

EVALUATIONS OF THREE SYMMETRIC TOEPLITZ DETERMINANTS

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ABSTRACT. In this paper we evaluate the following three symmetric Toeplitz determinants:

$$\det [|j - k| + \delta_{jk}]_{1 \leq j, k \leq n}, \det [F_{|j-k|} + \delta_{jk}]_{1 \leq j, k \leq n} \text{ and } \det \left[\binom{|j-k|}{3} - \delta_{jk} \right]_{1 \leq j, k \leq n},$$

where δ_{jk} is the Kronecker delta. $(F_i)_{i \geq 0}$ is the Fibonacci sequence, and $\binom{\cdot}{3}$ is the Legendre symbol.

1. INTRODUCTION

For a matrix $M = [a_{jk}]_{1 \leq j, k \leq n}$ over the field \mathbb{C} of complex numbers, we use $\det(M)$ or $\det[a_{jk}]_{1 \leq j, k \leq n}$ to denote its determinant. A Toeplitz matrix over \mathbb{C} has the form $[a_{j-k}]_{1 \leq j, k \leq n}$. In this paper we evaluate determinants of three symmetric Toeplitz matrices.

In 1934 the evaluation of the symmetric Toeplitz determinant $\det[|j-k|]_{1 \leq j, k \leq n}$ was proposed by R. Robinson as a problem in Amer. Math. Monthly, later its solutions appeared in [4]. As a result,

$$\det[|j-k|]_{1 \leq j, k \leq n} = (-1)^{n-1} (n-1) 2^{n-2}.$$

Moreover, the inverse of the matrix $[|j-k|]_{1 \leq j, k \leq n}$ was found by M. Fiedler (cf. J. Todd [5]).

For $j, k \in \mathbb{N} = \{0, 1, 2, \dots\}$, we adopt the usual Kronecker symbol δ_{jk} which takes 1 or 0 according as $j = k$ or not.

Now we state our first result.

Theorem 1.1. *For any positive integer n , we have*

$$\det[|j-k| + \delta_{jk}]_{1 \leq j, k \leq n} = \begin{cases} \frac{1+(-1)^{(n-1)/2n}}{2} & \text{if } 2 \nmid n, \\ \frac{1+(-1)^{n/2}}{2} & \text{if } 2 \mid n. \end{cases} \quad (1.1)$$

Remark 1.1. For $n = 1, 2, 3, \dots$, let $f(n)$ denote the right-hand side of (1.1). The sequence

$$(f(n))_{n \geq 1} = (1, 0, -1, 3, 0, -3, 1, 5, 0, -5, 1, 7, 0, -7, 1, \dots)$$

was generated by P. Barry [1] in 2009 as the Hankel transform $(\det[a_{j+k-1}]_{1 \leq j, k \leq n})_{n \geq 1}$ of the integer sequence

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, \dots) = (1, 0, 0, 1, 2, 4, 8, 17, 38, 88, \dots)$$

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satisfying the recurrence

$$a_n = \sum_{k=1}^{n-1} a_k a_{n-k} \quad (n = 5, 6, 7, \dots)$$

which has a combinatorial interpretation (cf. [2]). According to [5, pp. 32-33], H. Heilbronn proved that -1 is an eigenvalue of the matrix $[|j-k|]_{1 \leq j, k \leq n}$ in the case $n \equiv 2 \pmod{4}$.

Our second theorem involves the Fibonacci sequence $(F_n)_{n \geq 0}$ defined by $F_0 = 0$, $F_1 = 1$ and the recurrence

$$F_{i+1} = F_i + F_{i-1} \quad (i = 1, 2, 3, \dots).$$

Theorem 1.2. *For any positive integer n , we have*

$$\det[F_{|j-k|} + \delta_{jk}]_{1 \leq j, k \leq n} = \begin{cases} 1 & \text{if } n \equiv 0, \pm 1 \pmod{6}, \\ 0 & \text{otherwise,} \end{cases} \quad (1.2)$$

Our third theorem involves the Legendre symbol $\left(\frac{a}{3}\right)$ with $a \in \mathbb{Z}$, which coincides with the unique $r \in \{0, \pm 1\}$ such that $a \equiv r \pmod{3}$.

