Included in: M. B. Nathanson (ed.), Combinatorial and Additive Number Theory VI, Springer Proc. Math. Stat. 464, Springer, Cham, 2025, pp. 413–460.

SERIES WITH SUMMANDS INVOLVING HARMONIC NUMBERS

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ABSTRACT. For each positive integer m, the mth order harmonic numbers are given by

$$H_n^{(m)} = \sum_{0<k\leqslant n} \frac{1}{k^m} \quad (n = 0, 1, 2, \ldots).$$

We discover exact values of some series involving harmonic numbers of order not exceeding four. For example, we conjecture that

$$\sum_{k=0}^{\infty} (6k+1) \frac{\binom{2k}{k}^3}{256^k} \left(H_{2k}^{(3)} - \frac{7}{64} H_k^{(3)} \right) = \frac{25\zeta(3)}{8\pi} - G,$$

where G denotes the Catalan constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$. This paper contains 70 conjectures posed by the author during 2022–2023.

1. Introduction

The usual harmonic numbers are those rational numbers

$$H_n = \sum_{0<k\leqslant n} \frac{1}{k} \quad (n = 0, 1, 2, \ldots).$$

For each $m \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, the harmonic numbers of order m are defined by

$$H_n^{(m)} = \sum_{0<k\leqslant n} \frac{1}{k^m} \quad (n \in \mathbb{N} = \{0, 1, 2, \ldots\}).$$

For any $m, n \in \mathbb{Z}^+$, we clearly have

$$\sum_{k=1}^{n} \frac{1}{(2k-1)^m} = \sum_{k=1}^{2n} \frac{1}{k^m} - \sum_{j=1}^{n} \frac{1}{(2j)^m} = H_{2n}^{(m)} - \frac{1}{2^m} H_n^{(m)}.$$

J. Wolstenholme [48] established two fundamental congruences for harmonic numbers:

$$H_{p-1} \equiv 0 \pmod{p^2} \quad\text{and}\quad H_{p-1}^{(2)} \equiv 0 \pmod{p}$$

Key words and phrases. Harmonic numbers, series for π, Dirichlet L-functions, combinatorial identities, congruences.

2020 *Mathematics Subject Classification.* Primary 11B65, 05A19; Secondary 11A07, 11B68.

The initial version of this paper was posted to arXiv in Oct. 2022 as a preprint with the ID arXiv:2210.07238.

for any prime $p>3$. For series and congruences involving harmonic numbers, one may consult [29, 36, 42], [40, Section 10.5], and the recent preprint [5] solving various conjectures of the author.

In 2012, K. N. Boyadzhiev [7] proved that

$$\sum_{k=0}^\infty\binom{2k}{k}H_kx^k=\frac{2}{\sqrt{1-4x}}\log\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}\quad\text{for } x\in\left(-\frac14,\frac14\right). \tag{1}$$

In 2016, H. Chen [8] deduced that

$$\sum_{k=0}^\infty\binom{2k}{k}H_{2k}x^k=\frac{1}{\sqrt{1-4x}}\log\frac{1+\sqrt{1-4x}}{2(1-4x)}\quad\text{for } x\in\left(-\frac14,\frac14\right). \tag{2}$$

It is well known that

$$2\arcsin\frac x2=\sum_{k=0}^\infty\frac{\binom{2k}{k}x^{2k+1}}{(2k+1)16^k}\quad\text{for } |x|\leqslant 2;$$

in particular,

$$\sum_{k=0}^\infty\frac{\binom{2k}{k}}{(2k+1)16^k}=\frac{\pi}{3}\quad\text{and}\quad\sum_{k=0}^\infty\frac{\binom{2k}{k}}{(2k+1)8^k}=\frac{\pi}{2\sqrt2}.$$

The author [27, Theorem 1.1(ii)] determined

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)16^k}\quad\text{and}\quad\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}}{(2k+1)16^k}$$

modulo p^2 for any prime $p>3$. By [6], we have

$$\left(\arcsin\frac x2\right)^3=3\sum_{k=0}^\infty\frac{\binom{2k}{k}x^{2k+1}}{(2k+1)16^k}\sum_{0\leqslant j<k}\frac{1}{(2j+1)^2} \tag{3}$$

and

$$\left(\arcsin\frac x2\right)^4=\frac32\sum_{k=1}^\infty\frac{H_{k-1}^{(2)}x^{2k}}{k^2\binom{2k}{k}} \tag{4}$$

for $|x|\leqslant 2$. In particular,

$$\sum_{k=0}^\infty\frac{\binom{2k}{k}}{(2k+1)16^k}\left(H_{2k}^{(2)}-\frac14H_k^{(2)}\right)=\frac{\pi^3}{648},$$
$$\sum_{k=0}^\infty\frac{\binom{2k}{k}}{(2k+1)8^k}\left(H_{2k}^{(2)}-\frac14H_k^{(2)}\right)=\frac{\sqrt2\,\pi^3}{384}.$$

and

$$\sum_{k=1}^\infty\frac{H_{k-1}^{(2)}}{k^2\binom{2k}{k}}=\frac{\pi^4}{1944}.$$

In view of (3), we have the following result.

