Linear Multilinear Algebra 72 (2024), no. 15, 2504–2515.

EVALUATIONS OF THREE SYMMETRIC TOEPLITZ DETERMINANTS

HAN WANG AND ZHI-WEI SUN

ABSTRACT. In this paper we evaluate the following three symmetric Toeplitz determinants:

$$\det\,[|j-k|+\delta_{jk}]_{1\le j,k\le n},\ \det[F_{|j-k|}+\delta_{jk}]_{1\le j,k\le n}\ \text{and}\ \det\left[\left(\frac{|j-k|}{3}\right)-\delta_{jk}\right]_{1\le j,k\le n},$$

where δ_{jk} is the Kronecker delta. $(F_i)_{i\ge0}$ is the Fibonacci sequence, and $(\frac{\cdot}{3})$ is the Legendre symbol.

1. Introduction

For a matrix $M=[a_{jk}]_{1\le j,k\le n}$ over the field $\mathbb{C}$ of complex numbers, we use $\det(M)$ or $\det[a_{jk}]_{1\le j,k\le n}$ to denote its determinant. A Toeplitz matrix over $\mathbb{C}$ has the form $[a_{j-k}]_{1\le j,k\le n}$. In this paper we evaluate determinants of three symmetric Toeplitz matrices.

In 1934 the evaluation of the symmetric Toeplitz determinant $\det[|j-k|]_{1\le j,k\le n}$ was proposed by R. Robinson as a problem in Amer. Math. Monthly, later its solutions appeared in [4]. As a result,

$$\det[|j-k|]_{1\le j,k\le n}=(-1)^{n-1}(n-1)2^{n-2}.$$

Moreover, the inverse of the matrix $[|j-k|]_{1\le j,k\le n}$ was found by M. Fiedler (cf. J. Todd [5]).

For $j,k\in\mathbb{N}=\{0,1,2,\ldots\}$, we adopt the usual Kronecker symbol δ_{jk} which takes 1 or 0 according as $j=k$ or not.

Now we state our first result.

Theorem 1.1. *For any positive integer n, we have*

$$\det[|j-k|+\delta_{jk}]_{1\le j,k\le n}=\begin{cases}\frac{1+(-1)^{(n-1)/2}n}{2} & \text{if } 2\nmid n,\\ \frac{1+(-1)^{n/2}}{2} & \text{if } 2\mid n.\end{cases}\tag{1.1}$$

Remark 1.1. For $n=1,2,3,\cdots$, let $f(n)$ denote the right-hand side of (1.1). The sequence

$$(f(n))_{n\ge1}=(1,0,-1,3,0,-3,1,5,0,-5,1,7,0,-7,1,\ldots)$$

was generated by P. Barry [1] in 2009 as the Hankel transform $(\det[a_{j+k-1}]_{1\le j,k\le n})_{n\ge1}$ of the integer sequence

$$(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},\ldots)=(1,0,0,1,2,4,8,17,38,88,\ldots)$$

Key words and phrases. Determinant, Fibonacci sequence, Toeplitz matrix.

2020 *Mathematics Subject Classification.* Primary 15A15; Secondary 11B39, 11C20.

Supported by the National Natural Science Foundation of China (grant no. 11971222).

satisfying the recurrence

$$a_n = \sum_{k=1}^{n-1} a_k a_{n-k} \quad (n = 5, 6, 7, \ldots)$$

which has a combinatorial interpretation (cf. [2]). According to [5, pp. 32-33], H. Heilbronn proved that -1 is an eigenvalue of the matrix $[|j-k|]_{1\le j,k\le n}$ in the case $n \equiv 2 \pmod 4$.