Theorem 1.3. *For any positive integer n , we have*

$$\det \left[\left(\frac{|j-k|}{3} \right) - \delta_{jk} \right]_{1 \leq j, k \leq n} = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{6}, \\ -1 & \text{if } n \equiv \pm 1 \pmod{6}, \\ 0 & \text{otherwise.} \end{cases} \quad (1.3)$$

Theorems 1.1-1.3 are quite similar. We are unable to find any other results of this kind. Sections 2 and 3 are devoted to our proofs of Theorem 1.1 and Theorems 1.2-1.3, respectively.

2. PROOF OF THEOREM 1.1

Let n be a positive integer. For $j_0, k_0 \in \{1, \dots, n\}$ with $j_0 \neq k_0$, we define

$$T_{j_0, k_0} = [t_{jk}]_{1 \leq j, k \leq n} \quad \text{and} \quad \tilde{T}_{j_0, k_0} = [\tilde{t}_{jk}]_{1 \leq j, k \leq n},$$

where

$$t_{jk} = \begin{cases} 1 & \text{if } j = k, \\ -1 & \text{if } j = j_0 \text{ and } k = k_0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{t}_{jk} = \begin{cases} 1 & \text{if } j = j_0 \text{ and } k = k_0, \text{ or } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\det(T_{j_0, k_0}) = 1 = \det(\tilde{T}_{j_0, k_0}) \quad \text{for all } j_0, k_0 = 1, \dots, n \text{ with } j_0 \neq k_0. \quad (2.1)$$

We need this useful fact in our proofs of Theorems 1.1-1.3.

Proof of Theorem 1.1. Let A denote the matrix $[|j-k| + \delta_{jk}]_{1 \leq j, k \leq n}$. Clearly (1.1) holds trivially for $n = 1, 2$. (When $n = 2$ all the entries of A are 1.)

Now we assume that $n \geq 3$. Observe that

$$T_{21}T_{32} \cdots T_{n-1,n-2}T_{n,n-1}AT_{n-1,n}T_{n-2,n-1} \cdots T_{23}T_{12} = C,$$

where $C = [c_{jk}]_{1 \leq j, k \leq n}$ and

$$c_{jk} = \begin{cases} 1 & \text{if } 1 \in \{j, k\} \text{ and } jk \neq 2, \\ -1 & \text{if } |j - k| = 1 \text{ and } jk \neq 2, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\det(A) = \det(C)$ in view of (2.1).

Note that both the last column and the last row of $T_{n,n-2}CT_{n-2,n}$ contain a unique nonzero entry (which is 1). We illustrate this via the transformation from C to $T_{n,n-2}CT_{n-2,n}$:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \end{bmatrix}. \quad (2.2)$$

Thus, via expanding $\det(T_{n,n-2}CT_{n-2,n})$ by its last column and the last row, we obtain

$$\det(A) = \det(C) = \det(T_{n,n-2}CT_{n-2,n}) = -\det D_{n-2},$$

where the $(n-2) \times (n-2)$ matrix D_{n-2} has the form

$$\begin{bmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \end{bmatrix}$$

obtained by deleting the n th and the $(n-3)$ th columns and rows from the last matrix in (2.2).

By repeating the above procedure with $T_{n,n-2}CT_{n-2,n} \rightarrow D_{n-2}$ replaced by

$$T_{n-4k,n-2-4k}D_{n-2k}T_{n-2-4k,n-4k} \rightarrow D_{n-2-2k} \quad \left(0 < k < \left\lfloor \frac{n}{4} \right\rfloor\right),$$

we get that

$$\begin{aligned}\det(A) &= -\det(D_{n-2}) = \det(D_{n-4}) = \cdots \\ &= (-1)^{\lfloor n/4 \rfloor} \det\left(D_{n-2\lfloor \frac{n}{4} \rfloor}\right) = (-1)^{\lfloor n/4 \rfloor} \det\left(D_{\frac{n+\{n\}_4}{2}}\right)\end{aligned}$$

with the aid of (2.1), where $\{n\}_4$ is the least nonnegative residue of n modulo 4.