Theorem 1. *If $|x| < 2$, then*

$$\frac{(\arcsin(x/2))^2}{\sqrt{4-x^2}} = \sum_{k=1}^{\infty} \frac{\binom{2k}{k} x^{2k}}{16^k} \left(H_{2k}^{(2)} - \frac{1}{4} H_k^{(2)} \right). \tag{5}$$

Proof. By taking derivatives of both sides of (3), we get

$$3\left(\arcsin\frac{x}{2}\right)^2 \times \frac{1/2}{\sqrt{1-(x/2)^2}} = 3\sum_{k=0}^{\infty} \frac{\binom{2k}{k} x^{2k}}{16^k} \sum_{0\leqslant j<k} \frac{1}{(2j+1)^2}$$

and hence

$$\frac{(\arcsin(x/2))^2}{\sqrt{4-x^2}} = \sum_{k=1}^{\infty} \frac{\binom{2k}{k} x^k}{16^k} \sum_{j=1}^{k} \frac{1}{(2j-1)^2},$$

which is equivalent to (5). □

Motivated by the Ramanujan series

$$\sum_{k=0}^{\infty} (6k+1) \frac{\binom{2k}{k}^3}{(-512)^k} = \frac{2\sqrt{2}}{\pi} \tag{6}$$

(cf. [25]), L. Long [23] conjectured the congruence

$$\sum_{k=0}^{(p-1)/2} (6k+1) \frac{\binom{2k}{k}^3}{(-512)^k} \sum_{j=1}^{k} \left(\frac{1}{(2j-1)^2} - \frac{1}{16j^2} \right) \equiv 0 \pmod{p} \tag{7}$$

for any odd prime p, which was confirmed by H. Swisher [43] in 2015. Note that (7) can be rewritten as

$$\sum_{k=0}^{(p-1)/2} (6k+1) \frac{\binom{2k}{k}^3}{(-512)^k} \left(H_{2k}^{(2)} - \frac{5}{16} H_k^{(2)} \right) \equiv 0 \pmod{p}.$$

In 2022 C. Wei [46] deduced the two identities

$$\sum_{k=0}^{\infty} (6k+1) \frac{\binom{2k}{k}^3}{(-512)^k} \left(H_{2k}^{(2)} - \frac{5}{16} H_k^{(2)} \right) = -\frac{\sqrt{2}}{48}\pi$$

and

$$\sum_{k=0}^{\infty} (6k+1) \frac{\binom{2k}{k}^3}{256^k} \left(H_{2k}^{(2)} - \frac{5}{16} H_k^{(2)} \right) = \frac{\pi}{12}$$

conjectured by V.J.W. Guo and X. Lian [19], as well as their q-analogues.

Motivated by Bauer's series

$$\sum_{k=0}^{\infty} (4k+1) \frac{\binom{2k}{k}^3}{(-64)^k} = \frac{2}{\pi} \tag{8}$$

and Ramanujan's series

$$\sum_{k=0}^{\infty} (8k+1) \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{48^{2k}} = \frac{2\sqrt{3}}{\pi}, \tag{9}$$

Wei and G. Ruan [47] proved the two new identities

$$\sum_{k=1}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\sum_{j=1}^{2k}\frac{(-1)^j}{j^2}=\frac{\pi}{12}$$

and

$$\sum_{k=1}^{\infty}(8k+1)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\sum_{j=1}^{k}\left(\frac{1}{(2j-1)^2}-\frac{1}{36j^2}\right)=\frac{\sqrt{3}\,\pi}{54},$$

i.e.,

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\left(H_{2k}^{(2)}-\frac{1}{2}H_k^{(2)}\right)=-\frac{\pi}{12} \tag{10}$$

and

$$\sum_{k=0}^{\infty}(8k+1)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\left(H_{2k}^{(2)}-\frac{5}{18}H_k^{(2)}\right)=\frac{\sqrt{3}\,\pi}{54}. \tag{11}$$

In 1997 van Hamme [44] thought that series for powers of $\pi=\Gamma(1/2)^2$ should have their p-adic analogues involving the p-adic Gamma function $\Gamma_p(x)$, where p is an odd prime. Note that for any odd prime p we have

$$\Gamma_p\left(\frac{1}{2}\right)^2=(-1)^{(p+1)/2}=-\left(\frac{-1}{p}\right),$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol. The author [38, 39, 41] found many new series for powers of π motivated by related congruences. However, van Hamme's philosophy fails for some Ramanujan-type series for $1/\pi$. For example, T. Huber, D. Schultz and D. Ye [22] used modular forms to obtain that

$$\sum_{k=0}^{\infty}(6k+1)\frac{a_k}{16^k}=\frac{16}{\pi},$$

where $a_0=1,\ a_1=4,\ a_2=20$ and

$$(n+1)^3a_{n+1}=4(2n+1)(2n^2+2n+1)a_n-16n(4n^2+1)a_{n-1}+8(2n-1)^3a_{n-2}$$

for all $n=2,3,\ldots$; but for a general odd prime p we even cannot find any pattern for $\sum_{k=0}^{p-1}(6k+1)a_k/16^k$ modulo p.

The Bernoulli numbers $B_0,B_1,B_2,\ldots$ are defined by

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}\quad(0<|x|<2\pi).$$

Equivalently,

$$B_0=1,\ \text{and}\ \sum_{k=0}^{n}\binom{n+1}{k}B_k=0\ \text{ for } n=1,2,3,\ldots.$$

In 1900 J.W.L. Glaiser [12] proved that

$$H_{p-1}\equiv-\frac{p^2}{3}B_{p-3}\pmod{p^3}\quad\text{and}\quad H_{p-1}^{(2)}\equiv\frac{2}{3}pB_{p-3}\pmod{p^2}$$

for any prime $p > 3$. The Bernoulli polynomials are given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad (n \in \mathbb{N}).$$

The Euler numbers $E_0, E_1, E_2, \ldots$ are defined by

$$\frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} \quad \left(|x| < \frac{\pi}{2}\right).$$

Clearly $E_{2n+1} = 0$ for all $n \in \mathbb{N}$. It is also known that

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} = 0 \quad \text{for each } n \in \mathbb{N}.$$

The Euler polynomials are given by

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k} \quad (n \in \mathbb{N}).$$

The author [28, 32] first observed that many Ramanujan-type series have corresponding congruences involving Bernoulli or Euler polynomials.