Our second theorem involves the Fibonacci sequence $(F_n)_{n\ge0}$ defined by $F_0 = 0$, $F_1 = 1$ and the recurrence

$$F_{i+1} = F_i + F_{i-1} \quad (i = 1, 2, 3, \ldots).$$

Theorem 1.2. *For any positive integer n, we have*

$$\det[F_{|j-k|} + \delta_{jk}]_{1\le j,k\le n} = \begin{cases} 1 & \text{if } n \equiv 0, \pm1 \pmod 6, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

Our third theorem involves the Legendre symbol $(\frac{a}{3})$ with $a \in \mathbb{Z}$, which coincides with the unique $r \in \{0, \pm\}$ such that $a \equiv r \pmod 3$.

Theorem 1.3. *For any positive integer n, we have*

$$\det\left[\left(\frac{|j-k|}{3}\right) - \delta_{jk}\right]_{1\le j,k\le n} = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 6, \\ -1 & \text{if } n \equiv \pm1 \pmod 6, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Theorems 1.1-1.3 are quite similar. We are unable to find any other results of this kind. Sections 2 and 3 are devoted to our proofs of Theorem 1.1 and Theorems 1.2-1.3, respectively.

2. Proof of Theorem 1.1

Let n be a positive integer. For $j_0, k_0 \in \{1, \ldots, n\}$ with $j_0 \ne k_0$, we define

$$T_{j_0,k_0} = [t_{jk}]_{1\le j,k\le n} \quad \text{and} \quad \widetilde{T}_{j_0,k_0} = [\widetilde{t}_{jk}]_{1\le j,k\le n},$$

where

$$t_{jk} = \begin{cases} 1 & \text{if } j = k, \\ -1 & \text{if } j = j_0 \text{ and } k = k_0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \widetilde{t}_{jk} = \begin{cases} 1 & \text{if } j = j_0 \text{ and } k = k_0, \text{or } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\det(T_{j_0,k_0}) = 1 = \det(\widetilde{T}_{j_0,k_0}) \quad \text{for all } j_0, k_0 = 1, \ldots, n \text{ with } j_0 \ne k_0. \tag{2.1}$$

We need this useful fact in our proofs of Theorems 1.1-1.3.

Proof of Theorem 1.1. Let A denote the matrix $[|j-k|+\delta_{jk}]_{1\le j,k\le n}$. Clearly (1.1) holds trivially for $n = 1, 2$. (When $n = 2$ all the entries of A are 1.)

Now we assume that $n \geqslant 3$. Observe that

$$T_{21}T_{32}\cdots T_{n-1,n-2}T_{n,n-1}AT_{n-1,n}T_{n-2,n-1}\cdots T_{23}T_{12}=C,$$

where $C=[c_{jk}]_{1\le j,k\le n}$ and

$$c_{jk}=\begin{cases}1 & \text{if } 1\in\{j,k\} \text{ and } jk\neq 2,\\ -1 & \text{if } |j-k|=1 \text{ and } jk\neq 2,\\ 0 & \text{otherwise.}\end{cases}$$

It follows that $\det(A)=\det(C)$ in view of (2.1).

Note that both the last column and the last row of $T_{n,n-2}CT_{n-2,n}$ contain a unique nonzero entry (which is 1). We illustrate this via the transformation from C to $T_{n,n-2}CT_{n-2,n}$:

$$\begin{bmatrix}1&0&1&1&\cdots&1&1&1&1\\0&0&-1&0&\cdots&0&0&0&0\\1&-1&0&-1&\cdots&0&0&0&0\\1&0&-1&0&\cdots&0&0&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\1&0&0&0&\cdots&0&-1&0&0\\1&0&0&0&\cdots&-1&0&-1&0\\1&0&0&0&\cdots&0&-1&0&-1\\1&0&0&0&\cdots&0&0&-1&0\end{bmatrix}\to\begin{bmatrix}1&0&1&1&\cdots&1&1&1&0\\0&0&-1&0&\cdots&0&0&0&0\\1&-1&0&-1&\cdots&0&0&0&0\\1&0&-1&0&\cdots&0&0&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\1&0&0&0&\cdots&0&-1&0&1\\1&0&0&0&\cdots&-1&0&-1&0\\1&0&0&0&\cdots&0&-1&0&0\\0&0&0&0&\cdots&1&0&0&0\end{bmatrix}. \tag{2.2}$$