Case 1. $n \equiv 0 \pmod{2}$.

When $n \equiv 0 \pmod{4}$, the matrix $D_{\frac{n+\{n\}_4}{2}} = D_{\frac{n}{2}}$ has the form

$$\begin{bmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \end{bmatrix},$$

therefore

$$\det(D_{\frac{n}{2}}) = (-1)^{n/2} \times (-1)^{n/4} = (-1)^{n/4}$$

and hence $\det(A) = (-1)^{n/4} \det(D_{\frac{n}{2}}) = 1$.

When $n \equiv 2 \pmod{4}$, the matrix $D_{\frac{n+\{n\}_4}{2}} = D_{\frac{n}{2}+1}$ has the form

$$\begin{bmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \end{bmatrix}$$

(with the second row containing only zero entries), and hence

$$\det(A) = (-1)^{(n-2)/4} \det(D_{\frac{n}{2}+1}) = 0.$$

Below we assume that n is odd.

Case 2. $n \equiv 1 \pmod{4}$.

In this case, we have

$$D_{\frac{n+\{n\}_4}{2}} = D_{\frac{n+1}{2}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix},$$

and $M_{\frac{n+1}{2}} = \tilde{T}_{1, \frac{n+1}{2}} \tilde{T}_{1, \frac{n-1}{2}} \cdots \tilde{T}_{12} D_{\frac{n+1}{2}}$ has the form

$$\begin{bmatrix} \frac{n+1}{2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix}.$$

Thus

$$\det\left(D_{\frac{n+1}{2}}\right) = \det\left(M_{\frac{n+1}{2}}\right) = \frac{n+1}{2}(-1)^{(n-1)/2} \times (-1)^{(n-1)/4} = \frac{n+1}{2}(-1)^{(n-1)/4}$$

and hence

$$\det(A) = (-1)^{(n-1)/4} \det\left(D_{\frac{n+1}{2}}\right) = \frac{n+1}{2}.$$

Case 3. $n \equiv 3 \pmod{4}$.

In this case, we have

$$D_{\frac{n+\{n\}_4}{2}} = D_{\frac{n+3}{2}} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix},$$

and $M_{\frac{n+3}{2}} = \tilde{T}_{1, \frac{n+3}{2}} \cdots \tilde{T}_{12} D_{\frac{n+1}{2}} \tilde{T}_{21} \cdots \tilde{T}_{\frac{n+3}{2}, 1}$ has the form

$$\begin{bmatrix} \frac{n-1}{2} & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix}.$$

Therefore

$$\det\left(D_{\frac{n+3}{2}}\right) = \det\left(M_{\frac{n+3}{2}}\right) = \frac{n-1}{2}(-1)^{(n+1)/2} \times (-1)^{(n+1)/4} = \frac{n-1}{2}(-1)^{(n+1)/4}$$

and hence

$$\det(A) = (-1)^{(n-3)/4} \det\left(D_{\frac{n+3}{2}}\right) = \frac{1-n}{2}.$$

In view of the above, we have completed our proof Theorem 1.1. \square

3. PROOFS OF THEOREMS 1.2 AND 1.3

Proof of Theorem 1.2. Let B_n denote the matrix $[F_{|j-k|} + \delta_{jk}]_{1 \leq j, k \leq n}$. It is easy to verify (1.2) for $1 \leq n \leq 6$.