Now we introduce some notations throughout this paper. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{with } \Re(s) > 1.$$

The Dirichlet beta function is given by

$$\beta(m) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^m} \quad (m = 1, 2, 3, \ldots).$$

Note that $G = \beta(2)$ is the Catalan constant. We also adopt the notation

$$K := L\left(2, \left(\frac{-3}{\cdot}\right)\right) = \sum_{n=1}^{\infty} \frac{\left(\frac{k}{3}\right)}{k^2}$$

with $(\frac{-3}{\cdot})$ the Kronecker symbol. For a prime p and an integer $a \not\equiv 0 \pmod{p}$, we use $q_p(a)$ to denote the Fermat quotient $(a^{p-1} - 1)/p$. Many congruences in later sections involve Fermat quotients.

In Sections 2–4, we will propose 70 new conjectures on series and related congruences with summands involving not only harmonic numbers of order at most four, but also products of several binomial coefficients. All the conjectures have been checked via `Mathematica`.

2. Series with summands containing one or two binomial coefficients

Conjecture 1 (2022-10-12)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\left(H_{2k-1}^{(2)}-\frac{123}{16}H_{k-1}^{(2)}\right)=\frac{451}{40}\zeta(5)-\frac{14}{15}\pi^2\zeta(3). \tag{12}$$

(ii) *For any prime $p>5$, we have*

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^3\binom{2k}{k}}\left(16H_{2k-1}^{(2)}-123H_{k-1}^{(2)}\right)\equiv -542B_{p-5} \pmod{p} \tag{13}$$

and

$$p\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}\left(16H_{2k}^{(2)}-123H_{k}^{(2)}\right)\equiv 192\frac{H_{p-1}}{p^2} \pmod{p^2}. \tag{14}$$

Remark 1. In 1979 R. Apéry [3] proved the irrationality of $\zeta(3)=\sum_{n=1}^{\infty}1/n^3$ via the identity

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}=\frac{2}{5}\zeta(3).$$

In 2014 the author [33] proved the congruence

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\equiv 2B_{p-3} \pmod{p}$$

for any prime $p>5$. The author's conjectural identity (cf. [34])

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}(H_{2k}+4H_k)=\frac{2\pi^4}{75}$$

was proved by W. Chu [9] as well as K. C. Au [4, Prop. 7.14]. After seeing an earlier arXiv version of this paper, Au [5, Corollary 2.9] confirmed the author's conjectural identity (12).

Conjecture 2 (2022-11-14)**.** *We have the identity*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8^k}\left(H_{2k}^{(3)}-\frac{1}{8}H_{k}^{(3)}\right)=\frac{35\sqrt{2}}{64}\zeta(3)-\frac{\sqrt{2}}{8}\pi G. \tag{15}$$

Remark 2. Applying (1) and (2) with $x=1/8$, we see that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8^k}H_k=-\sqrt{2}\log(12-8\sqrt{2})\ \text{and}\ \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8^k}H_{2k}=\frac{\log(3/2+\sqrt{2})}{\sqrt{2}}.$$

In contrast with (15), we have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8^k}\left(H_{2k}^{(2)}-\frac{1}{4}H_{k}^{(2)}\right)=\frac{\pi^2}{16\sqrt{2}}$$

by applying (5) with $x = \sqrt{2}$.

Conjecture 3 (2022-11-14)**.** *We have the identity*

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k} \left(H_{2k}^{(3)} - \frac{1}{8} H_k^{(3)} \right) = \frac{2\zeta(3)}{3\sqrt{3}} - \frac{\pi K}{8}. \tag{16}$$

Remark 3. Applying (1) and (2) with $x = 1/8$, we see that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k} H_k = -\frac{2}{\sqrt{3}} \log(84 - 48\sqrt{3}) \text{ and } \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k} H_{2k} = \frac{\log((7 + 4\sqrt{3})/9)}{\sqrt{3}}.$$

In contrast with (16), we have

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k} \left(H_{2k}^{(2)} - \frac{1}{4} H_k^{(2)} \right) = \frac{\pi^2}{36\sqrt{3}}$$

by applying (5) with $x = 1$.

Conjecture 4 (2023-05-28)**.** (i) *We have*

$$\sum_{k=1}^{\infty} \frac{H_{3k} - H_k}{k 2^k \binom{3k}{k}} = \frac{2}{5}(G + \log^2 2) - \frac{\pi^2}{24} \tag{17}$$

and

$$\sum_{k=1}^{\infty} \frac{H_{3k} - H_k}{k^2 2^k \binom{3k}{k}} = \frac{11}{4}\zeta(3) - \frac{\pi^2}{24} \log 2 - \pi G. \tag{18}$$

(ii) *For any prime $p > 5$ with $p \equiv 1 \pmod{4}$, we have*

$$p \sum_{k=1}^{p-1} \frac{H_{3k} - H_k}{k 2^k \binom{3k}{k}} \equiv \frac{7}{10} q_p(2) \pmod{p}. \tag{19}$$

Also, for each odd prime p we have

$$p^2 \sum_{k=1}^{p-1} \frac{H_{3k} - H_k}{k^2 2^k \binom{3k}{k}} \equiv -\frac{q_p(2)}{4} \pmod{p}. \tag{20}$$

Remark 4. The author's conjectural identities

$$\sum_{k=1}^{\infty} \frac{H_{2k} - H_k}{k 2^k \binom{3k}{k}} = \frac{3}{10} \log^2 2 + \frac{\pi}{20} \log 2 - \frac{\pi^2}{60}$$

and

$$\sum_{k=1}^{\infty} \frac{H_{2k} - H_k}{k^2 2^k \binom{3k}{k}} = \frac{33}{32}\zeta(3) + \frac{\pi^2}{24} \log 2 - \frac{\pi G}{2}$$

(cf. [40, Conjecture 10.61]) were confirmed by Au [4] in 2022.