Thus, via expanding $\det(T_{n,n-2}CT_{n-2,n})$ by its last column and the last row, we obtain

$$\det(A)=\det(C)=\det(T_{n,n-2}CT_{n-2,n})=-\det D_{n-2},$$

where the $(n-2)\times(n-2)$ matrix D_{n-2} has the form

$$\begin{bmatrix}1&0&1&1&\cdots&1&1&1&1\\0&0&-1&0&\cdots&0&0&0&0\\1&-1&0&-1&\cdots&0&0&0&0\\1&0&-1&0&\cdots&0&0&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\1&0&0&0&\cdots&0&-1&0&0\\1&0&0&0&\cdots&-1&0&0&0\\1&0&0&0&\cdots&0&0&0&-1\\1&0&0&0&\cdots&0&0&-1&0\end{bmatrix}$$

obtained by deleting the nth and the $(n-3)$th columns and rows from the last matrix in (2.2).

By repeating the above procedure with $T_{n,n-2}CT_{n-2,n}\to D_{n-2}$ replaced by

$$T_{n-4k,n-2-4k}D_{n-2k}T_{n-2-4k,n-4k}\to D_{n-2-2k}\quad\left(0<k<\left\lfloor\frac{n}{4}\right\rfloor\right),$$

we get that

$$\begin{aligned}\det(A) &= -\det(D_{n-2}) = \det(D_{n-4}) = \cdots \\ &= (-1)^{\lfloor n/4 \rfloor} \det\left(D_{n-2\lfloor \frac{n}{4} \rfloor}\right) = (-1)^{\lfloor n/4 \rfloor} \det\left(D_{\frac{n+\{n\}_4}{2}}\right)\end{aligned}$$

with the aid of (2.1), where $\{n\}_4$ is the least nonnegative residue of n modulo 4.

Case 1. $n \equiv 0 \pmod 2$.

When $n \equiv 0 \pmod 4$, the matrix $D_{\frac{n+\{n\}_4}{2}} = D_{\frac{n}{2}}$ has the form

$$\begin{bmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \end{bmatrix},$$

therefore

$$\det\left(D_{\frac{n}{2}}\right) = (-1)^{n/2} \times (-1)^{n/4} = (-1)^{n/4}$$

and hence $\det(A) = (-1)^{n/4} \det(D_{\frac{n}{2}}) = 1$.

When $n \equiv 2 \pmod 4$, the matrix $D_{\frac{n+\{n\}_4}{2}} = D_{\frac{n}{2}+1}$ has the form

$$\begin{bmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \end{bmatrix}$$

(with the second row containing only zero entries), and hence

$$\det(A) = (-1)^{(n-2)/4} \det\left(D_{\frac{n}{2}+1}\right) = 0.$$

Below we assume that n is odd.

Case 2. $n \equiv 1 \pmod 4$.

In this case, we have

$$D_{\frac{n+\{n\}_4}{2}} = D_{\frac{n+1}{2}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix},$$

and $M_{\frac{n+1}{2}} = \widetilde{T}_{1,\frac{n+1}{2}} \widetilde{T}_{1,\frac{n-1}{2}} \cdots \widetilde{T}_{12} D_{\frac{n+1}{2}}$ has the form

$$\begin{bmatrix} \frac{n+1}{2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix}.$$

Thus

$$\det\left(D_{\frac{n+1}{2}}\right) = \det\left(M_{\frac{n+1}{2}}\right) = \frac{n+1}{2}(-1)^{(n-1)/2} \times (-1)^{(n-1)/4} = \frac{n+1}{2}(-1)^{(n-1)/4}$$

and hence

$$\det(A) = (-1)^{(n-1)/4} \det\left(D_{\frac{n+1}{2}}\right) = \frac{n+1}{2}.$$

Case 3. $n \equiv 3 \pmod 4$.
In this case, we have

$$D_{\frac{n+\{n\}_4}{2}} = D_{\frac{n+3}{2}} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix},$$

and $M_{\frac{n+3}{2}} = \widetilde{T}_{1,\frac{n+3}{2}} \cdots \widetilde{T}_{12} D_{\frac{n+1}{2}} \widetilde{T}_{21} \cdots \widetilde{T}_{\frac{n+3}{2},1}$ has the form

$$\begin{bmatrix} \frac{n-1}{2} & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix}.$$