Now, let $n \geq 7$. It suffices to prove that $\det(B_n) = \det(B_{n-6})$. It is easy to verify that

$$T_{n, n-2} T_{n, n-1} B_n T_{n-1, n} T_{n-2, n} = \begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-4} & F_{n-3} & F_{n-2} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-5} & F_{n-4} & F_{n-3} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 1 & 1 & 0 \\ F_{n-3} & F_{n-4} & F_{n-5} & F_{n-6} & \cdots & 1 & 1 & 1 & -1 \\ F_{n-2} & F_{n-3} & F_{n-4} & F_{n-5} & \cdots & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & -1 & 1 \end{bmatrix}.$$

It follows that the matrix $\tilde{T}_{n-2,n}\tilde{T}_{n-1,n}(T_{n,n-2}T_{n,n-1}B_nT_{n-1,n}T_{n-2,n})\tilde{T}_{n,n-1}\tilde{T}_{n,n-2}$ has the form

$$\begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-4} & F_{n-3} & F_{n-2} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-5} & F_{n-4} & F_{n-3} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 1 & 1 & 0 \\ F_{n-3} & F_{n-4} & F_{n-5} & F_{n-6} & \cdots & 1 & 0 & 0 & 0 \\ F_{n-2} & F_{n-3} & F_{n-4} & F_{n-5} & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

As there is a unique nonzero entry (which is 1) in the last row of the last matrix, and

$$\det\left(\tilde{T}_{n-2,n}\tilde{T}_{n-1,n}T_{n,n-2}T_{n,n-1}B_nT_{n-1,n}T_{n-2,n}\tilde{T}_{n,n-1}\tilde{T}_{n,n-2}\right) = \det(B_n)$$

in light of (2.1), we see that $\det(B_n) = \det(B^{(1)})$, where

$$B^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-5} & F_{n-4} & F_{n-3} & F_{n-2} \\ 1 & 1 & 1 & 1 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & F_{n-3} \\ 1 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & F_{n-4} \\ 2 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & F_{n-6} & F_{n-5} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-5} & F_{n-6} & F_{n-7} & F_{n-8} & \cdots & 1 & 1 & 1 & 2 \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 1 & 1 & 1 \\ F_{n-3} & F_{n-4} & F_{n-5} & F_{n-6} & \cdots & 1 & 1 & 0 & 0 \\ F_{n-2} & F_{n-3} & F_{n-4} & F_{n-5} & \cdots & 2 & 1 & 0 & 0 \end{bmatrix}.$$

Similarly, $\tilde{T}_{n-3,n-1}\tilde{T}_{n-2,n-1}T_{n-1,n-3}T_{n-1,n-2}B^{(1)}T_{n-2,n-1}T_{n-3,n-1}\tilde{T}_{n-1,n-2}\tilde{T}_{n-1,n-3}$ has the form

$$\begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-5} & F_{n-4} & F_{n-3} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & F_{n-6} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-5} & F_{n-6} & F_{n-7} & F_{n-8} & \cdots & 1 & 1 & 1 & 0 \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 0 & 0 & 0 \\ F_{n-3} & F_{n-4} & F_{n-5} & F_{n-6} & \cdots & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix},$$

and hence $\det(B^{(1)})$ equals the determinant of the matrix

$$B^{(2)} = \begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & F_{n-3} \\ 1 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & F_{n-4} \\ 1 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & F_{n-6} & F_{n-5} \\ 2 & 1 & 1 & 1 & \cdots & F_{n-9} & F_{n-8} & F_{n-7} & F_{n-6} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-6} & F_{n-7} & F_{n-8} & F_{n-9} & \cdots & 1 & 1 & 1 & 2 \\ F_{n-5} & F_{n-6} & F_{n-7} & F_{n-8} & \cdots & 1 & 1 & 1 & 1 \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 1 & 0 & 0 \\ F_{n-3} & F_{n-4} & F_{n-5} & F_{n-6} & \cdots & 2 & 1 & 0 & -1 \end{bmatrix}.$$