Conjecture 5 (2023-05-28)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{25k-3}{2^k\binom{3k}{k}}(H_{3k}-8H_{2k}+7H_k)=2G-2(\pi+9)\log 2. \tag{21}$$

(ii) *For any odd prime p, we have the congruence*

$$p^2\sum_{k=1}^{p-1}\frac{25k-3}{2^k\binom{3k}{k}}(H_{3k}-8H_{2k}+7H_k)\equiv-\left(\frac{-1}{p}\right)\frac{9}{4} \pmod{p}. \tag{22}$$

Remark 5. In 1974 R. W. Gosper announced the identity

$$\sum_{k=0}^{\infty}\frac{25k-3}{2^k\binom{3k}{k}}=\frac{\pi}{2},$$

an elegant proof of which can be found in [1].

Conjecture 6 (2023-05-28)**.** (i) *We have*

$$\sum_{k=0}^{\infty}\frac{2^k\binom{3k}{k}}{27^k}H_{2k}=\frac{3}{2}(1+\sqrt{3})\log\left(1+\frac{\sqrt{3}}{3}\right)-\sqrt{3}\log 2 \tag{23}$$

and

$$\sum_{k=0}^{\infty}\frac{2^k\binom{3k}{k}}{27^k}H_{3k}=\frac{1+\sqrt{3}}{2}\left(2\log(1+\sqrt{3})-\frac{\log 3}{2}\right)-\sqrt{3}\log 2. \tag{24}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=(p+1)/2}^{p-1}\binom{3k}{k}\left(\frac{2}{27}\right)^k\equiv\frac{1-(\frac{3}{p})}{3} \pmod{p}. \tag{25}$$

Remark 6. For any positive integer n, we clearly have

$$H_n=\sum_{k=0}^{n-1}\frac{1}{k+1}=\sum_{k=0}^{n-1}\int_0^1 t^k dt=\int_0^1\sum_{k=0}^{n-1}t^k dt=\int_0^1\frac{1-t^n}{1-t}dt.$$

Using this trick we can deduce that

$$\sum_{k=0}^{\infty}\frac{2^k\binom{3k}{k}}{27^k}H_k=\frac{3}{4}\left((1-\sqrt{3})\log 4-(1+\sqrt{3})\log 3\right)+2\sqrt{3}\log(1+\sqrt{3}).$$

Conjecture 7 (2023-05-28)**.** *We have*

$$\sum_{k=0}^{\infty}\binom{3k}{k}\left(\frac{3+\sqrt{5}}{54}\right)^k(H_{3k}-H_{2k})=\phi(\log 3-2\log\phi), \tag{26}$$

where ϕ denotes the golden ratio $(1+\sqrt{5})/2\approx 1.618\ldots$.

Remark 7. `Mathematica` yields that

$$\sum_{k=0}^{\infty}\binom{3k}{k}\left(\frac{4x}{27}\right)^k=\frac{1}{\sqrt{1-x}}\cos\frac{\arcsin\sqrt{x}}{3}$$

for any $x\in(-1,1)$. Applying this with $x=((1+\sqrt{5})/4)^2$ we obtain that

$$\sum_{k=0}^{\infty}\binom{3k}{k}\left(\frac{3+\sqrt{5}}{54}\right)^k=\frac{\cos(\pi/10)}{\sqrt{(5-\sqrt{5})/8}}=\frac{\sqrt{(5+\sqrt{5})/8}}{\sqrt{(5-\sqrt{5})/8}}=\phi.$$

Conjecture 8 (2023-05-30)**.** *If* $(1-\sqrt{2})/2\leqslant x<1/2$*, then*

$$\sum_{k=0}^{\infty}\binom{4k}{2k}\left(\frac{x(1-x)}{4}\right)^k(2H_{4k}-3H_{2k}+H_k)=\frac{\sqrt{1-x}}{2x-1}\log(1-x). \tag{27}$$

Remark 8. For any $x\in(-1,1)$, we have

$$\sum_{k=0}^{\infty}\binom{4k}{2k}\left(\frac{x}{16}\right)^k=\sqrt{\frac{1+\sqrt{1-x}}{2(1-x)}}$$

which can be proved directly or via `Mathematica`. In particular,

$$\sum_{k=0}^{\infty}\binom{4k}{2k}\left(\frac{3}{64}\right)^k=\sqrt{3}.$$

For $x\in(-1,1)$, we obviously have

$$\sum_{k=0}^{\infty}\binom{2k}{k}H_k\left(\left(\frac{x}{4}\right)^k+\left(-\frac{x}{4}\right)^k\right)=2\sum_{k=0}^{\infty}\binom{4k}{2k}H_{2k}\left(\frac{x}{4}\right)^{2k}$$

and

$$\sum_{k=0}^{\infty}\binom{2k}{k}H_{2k}\left(\left(\frac{x}{4}\right)^k+\left(-\frac{x}{4}\right)^k\right)=2\sum_{k=0}^{\infty}\binom{4k}{2k}H_{4k}\left(\frac{x}{4}\right)^{2k},$$

and hence we may find closed formulas for the two series

$$\sum_{k=0}^{\infty}\binom{4k}{2k}H_{2k}\left(\frac{x}{4}\right)^{2k}\quad\text{and}\quad\sum_{k=0}^{\infty}\binom{2k}{k}H_{4k}\left(\frac{x}{4}\right)^{2k}$$

by using (1) and (2). In particular,

$$\sum_{k=0}^{\infty}\binom{4k}{2k}\left(\frac{3}{64}\right)^kH_{2k}=2(1+\sqrt{3})\log(1+\sqrt{3})-(1+3\sqrt{3})\log 2+\frac{\sqrt{3}-1}{2}\log 3$$

and

$$\sum_{k=0}^{\infty}\binom{4k}{2k}\left(\frac{3}{64}\right)^kH_{4k}=(2+\sqrt{3})\log(1+\sqrt{3})-(1+\sqrt{3})\log 2+\frac{\sqrt{3}-1}{4}\log 3.$$