Therefore

$$\det\left(D_{\frac{n+3}{2}}\right) = \det\left(M_{\frac{n+3}{2}}\right) = \frac{n-1}{2}(-1)^{(n+1)/2} \times (-1)^{(n+1)/4} = \frac{n-1}{2}(-1)^{(n+1)/4}$$

and hence

$$\det(A) = (-1)^{(n-3)/4} \det\left(D_{\frac{n+3}{2}}\right) = \frac{1-n}{2}.$$

In view of the above, we have completed our proof Theorem 1.1. □

3. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Let B_n denote the matrix $[F_{|j-k|} + \delta_{jk}]_{1 \le j,k \le n}$. It is easy to verify (1.2) for $1 \le n \le 6$.

Now, let $n \ge 7$. It suffices to prove that $\det(B_n) = \det(B_{n-6})$. It is easy to verify that

$$T_{n,n-2} T_{n,n-1} B_n T_{n-1,n} T_{n-2,n} = \begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-4} & F_{n-3} & F_{n-2} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-5} & F_{n-4} & F_{n-3} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 1 & 1 & 0 \\ F_{n-3} & F_{n-4} & F_{n-5} & F_{n-6} & \cdots & 1 & 1 & 1 & -1 \\ F_{n-2} & F_{n-3} & F_{n-4} & F_{n-5} & \cdots & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & -1 & 1 \end{bmatrix}.$$

It follows that the matrix $\widetilde{T}_{n-2,n}\widetilde{T}_{n-1,n}(T_{n,n-2}T_{n,n-1}B_nT_{n-1,n}T_{n-2,n})\widetilde{T}_{n,n-1}\widetilde{T}_{n,n-2}$ has the form

$$\begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-4} & F_{n-3} & F_{n-2} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-5} & F_{n-4} & F_{n-3} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 1 & 1 & 0 \\ F_{n-3} & F_{n-4} & F_{n-5} & F_{n-6} & \cdots & 1 & 0 & 0 & 0 \\ F_{n-2} & F_{n-3} & F_{n-4} & F_{n-5} & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}.$$

As there is a unique nonzero entry (which is 1) in the last row of the last matrix, and

$$\det\left(\widetilde{T}_{n-2,n}\widetilde{T}_{n-1,n}T_{n,n-2}T_{n,n-1}B_nT_{n-1,n}T_{n-2,n}\widetilde{T}_{n,n-1}\widetilde{T}_{n,n-2}\right) = \det(B_n)$$

in light of (2.1), we see that $\det(B_n) = \det(B^{(1)})$, where

$$B^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-5} & F_{n-4} & F_{n-3} & F_{n-2} \\ 1 & 1 & 1 & 1 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & F_{n-3} \\ 1 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & F_{n-4} \\ 2 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & F_{n-6} & F_{n-5} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-5} & F_{n-6} & F_{n-7} & F_{n-8} & \cdots & 1 & 1 & 1 & 2 \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 1 & 1 & 1 \\ F_{n-3} & F_{n-4} & F_{n-5} & F_{n-6} & \cdots & 1 & 1 & 0 & 0 \\ F_{n-2} & F_{n-3} & F_{n-4} & F_{n-5} & \cdots & 2 & 1 & 0 & 0 \end{bmatrix}.$$

