=4. \tag{5.32}$$

(ii) *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^7\left((92k^3+54k^2+12k+1)(20H_{2k}^{(2)}-7H_k^{(2)})+44k+10\right)}{(6k+1)256^k\binom{3k}{k}\binom{6k}{3k}} \equiv 14p^3B_{p-3} \pmod{p^4}.$$

If $p>3$, then

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^7\left((92k^3+54k^2+12k+1)(20H_{2k}^{(2)}-7H_k^{(2)})+44k+10\right)}{(6k+1)256^k\binom{3k}{k}\binom{6k}{3k}} \equiv \frac{93}{4}p^5B_{p-5} \pmod{p^6}.$$

Remark 5.21. We haven't found identities similar to (5.32) with summands involving harmonic numbers of order at least three.

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