With the aid of Mathematica, we obtain that

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{4k}{2k}\left(\frac{3}{64}\right)^k H_k &= \sum_{k=1}^{\infty}\binom{4k}{2k}\left(\frac{3}{64}\right)^k\int_0^1\frac{1-t^k}{1-t}dt\\ &=\int_0^1\frac{1}{1-t}\sum_{k=1}^{\infty}\binom{4k}{2k}\left(\frac{3}{64}\right)^k(1-t^k)dt\\ &=\int_0^1\frac{1}{1-t}\left(\sqrt{3}-\frac{\sqrt{2+\sqrt{4-3t}}}{\sqrt{4-3t}}\right)dt\\ &=\log\frac{2+\sqrt{3}}{3}+\sqrt{3}\log\frac{7+4\sqrt{3}}{8}\\ &=(2+4\sqrt{3})\log(1+\sqrt{3})-\log 3-(1+5\sqrt{3})\log 2.\end{aligned}$$

Conjecture 9 (2022-11-14). *We have the identity*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2}{32^k}\left(H_{2k}^{(2)}-\frac{1}{4}H_k^{(2)}\right)=\Gamma\left(\frac14\right)^2\frac{\pi^2-8G}{32\pi\sqrt{\pi}},\tag{28}$$

where $\Gamma(x)$ *is the well-known Gamma function.*

Remark 9. In contrast with (28), Mathematica yields that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2}{32^k}=\frac{\Gamma(1/4)}{\sqrt{2\pi}\,\Gamma(3/4)}=\frac{\Gamma(1/4)^2}{2\pi\sqrt{\pi}}.$$

For any odd prime p, Z.-H. Sun [26] proved that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\equiv\begin{cases}2x-p/(2x)\pmod{p^2} & \text{if } p=x^2+y^2\ (x,y\in\mathbb{Z}) \text{ with } 4\mid x-1,\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4),\end{cases}$$

which was previously conjectured by the author (cf. [27, Conjecture 5.5]).

Conjecture 10 (2022-12-30). *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}(3H_{3k}-H_k)=\left(\log\frac89\right)\sum_{k=0}^{\infty}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}.\tag{29}$$

Remark 10. For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\pmod{p^2}$$

by [31, Corollary 1.4], and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv\begin{cases}\binom{(2p-2)/3}{(p-1)/3}\pmod{p^2} & \text{if } p\equiv 1\pmod 3),\\ p/\binom{(2p+2)/3}{(p+1)/3}\pmod{p^2} & \text{if } p\equiv 2\pmod 3),\end{cases}$$

as conjectured by the author [28, Conjecture 5.13] and proved by C. Wang and Sun [45, Theorem 1.2].

Conjecture 11 (2023-09-12)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{48^k}{k(2k-1)\binom{2k}{k}\binom{4k}{2k}}\left(6H_{4k-1}-9H_{2k-1}+2H_{k-1}+\frac{6}{2k-1}\right)=\frac{2\pi^3}{\sqrt{3}}. \tag{30}$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{(p-1)/2}\frac{k\binom{2k}{k}\binom{4k}{2k}}{(2k+1)48^k}\left(6H_{4k}-9H_{2k}+2H_k-\frac{6}{2k+1}\right)\equiv 0 \pmod{p} \tag{31}$$

and

$$\sum_{k=1}^{p-1}\frac{k\binom{2k}{k}\binom{4k}{2k}}{(2k+1)48^k}\left(6H_{4k}-9H_{2k}+2H_k-\frac{6}{2k+1}\right)\equiv \frac{5p}{12}B_{p-2}\left(\frac{1}{3}\right) \pmod{p^2}. \tag{32}$$

Remark 11. The author's conjectural identity

$$\sum_{k=1}^{\infty}\frac{48^k}{k(2k-1)\binom{2k}{k}\binom{4k}{2k}}=\frac{15}{2}K$$

(cf. [34]) was confirmed by Au [5].

Conjecture 12 (2022-12-30)**.** *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{4k}{2k}\binom{2k}{k}}{128^k}(2H_{4k}-H_{2k})=\frac{(\log 2)\sqrt{\pi}}{2\Gamma(5/8)\Gamma(7/8)}. \tag{33}$$

Remark 12. `Mathematica` yields the identity

$$\sum_{k=0}^{\infty}\frac{\binom{4k}{2k}\binom{2k}{k}}{128^k}=\frac{\sqrt{\pi}}{\Gamma(5/8)\Gamma(7/8)}.$$

By [31, Corollary 1.3], for any prime $p\equiv 5,7 \pmod{8}$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{128^k}\equiv 0 \pmod{p^2}.$$

Conjecture 13 (2022-12-30)**.** *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{4k}{2k}\binom{2k}{k}}{72^k}(2H_{4k}-H_{2k})=(\log 3)\sum_{k=0}^{\infty}\frac{\binom{4k}{2k}\binom{2k}{k}}{72^k} \tag{34}$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{4k}{2k}\binom{2k}{k}}{576^k}(2H_{4k}-H_{2k})=\frac{1}{2}\left(\log\frac{9}{8}\right)\sum_{k=0}^{\infty}\frac{\binom{4k}{2k}\binom{2k}{k}}{576^k}. \tag{35}$$