Similarly, $\widetilde{T}_{n-3,n-1}\widetilde{T}_{n-2,n-1}T_{n-1,n-3}T_{n-1,n-2}B^{(1)}T_{n-2,n-1}T_{n-3,n-1}\widetilde{T}_{n-1,n-2}\widetilde{T}_{n-1,n-3}$ has the form

$$\begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-5} & F_{n-4} & F_{n-3} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & F_{n-6} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-5} & F_{n-6} & F_{n-7} & F_{n-8} & \cdots & 1 & 1 & 1 & 0 \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 0 & 0 & 0 \\ F_{n-3} & F_{n-4} & F_{n-5} & F_{n-6} & \cdots & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix},$$

and hence $\det(B^{(1)})$ equals the determinant of the matrix

$$B^{(2)} = \begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & F_{n-3} \\ 1 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & F_{n-4} \\ 1 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & F_{n-6} & F_{n-5} \\ 2 & 1 & 1 & 1 & \cdots & F_{n-9} & F_{n-8} & F_{n-7} & F_{n-6} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-6} & F_{n-7} & F_{n-8} & F_{n-9} & \cdots & 1 & 1 & 1 & 2 \\ F_{n-5} & F_{n-6} & F_{n-7} & F_{n-8} & \cdots & 1 & 1 & 1 & 1 \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 1 & 0 & 0 \\ F_{n-3} & F_{n-4} & F_{n-5} & F_{n-6} & \cdots & 2 & 1 & 0 & -1 \end{bmatrix}.$$

Note also that

$$\begin{aligned} &\widetilde{T}_{n-4,n-2}\widetilde{T}_{n-3,n-2}T_{n-2,n-4}T_{n-2,n-3}B^{(2)}T_{n-3,n-2}T_{n-4,n-2} \\ =& \begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-6} & F_{n-5} & F_{n-4} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & F_{n-5} & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & F_{n-6} & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-9} & F_{n-8} & F_{n-7} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-6} & F_{n-7} & F_{n-8} & F_{n-9} & \cdots & 1 & 1 & 1 & 0 \\ F_{n-5} & F_{n-6} & F_{n-7} & F_{n-8} & \cdots & 1 & 0 & 0 & -1 \\ F_{n-4} & F_{n-5} & F_{n-6} & F_{n-7} & \cdots & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & -1 & 0 \end{bmatrix} \end{aligned}$$

and hence

$$T_{n-3,n-5}T_{n-3,n-4}(\widetilde{T}_{n-4,n-2}\widetilde{T}_{n-3,n-2}T_{n-2,n-4}T_{n-2,n-3}B^{(2)}T_{n-3,n-2}T_{n-4,n-2})T_{n-4,n-3}T_{n-5,n-3}$$

has the form

$$\begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-6} & F_{n-5} & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-7} & F_{n-6} & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & 0 & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-9} & F_{n-8} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ F_{n-6} & F_{n-7} & F_{n-8} & F_{n-9} & \cdots & 1 & 1 & -1 & 0 \\ F_{n-5} & F_{n-6} & F_{n-7} & F_{n-8} & \cdots & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Note that both the last row and the last column contain only one nonzero entry (which is -1). Therefore, $\det(B^{(2)})$ equals the determinant of the $(n-2)\times(n-2)$ matrix

$$\begin{bmatrix} 1 & 1 & 1 & 2 & \cdots & F_{n-7} & F_{n-6} & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-8} & F_{n-7} & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & F_{n-9} & F_{n-8} & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & \cdots & F_{n-10} & F_{n-9} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{n-7} & F_{n-8} & F_{n-9} & F_{n-10} & \cdots & 1 & 1 & 0 & 0 & 0 \\ F_{n-6} & F_{n-7} & F_{n-8} & F_{n-9} & \cdots & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

which has a unique nonzero entry (which is -1) in the $(n-3)$-th row and the $(n-3)$-th column. As B_{n-6} is exactly the submatrix of the last matrix formed by the first $n-6$ rows and the first $n-6$ columns, we see that $\det(B^{(2)}) = \det(B_{n-6})$.