Note also that

$$\begin{aligned} & \widetilde{T}_{n-4,n-2} \widetilde{T}_{n-3,n-2} T_{n-2,n-4} T_{n-2,n-3} B^{(2)} T_{n-3,n-2} T_{n-4,n-2} \\ &= \begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & F_{n-6} & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-9} & F_{n-8} & F_{n-7} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-6} & F_{n-7} & F_{n-8} & F_{n-9} & \cdots & 1 & 1 & 1 & 0 \\ F_{n-5} & F_{n-6} & F_{n-7} & F_{n-8} & \cdots & 1 & 0 & 0 & -1 \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & -1 & 0 \end{bmatrix} \end{aligned}$$

and hence

$$T_{n-3,n-5} T_{n-3,n-4} (\widetilde{T}_{n-4,n-2} \widetilde{T}_{n-3,n-2} T_{n-2,n-4} T_{n-2,n-3} B^{(2)} T_{n-3,n-2} T_{n-4,n-2}) T_{n-4,n-3} T_{n-5,n-3}$$

has the form

$$\begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-6} & F_{n-5} & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & 0 & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-9} & F_{n-8} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-6} & F_{n-7} & F_{n-8} & F_{n-9} & \cdots & 1 & 1 & -1 & 0 \\ F_{n-5} & F_{n-6} & F_{n-7} & F_{n-8} & \cdots & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Note that both the last row and the last column contain only one nonzero entry (which is -1). Therefore, $\det(B^{(2)})$ equals the determinant of the $(n-2) \times (n-2)$ matrix

$$\begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-7} & F_{n-6} & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-9} & F_{n-8} & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-10} & F_{n-9} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{n-7} & F_{n-8} & F_{n-9} & F_{n-10} & \cdots & 1 & 1 & 0 & 0 & 0 \\ F_{n-6} & F_{n-7} & F_{n-8} & F_{n-9} & \cdots & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

which has a unique nonzero entry (which is -1) in the $(n-3)$ -th row and the $(n-3)$ -th column. As B_{n-6} is exactly the submatrix of the last matrix formed by the first $n-6$ rows and the first $n-6$ columns, we see that $\det(B^{(2)}) = \det(B_{n-6})$.

In view of the above,

$$\det(B_n) = \det(B^{(1)}) = \det(B^{(2)}) = \det(B_{n-6}).$$

This concludes our proof of Theorem 1.2. \square

Proof of Theorem 1.3. Let C_n denote the matrix $[(\frac{|j-k|}{3}) - \delta_{jk}]_{1 \leq j, k \leq n}$. It is easy to verify (1.3) for $1 \leq n \leq 6$.

Now, let $n \geq 7$. It suffices to prove that $\det(C_n) = \det(C_{n-6})$. It's routine to verify that $T_{n-2, n} \tilde{T}_{n-1, n} (\tilde{T}_{n, n-2} \tilde{T}_{n, n-1} C_n \tilde{T}_{n-1, n} \tilde{T}_{n-2, n}) \tilde{T}_{n, n-1} T_{n, n-2}$ has the form

$$\begin{bmatrix} -1 & 1 & -1 & 0 & \cdots & \binom{n-4}{3} & \binom{n-3}{3} & \binom{n-2}{3} & 0 \\ 1 & -1 & 1 & -1 & \cdots & \binom{n-5}{3} & \binom{n-4}{3} & \binom{n-3}{3} & 0 \\ -1 & 1 & -1 & 1 & \cdots & \binom{n-6}{3} & \binom{n-5}{3} & \binom{n-4}{3} & 0 \\ 0 & -1 & 1 & -1 & \cdots & \binom{n-7}{3} & \binom{n-6}{3} & \binom{n-5}{3} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-4}{3} & \binom{n-5}{3} & \binom{n-6}{3} & \binom{n-7}{3} & \cdots & -1 & 1 & -1 & 0 \\ \binom{n-3}{3} & \binom{n-4}{3} & \binom{n-5}{3} & \binom{n-6}{3} & \cdots & 1 & 0 & 0 & 0 \\ \binom{n-2}{3} & \binom{n-3}{3} & \binom{n-4}{3} & \binom{n-5}{3} & \cdots & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \end{bmatrix}.$$