Remark 13. Let $p>3$ be a prime. By [31, Corollary 1.4],

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{576^k}\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{72^k} \pmod{p^2}.$$

We also have

$$\left(\frac{6}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{72^k}$$
$$\equiv\begin{cases}2x-\frac{p}{2x}\pmod{p^2}&\text{if }p=x^2+y^2\ (x,y\in\mathbb{Z}\ \&\ 4\mid x-1),\\ \frac{2p}{3\binom{(p+1)/2}{(p+1)/4}}\pmod{p^2}&\text{if }p\equiv 3\pmod 4),\end{cases}$$

as conjectured by the author [28, Conjecture 5.14(iii)] and proved by Wang and Sun [45, Theorem 5.2 and Remark 5.2].

Conjecture 14 (2022-12-30)**.** *We have*

$$\sum_{k=0}^\infty\frac{\binom{4k}{2k}\binom{2k}{k}}{(-192)^k}(2H_{4k}-H_{2k})=\frac{1}{2}\left(\log\frac{3}{4}\right)\sum_{k=0}^\infty\frac{\binom{4k}{2k}\binom{2k}{k}}{(-192)^k}.\tag{36}$$

Remark 14. Let $p>3$ be a prime. By [31, Corollary 1.4], we have

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(-192)^k}\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{48^k}\pmod{p^2}.$$

If $p=x^2+3y^2$ with $x,y\in\mathbb{Z}$ and $x\equiv 1\pmod 3$, then

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{48^k}\equiv 2x-\frac{p}{2x}\pmod{p^2},$$

as conjectured by the author [28, Conjecture 5.14] and confirmed by G.-S. Mao and H. Pan [24]. If $p\equiv 2\pmod 3$, then

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{48^k}\equiv\frac{3p}{2\binom{(p+1)/2}{(p+1)/6}}\pmod{p^2}$$

as conjectured by the author [28, Conjecture 5.14] and confirmed by Wang and Sun [45].

Conjecture 15 (2022-12-30)**.** *We have*

$$\sum_{k=0}^\infty\frac{\binom{4k}{2k}\binom{2k}{k}}{(-4032)^k}(2H_{4k}-H_{2k})=\frac{1}{2}\left(\log\frac{63}{64}\right)\sum_{k=0}^\infty\frac{\binom{4k}{2k}\binom{2k}{k}}{(-4032)^k}.\tag{37}$$

Remark 15. Let $p>3$ be a prime with $p\neq 7$. By [31, Corollary 1.4],

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(-4032)^k}\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{63^k}\pmod{p^2}.$$

The author [28, Conjecture 5.14(ii)] conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{63^k}$$
$$\equiv\begin{cases}(\frac{p}{3})(2x-\frac{p}{2x}) \pmod{p^2} & \text{if } p=x^2+7y^2 \text{ with } x,y\in\mathbb{Z} \text{ and } (\frac{x}{7})=1,\\ 0 \pmod{p} & \text{if } (\frac{p}{7})=-1.\end{cases}$$

Conjecture 16 (2022-12-30)**.** *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k}(6H_{6k}-3H_{3k}-2H_{2k}+H_k)=\frac{(\log 2)\sqrt{\pi}}{\Gamma(7/12)\Gamma(11/12)}. \tag{38}$$

Remark 16. `Mathematica` yields the identity

$$\sum_{k=0}^{\infty}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k}=\frac{\sqrt{\pi}}{\Gamma(7/12)\Gamma(11/12)}.$$

By [31, Corollary 1.3], for any prime $p>3$ with $p\equiv 3 \pmod{4}$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k}\equiv 0 \pmod{p^2}.$$

3. Series and congruences with summands containing 3 or 4 binomial coefficients

In 1993 D. Zeilberger [49] used the WZ method to establish the identity

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{6}.$$

The author [35] proved that

$$\sum_{k=1}^{(p-1)/2}\frac{21k-8}{k^3\binom{2k}{k}^3}\equiv-\left(\frac{-1}{p}\right)4E_{p-3} \pmod{p}$$

for any prime $p>3$.

Conjecture 17 (2022-10-11)**.** (i) *We have the identity*

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(2)}-\frac{25}{8}H_{k-1}^{(2)}\right)=\frac{47\pi^4}{2880}. \tag{39}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=1}^{(p-1)/2}(21k+8)\binom{2k}{k}^3\left(H_{2k}^{(2)}-\frac{25}{8}H_k^{(2)}\right)\equiv 32p\left(\frac{-1}{p}\right)E_{p-3} \pmod{p^2} \tag{40}$$

and

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3\left(H_{2k}^{(2)}-\frac{25}{8}H_k^{(2)}\right)\equiv-48H_{p-1}+\frac{246}{5}p^4B_{p-5}\pmod{p^5}. \tag{41}$$

(iii) *For each prime $p>5$, we have*

$$\sum_{k=1}^{p-1}\binom{2k}{k}^3(2(21k+8)(H_{2k}-H_k)+7)\equiv-112pH_{p-1}\pmod{p^5}. \tag{42}$$

Remark 17. After seeing an earlier arXiv version of this paper, Au [5, Corollary 2.3] confirmed the author's conjectural identity (39), and proved an identity (after the proof of [5, Theorem 2.2]) which has the equivalent form

$$\sum_{k=1}^\infty\frac{(21k-8)(H_{2k-1}-H_{k-1})-7/2}{k^3\binom{2k}{k}^3}=\zeta(3).$$