In view of the above,

$$\det(B_n) = \det(B^{(1)}) = \det(B^{(2)}) = \det(B_{n-6}).$$

This concludes our proof of Theorem 1.2. □

Proof of Theorem 1.3. Let C_n denote the matrix $[(\frac{|j-k|}{3}) - \delta_{jk}]_{1\le j,k\le n}$. It is easy to verify (1.3) for $1 \le n \le 6$.

Now, let $n \ge 7$. It suffices to prove that $\det(C_n) = \det(C_{n-6})$. It's routine to verify that $T_{n-2,n}\widetilde{T}_{n-1,n}(\widetilde{T}_{n,n-2}\widetilde{T}_{n,n-1}C_n\widetilde{T}_{n-1,n}\widetilde{T}_{n-2,n})\widetilde{T}_{n,n-1}T_{n,n-2}$ has the form

$$\begin{bmatrix} -1 & 1 & -1 & 0 & \cdots & (\frac{n-4}{3}) & (\frac{n-3}{3}) & (\frac{n-2}{3}) & 0 \\ 1 & -1 & 1 & -1 & \cdots & (\frac{n-5}{3}) & (\frac{n-4}{3}) & (\frac{n-3}{3}) & 0 \\ -1 & 1 & -1 & 1 & \cdots & (\frac{n-6}{3}) & (\frac{n-5}{3}) & (\frac{n-4}{3}) & 0 \\ 0 & -1 & 1 & -1 & \cdots & (\frac{n-7}{3}) & (\frac{n-6}{3}) & (\frac{n-5}{3}) & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ (\frac{n-4}{3}) & (\frac{n-5}{3}) & (\frac{n-6}{3}) & (\frac{n-7}{3}) & \cdots & -1 & 1 & -1 & 0 \\ (\frac{n-3}{3}) & (\frac{n-4}{3}) & (\frac{n-5}{3}) & (\frac{n-6}{3}) & \cdots & 1 & 0 & 0 & 0 \\ (\frac{n-2}{3}) & (\frac{n-3}{3}) & (\frac{n-4}{3}) & (\frac{n-5}{3}) & \cdots & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \end{bmatrix}.$$

As there is a unique nonzero entry (which is -1) in the last row of the last matrix, and

$$\det\left(T_{n-2,n}\widetilde{T}_{n-1,n}\widetilde{T}_{n,n-2}\widetilde{T}_{n,n-1}C_n\widetilde{T}_{n-1,n}\widetilde{T}_{n-2,n}\widetilde{T}_{n,n-1}T_{n,n-2}\right) = \det(C_n)$$

in light of (2.1), we have $-\det(C_n)=\det(C^{(1)})$, where

$$C^{(1)}=\begin{bmatrix} -1 & 1 & -1 & 0 & \cdots & \binom{\frac{n-5}{3}} & \binom{\frac{n-4}{3}} & \binom{\frac{n-3}{3}} & \binom{\frac{n-2}{3}} \\ 1 & -1 & 1 & -1 & \cdots & \binom{\frac{n-6}{3}} & \binom{\frac{n-5}{3}} & \binom{\frac{n-4}{3}} & \binom{\frac{n-3}{3}} \\ -1 & 1 & -1 & 1 & \cdots & \binom{\frac{n-7}{3}} & \binom{\frac{n-6}{3}} & \binom{\frac{n-5}{3}} & \binom{\frac{n-4}{3}} \\ 0 & -1 & 1 & -1 & \cdots & \binom{\frac{n-8}{3}} & \binom{\frac{n-7}{3}} & \binom{\frac{n-6}{3}} & \binom{\frac{n-5}{3}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \binom{\frac{n-5}{3}} & \binom{\frac{n-6}{3}} & \binom{\frac{n-7}{3}} & \binom{\frac{n-8}{3}} & \cdots & -1 & 1 & -1 & 0 \\ \binom{\frac{n-4}{3}} & \binom{\frac{n-5}{3}} & \binom{\frac{n-6}{3}} & \binom{\frac{n-7}{3}} & \cdots & 1 & -1 & 1 & -1 \\ \binom{\frac{n-3}{3}} & \binom{\frac{n-4}{3}} & \binom{\frac{n-5}{3}} & \binom{\frac{n-6}{3}} & \cdots & -1 & 1 & 0 & 0 \\ \binom{\frac{n-2}{3}} & \binom{\frac{n-3}{3}} & \binom{\frac{n-4}{3}} & \binom{\frac{n-5}{3}} & \cdots & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Similarly, $T_{n-3,n-1}\widetilde{T}_{n-2,n-1}(\widetilde{T}_{n-1,n-3}\widetilde{T}_{n-1,n-2}C^{(1)}\widetilde{T}_{n-2,n-1}\widetilde{T}_{n-3,n-1})\widetilde{T}_{n-1,n-2}T_{n-1,n-3}$ has the form