As there is a unique nonzero entry (which is -1) in the last row of the last matrix, and

$$\det \left(T_{n-2, n} \tilde{T}_{n-1, n} \tilde{T}_{n, n-2} \tilde{T}_{n, n-1} C_n \tilde{T}_{n-1, n} \tilde{T}_{n-2, n} \tilde{T}_{n, n-1} T_{n, n-2} \right) = \det(C_n)$$

in light of (2.1), we have $-\det(C_n) = \det(C^{(1)})$, where

$$C^{(1)} = \begin{bmatrix} -1 & 1 & -1 & 0 & \cdots & \binom{n-5}{3} & \binom{n-4}{3} & \binom{n-3}{3} & \binom{n-2}{3} \\ 1 & -1 & 1 & -1 & \cdots & \binom{n-6}{3} & \binom{n-5}{3} & \binom{n-4}{3} & \binom{n-3}{3} \\ -1 & 1 & -1 & 1 & \cdots & \binom{n-7}{3} & \binom{n-6}{3} & \binom{n-5}{3} & \binom{n-4}{3} \\ 0 & -1 & 1 & -1 & \cdots & \binom{n-8}{3} & \binom{n-7}{3} & \binom{n-6}{3} & \binom{n-5}{3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-5}{3} & \binom{n-6}{3} & \binom{n-7}{3} & \binom{n-8}{3} & \cdots & -1 & 1 & -1 & 0 \\ \binom{n-4}{3} & \binom{n-5}{3} & \binom{n-6}{3} & \binom{n-7}{3} & \cdots & 1 & -1 & 1 & -1 \\ \binom{n-3}{3} & \binom{n-4}{3} & \binom{n-5}{3} & \binom{n-6}{3} & \cdots & -1 & 1 & 0 & 0 \\ \binom{n-2}{3} & \binom{n-3}{3} & \binom{n-4}{3} & \binom{n-5}{3} & \cdots & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Similarly, $T_{n-3,n-1}\tilde{T}_{n-2,n-1}(\tilde{T}_{n-1,n-3}\tilde{T}_{n-1,n-2}C^{(1)}\tilde{T}_{n-2,n-1}\tilde{T}_{n-3,n-1})\tilde{T}_{n-1,n-2}T_{n-1,n-3}$ has the form

$$\begin{bmatrix} -1 & 1 & -1 & 0 & \cdots & \binom{n-5}{3} & \binom{n-4}{3} & \binom{n-3}{3} & 0 \\ 1 & -1 & 1 & -1 & \cdots & \binom{n-6}{3} & \binom{n-5}{3} & \binom{n-4}{3} & 0 \\ -1 & 1 & -1 & 1 & \cdots & \binom{n-7}{3} & \binom{n-6}{3} & \binom{n-5}{3} & 0 \\ 0 & -1 & 1 & -1 & \cdots & \binom{n-8}{3} & \binom{n-7}{3} & \binom{n-6}{3} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-5}{3} & \binom{n-6}{3} & \binom{n-7}{3} & \binom{n-8}{3} & \cdots & -1 & 1 & -1 & 0 \\ \binom{n-4}{3} & \binom{n-5}{3} & \binom{n-6}{3} & \binom{n-7}{3} & \cdots & 1 & 0 & 0 & 0 \\ \binom{n-3}{3} & \binom{n-4}{3} & \binom{n-5}{3} & \binom{n-6}{3} & \cdots & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \end{bmatrix},$$