Conjecture 18 (2022-10-11)**.** (i) *We have*

$$\sum_{k=1}^\infty\frac{21k-8}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(3)}+\frac{43}{8}H_{k-1}^{(3)}\right)=\frac{711}{28}\zeta(5)-\frac{29}{14}\pi^2\zeta(3). \tag{43}$$

(ii) *For any prime $p>7$, we have*

$$\sum_{k=0}^{(p-1)/2}(21k+8)\binom{2k}{k}^3\left(H_{2k}^{(3)}+\frac{43}{8}H_k^{(3)}\right)\equiv32\left(\frac{-1}{p}\right)E_{p-3}\pmod{p} \tag{44}$$

and

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3\left(H_{2k}^{(3)}+\frac{43}{8}H_k^{(3)}\right)\equiv-\frac{120}{7}pB_{p-3}\pmod{p^2}. \tag{45}$$

Remark 18. The identity (43) looks quite challenging.

Conjecture 19 (2022-10-13)**.** (i) *We have*

$$\sum_{k=1}^\infty\frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(2)}-\frac{5}{4}H_{k-1}^{(2)}\right)=-2\beta(4). \tag{46}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{(p-1)/2}(3k+1)\frac{\binom{2k}{k}^3}{(-8)^k}\left(H_{2k}^{(2)}-\frac{5}{4}H_k^{(2)}\right)\equiv\left(\frac{2}{p}\right)\frac{p}{4}E_{p-3}\left(\frac14\right)\pmod{p^2} \tag{47}$$

and

$$\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}^3}{(-8)^k}\left(H_{2k}^{(2)}-\frac{5}{4}H_k^{(2)}\right)\equiv pE_{p-3}\pmod{p^2}. \tag{48}$$

(iii) *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-8)^k}(2(3k+1)(H_{2k}-H_k)+1)\equiv\left(\frac{-1}{p}\right)2^{p-1}\pmod{p^2}\qquad(49)$$

and

$$\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}^3}{(-8)^k}\left(H_{2k}^{(3)}+\frac{7}{8}H_k^{(3)}\right)\equiv 0\pmod{p}.\qquad(50)$$

Remark 19. In 2008, J. Guillera [14] used the WZ method to find the identity

$$\sum_{k=1}^{\infty}\frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3}=-2G.$$

The identity (46) provides a fast converging series for computing the constant $\beta(4)$. We are unable to find the exact values of the series

$$\sum_{k=1}^{\infty}\frac{(-8)^k}{k^3\binom{2k}{k}^3}(2(3k-1)(H_{2k-1}-H_{k-1})-1)$$

and

$$\sum_{k=1}^{\infty}\frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(3)}+\frac{7}{8}H_{k-1}^{(3)}\right).$$

Conjecture 20 (2022-10-11)**.** (i) *We have the identity*

$$\sum_{k=1}^{\infty}\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(2)}-\frac{5}{4}H_{k-1}^{(2)}\right)=\frac{\pi^4}{24}.\qquad(51)$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}(3k+1)\frac{\binom{2k}{k}^3}{16^k}\left(H_{2k}^{(2)}-\frac{5}{4}H_k^{(2)}\right)\equiv 2p\left(\frac{-1}{p}\right)E_{p-3}\pmod{p^2}\qquad(52)$$

and

$$\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}^3}{16^k}\left(H_{2k}^{(2)}-\frac{5}{4}H_k^{(2)}\right)\equiv\frac{7}{6}p^2B_{p-3}\pmod{p^3}.\qquad(53)$$

(iii) *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^3}{16^k}(2(3k+1)(H_{2k}-H_k)+1)\equiv\frac{4}{3}p\,q_p(2)-\frac{2}{3}p^2q_p(2)^2\pmod{p^3}.\qquad(54)$$

If $p\equiv 2\pmod 3$, then

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{16^k}\left(H_{2k}^{(2)}-\frac{5}{4}H_k^{(2)}\right)\equiv 0\pmod{p}.\qquad(55)$$

Remark 20. In 2008 Guillera [14, Identity 1] used the WZ method to establish the identity
$$\sum_{k=1}^\infty\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{2}.$$
Two q-analogues of this identity were given by Q.-H. Hou, C. Krattenthaler and Z.-W. Sun [21]. Guo and Lian [19] proved that the two sides of (53) are congruent modulo p^2 for any prime $p>3$. After seeing an earlier arXiv version of this paper, Au [5] confirmed the author's conjectural identity (51), and proved an identity (after the proof of [5, Corollary 2.3]) which has the equivalent form
$$\sum_{k=1}^\infty\frac{16^k}{k^3\binom{2k}{k}^3}\left((3k-1)(H_{2k-1}-H_{k-1})-\frac12\right)=\frac{\pi^2}{3}\log 2+\frac76\zeta(3).$$

Conjecture 21 (2022-10-11)**.** (i) *We have*
$$\sum_{k=1}^\infty\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(3)}+\frac78H_{k-1}^{(3)}\right)=\frac{\pi^2}{2}\zeta(3).\tag{56}$$

(ii) *For any odd prime p, we have*
$$\sum_{k=1}^{(p-1)/2}(3k+1)\frac{\binom{2k}{k}^3}{16^k}\left(H_{2k}^{(3)}+\frac78H_k^{(3)}\right)\equiv 2\left(\frac{-1}{p}\right)E_{p-3}\pmod{p}\tag{57}$$
and
$$\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}^3}{16^k}\left(H_{2k}^{(3)}+\frac78H_k^{(3)}\right)\equiv\frac32pB_{p-3}\pmod{p^2}.\tag{58}$$

Remark 21. Conjecture 21 looks more challenging than Conjecture 20.