$$\begin{bmatrix} -1 & 1 & -1 & 0 & \cdots & \binom{\frac{n-5}{3}} & \binom{\frac{n-4}{3}} & \binom{\frac{n-3}{3}} & 0 \\ 1 & -1 & 1 & -1 & \cdots & \binom{\frac{n-6}{3}} & \binom{\frac{n-5}{3}} & \binom{\frac{n-4}{3}} & 0 \\ -1 & 1 & -1 & 1 & \cdots & \binom{\frac{n-7}{3}} & \binom{\frac{n-6}{3}} & \binom{\frac{n-5}{3}} & 0 \\ 0 & -1 & 1 & -1 & \cdots & \binom{\frac{n-8}{3}} & \binom{\frac{n-7}{3}} & \binom{\frac{n-6}{3}} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \binom{\frac{n-5}{3}} & \binom{\frac{n-6}{3}} & \binom{\frac{n-7}{3}} & \binom{\frac{n-8}{3}} & \cdots & -1 & 1 & -1 & 0 \\ \binom{\frac{n-4}{3}} & \binom{\frac{n-5}{3}} & \binom{\frac{n-6}{3}} & \binom{\frac{n-7}{3}} & \cdots & 1 & 0 & 0 & 0 \\ \binom{\frac{n-3}{3}} & \binom{\frac{n-4}{3}} & \binom{\frac{n-5}{3}} & \binom{\frac{n-6}{3}} & \cdots & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \end{bmatrix},$$

and hence $\det(C^{(1)})=-\det(C^{(2)})$, where

$$C^{(2)}=\begin{bmatrix} -1 & 1 & -1 & 0 & \cdots & \binom{\frac{n-6}{3}} & \binom{\frac{n-5}{3}} & \binom{\frac{n-4}{3}} & \binom{\frac{n-3}{3}} \\ 1 & -1 & 1 & -1 & \cdots & \binom{\frac{n-7}{3}} & \binom{\frac{n-6}{3}} & \binom{\frac{n-5}{3}} & \binom{\frac{n-4}{3}} \\ -1 & 1 & -1 & 1 & \cdots & \binom{\frac{n-8}{3}} & \binom{\frac{n-7}{3}} & \binom{\frac{n-6}{3}} & \binom{\frac{n-5}{3}} \\ 0 & -1 & 1 & -1 & \cdots & \binom{\frac{n-9}{3}} & \binom{\frac{n-8}{3}} & \binom{\frac{n-7}{3}} & \binom{\frac{n-6}{3}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \binom{\frac{n-6}{3}} & \binom{\frac{n-7}{3}} & \binom{\frac{n-8}{3}} & \binom{\frac{n-9}{3}} & \cdots & -1 & 1 & -1 & 0 \\ \binom{\frac{n-5}{3}} & \binom{\frac{n-6}{3}} & \binom{\frac{n-7}{3}} & \binom{\frac{n-8}{3}} & \cdots & 1 & -1 & 1 & -1 \\ \binom{\frac{n-4}{3}} & \binom{\frac{n-5}{3}} & \binom{\frac{n-6}{3}} & \binom{\frac{n-7}{3}} & \cdots & -1 & 1 & 0 & 0 \\ \binom{\frac{n-3}{3}} & \binom{\frac{n-4}{3}} & \binom{\frac{n-5}{3}} & \binom{\frac{n-6}{3}} & \cdots & 0 & -1 & 0 & 1 \end{bmatrix}.$$