and hence $\det(C^{(1)}) = -\det(C^{(2)})$, where

$$C^{(2)} = \begin{bmatrix} -1 & 1 & -1 & 0 & \cdots & \binom{n-6}{3} & \binom{n-5}{3} & \binom{n-4}{3} & \binom{n-3}{3} \\ 1 & -1 & 1 & -1 & \cdots & \binom{n-7}{3} & \binom{n-6}{3} & \binom{n-5}{3} & \binom{n-4}{3} \\ -1 & 1 & -1 & 1 & \cdots & \binom{n-8}{3} & \binom{n-7}{3} & \binom{n-6}{3} & \binom{n-5}{3} \\ 0 & -1 & 1 & -1 & \cdots & \binom{n-9}{3} & \binom{n-8}{3} & \binom{n-7}{3} & \binom{n-6}{3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-6}{3} & \binom{n-7}{3} & \binom{n-8}{3} & \binom{n-9}{3} & \cdots & -1 & 1 & -1 & 0 \\ \binom{n-5}{3} & \binom{n-6}{3} & \binom{n-7}{3} & \binom{n-8}{3} & \cdots & 1 & -1 & 1 & -1 \\ \binom{n-4}{3} & \binom{n-5}{3} & \binom{n-6}{3} & \binom{n-7}{3} & \cdots & -1 & 1 & 0 & 0 \\ \binom{n-3}{3} & \binom{n-4}{3} & \binom{n-5}{3} & \binom{n-6}{3} & \cdots & 0 & -1 & 0 & 1 \end{bmatrix}.$$

It's routine to verify that

$$\tilde{T}_{n-3,n-5}\tilde{T}_{n-3,n-4}(T_{n-4,n-2}\tilde{T}_{n-3,n-2}\tilde{T}_{n-2,n-4}\tilde{T}_{n-2,n-3}C^{(2)}\tilde{T}_{n-3,n-2}\tilde{T}_{n-4,n-2})\tilde{T}_{n-4,n-3}\tilde{T}_{n-5,n-3}$$

has the form

$$\begin{bmatrix} -1 & 1 & -1 & 0 & \cdots & \binom{n-6}{3} & \binom{n-5}{3} & 0 & 0 \\ 1 & -1 & 1 & -1 & \cdots & \binom{n-7}{3} & \binom{n-6}{3} & 0 & 0 \\ -1 & 1 & -1 & 1 & \cdots & \binom{n-8}{3} & \binom{n-7}{3} & 0 & 0 \\ 0 & -1 & 1 & -1 & \cdots & \binom{n-9}{3} & \binom{n-8}{3} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-6}{3} & \binom{n-7}{3} & \binom{n-8}{3} & \binom{n-9}{3} & \cdots & -1 & 1 & -1 & 0 \\ \binom{n-5}{3} & \binom{n-6}{3} & \binom{n-7}{3} & \binom{n-8}{3} & \cdots & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Since there is only one nonzero entry (which is -1) in the last row of the last matrix, with the aid of (2.1) we have

$$\det(C^{(2)}) = \begin{vmatrix} -1 & 1 & -1 & 0 & \cdots & \binom{n-7}{3} & \binom{n-6}{3} & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & \cdots & \binom{n-8}{3} & \binom{n-7}{3} & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & \cdots & \binom{n-9}{3} & \binom{n-8}{3} & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & \cdots & \binom{n-10}{3} & \binom{n-9}{3} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-7}{3} & \binom{n-8}{3} & \binom{n-9}{3} & \binom{n-10}{3} & \cdots & -1 & 1 & 0 & 0 & 0 \\ \binom{n-6}{3} & \binom{n-7}{3} & \binom{n-8}{3} & \binom{n-9}{3} & \cdots & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 \end{vmatrix},$$

which has a unique nonzero entry (which is -1) in the $(n-3)$ -th row and in the $(n-3)$ -th column. As C_{n-6} is exactly the matrix formed by the first $n-6$ rows and the first $n-6$ columns in the last determinant, we have $\det(C^{(2)}) = \det(C_{n-6})$. Therefore

$$\det(C_n) = -\det(C^{(1)}) = \det(C^{(2)}) = \det(C_{n-6}).$$

This ends our proof of Theorem 1.3. \square

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