It's routine to verify that

$$\widetilde{T}_{n-3,n-5}\widetilde{T}_{n-3,n-4}(T_{n-4,n-2}\widetilde{T}_{n-3,n-2}\widetilde{T}_{n-2,n-4}\widetilde{T}_{n-2,n-3}C^{(2)}\widetilde{T}_{n-3,n-2}\widetilde{T}_{n-4,n-2})\widetilde{T}_{n-4,n-3}\widetilde{T}_{n-5,n-3}$$

has the form

$$\begin{bmatrix} -1 & 1 & -1 & 0 & \cdots & \binom{n-6}{3} & \binom{n-5}{3} & 0 & 0 \\ 1 & -1 & 1 & -1 & \cdots & \binom{n-7}{3} & \binom{n-6}{3} & 0 & 0 \\ -1 & 1 & -1 & 1 & \cdots & \binom{n-8}{3} & \binom{n-7}{3} & 0 & 0 \\ 0 & -1 & 1 & -1 & \cdots & \binom{n-9}{3} & \binom{n-8}{3} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-6}{3} & \binom{n-7}{3} & \binom{n-8}{3} & \binom{n-9}{3} & \cdots & -1 & 1 & -1 & 0 \\ \binom{n-5}{3} & \binom{n-6}{3} & \binom{n-7}{3} & \binom{n-8}{3} & \cdots & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Since there is only one nonzero entry (which is -1) in the last row of the last matrix, with the aid of (2.1) we have

$$\det(C^{(2)}) = \begin{vmatrix} -1 & 1 & -1 & 0 & \cdots & \binom{n-7}{3} & \binom{n-6}{3} & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & \cdots & \binom{n-8}{3} & \binom{n-7}{3} & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & \cdots & \binom{n-9}{3} & \binom{n-8}{3} & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & \cdots & \binom{n-10}{3} & \binom{n-9}{3} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n-7}{3} & \binom{n-8}{3} & \binom{n-9}{3} & \binom{n-10}{3} & \cdots & -1 & 1 & 0 & 0 & 0 \\ \binom{n-6}{3} & \binom{n-7}{3} & \binom{n-8}{3} & \binom{n-9}{3} & \cdots & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 \end{vmatrix},$$

which has a unique nonzero entry (which is -1) in the $(n-3)$-th row and in the $(n-3)$-th column. As C_{n-6} is exactly the matrix formed by the first $n-6$ rows and the first $n-6$ columns in the last determinant, we have $\det(C^{(2)}) = \det(C_{n-6})$. Therefore

$$\det(C_n) = -\det(C^{(1)}) = \det(C^{(2)}) = \det(C_{n-6}).$$

This ends our proof of Theorem 1.3. □

Acknowledgments. The authors would like to thank the two anonymous referees for helpful comments.

References

[1] P. Barry, Sequence A166645 at OEIS (On-Line Encyclopedia of Integer Sequences), https://oeis.org/A166645.

[2] C. Kimberling, Sequence A025276 at OEIS (On-Line Encyclopedia of Integer Sequences), https://oeis.org/A025276.

[3] C. Krattenthaler, *Advanced determinant calculus: a complement*, Linear Algebra Appl. **411** (2005), 68–166.

[4] R. Robinson and G. Szegö, *Solutions to problem 3705*, Amer. Math. Monthly **43** (1936), no. 4, 246–259.

[5] J. Todd, *The problem of error in digital computation*, in: L.B. Rall (Ed.), Error in Digital Computation, Vol. I, John Wiley & Sons, Inc., 1965, pp. 3-41.

(Han Wang) Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China
E-mail address: hWang@smail.nju.edu.cn

(Zhi-Wei Sun, corresponding author) Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China
E-mail address: zwsun@nju.edu.cn