Conjecture 22 (2022-10-16)**.** (i) *We have*
$$\sum_{k=1}^\infty\frac{(4k-1)(-64)^k}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(2)}-\frac12H_{k-1}^{(2)}\right)=-16\beta(4).\tag{59}$$

(ii) *For any prime $p>3$, we have*
$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\left(H_{2k}^{(2)}-\frac12H_k^{(2)}\right)\equiv pE_{p-3}\pmod{p^2},\tag{60}$$
$$\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\left(H_{2k}^{(2)}-\frac12H_k^{(2)}\right)\equiv pE_{p-3}\pmod{p^2}.\tag{61}$$

Remark 22. Guillera [14] proved that
$$\sum_{k=1}^\infty\frac{(4k-1)(-64)^k}{k^3\binom{2k}{k}^3}=-16G.$$

The congruence (60) is motivated by (10). After seeing an earlier arXiv version of this paper, Au [5, Corollary 2.11] confirmed the author's conjectural identity (59). It seems that for any $m,n\in\mathbb{Z}^+$ and odd prime p, we have

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^n}{(-4)^{kn}}\left(H_{2k}^{(2m)}-\frac{H_k^{(2m)}}{2^{2m-1}}\right)\equiv 0 \pmod{p}. \tag{62}$$

Conjecture 23 (2022-12-05)**.** (i) *We have*

$$\sum_{k=1}^\infty\frac{8^k((10k-3)(H_{2k-1}-H_{k-1})-1)}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{7}{2}\zeta(3) \tag{63}$$

and

$$\sum_{k=1}^\infty\frac{8^k((10k-3)(H_{3k-1}-H_{k-1})-8/3)}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{2\pi^2\log 2+7\zeta(3)}{4}. \tag{64}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k}((10k+3)(H_{2k}-H_k)+1)\equiv\frac{63}{8}p^3B_{p-3} \pmod{p^4} \tag{65}$$

and

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k}(3(10k+3)(H_{3k}-H_k)+8)\equiv 9p\,q_p(2)-\frac{9}{2}p^2q_p(2)^2 \pmod{p^3}. \tag{66}$$

Remark 23. As conjectured by the author [28] and confirmed by Guillera and M. Rogers [18], we have

$$\sum_{k=1}^\infty\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2}.$$

Conjecture 24 (2022-12-05)**.** (i) *We have*

$$\sum_{k=1}^\infty\frac{(-27)^k((15k-4)(3H_{3k-1}-H_{k-1})-9)}{k^3\binom{2k}{k}^2\binom{3k}{k}}=-\frac{4\pi^3}{\sqrt{3}}. \tag{67}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}((15k+4)(3H_{3k}-H_k)+9)\equiv 9\left(\frac{p}{3}\right)+6p^2B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}. \tag{68}$$

Remark 24. As conjectured by the author [28] and confirmed by Kh. Hessami Pilehrood and T. Hessami Pilehrood [20], we have

$$\sum_{k=1}^\infty\frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}}=K.$$

Conjecture 25 (2023-08-26). (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-27)^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}\left((15k-4)(H_{2k-1}-H_{k-1})+\frac{1}{2k-1}\right)=-\frac{4\pi^3}{3\sqrt{3}}. \tag{69}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}\left((15k+4)(H_{2k}-H_k)+\frac{1}{2k+1}\right)\equiv\left(\frac{p}{3}\right)+\frac{4}{3}p^2B_{p-2}\left(\frac{1}{3}\right)\pmod{p^3}. \tag{70}$$

Remark 25. This conjecture is similar to Conjecture 24.

Conjecture 26 (2022-12-05). (i) *We have*

$$\sum_{k=1}^{\infty}\frac{64^{k-1}((11k-3)(2H_{2k-1}+H_{k-1})-4)}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{7}{2}\zeta(3) \tag{71}$$

and

$$\sum_{k=1}^{\infty}\frac{64^{k-1}((11k-3)(3H_{3k-1}-6H_{k-1})-7)}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{6\pi^2\log 2-21\zeta(3)}{8}. \tag{72}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}((11k+3)(2H_{2k}+H_k)+4)\equiv 21p^3B_{p-3}\pmod{p^4} \tag{73}$$

and

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}((11k+3)(3H_{3k}-6H_k)+7)\equiv 18p\,q_p(2)+9p^2q_p(2)^2\pmod{p^3}. \tag{74}$$

Remark 26. As conjectured by the author [28] and confirmed by Guillera [17], we have

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2.$$

Conjecture 27 (2022-12-09). (i) *We have*

$$\sum_{k=1}^{\infty}\frac{81^k((35k-8)(H_{4k-1}-H_{k-1})-35/4)}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2\log 3+39\zeta(3). \tag{75}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k}(4(35k+8)(H_{4k}-H_k)+35)\equiv 32(3^{p-1}-1)-16(3^{p-1}-1)^2\pmod{p^3}. \tag{76}$$

Remark 27. As conjectured by the author [28] and confirmed in [18], we have

$$\sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2.$$

Conjecture 28 (2023-01-14)**.** *For any prime $p>3$, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-144)^k}\left(4(5k+1)(H_{4k}-H_k)+5\right)\equiv\left(\frac{p}{3}\right)\left(5+2p(2q_p(2)+q_p(3))\right)\pmod{p^2}.\tag{77}$$

Remark 28. As conjectured by the author [28] and confirmed in [18], we have

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}K.$$

We are unable to find the exact value of the series

$$\sum_{k=1}^{\infty}\frac{(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}(4(5k-1)(H_{4k-1}-H_{k-1})-5).$$

The classical rational Ramanujan-type series for $1/\pi$ have the following four forms:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k}=\frac{c\sqrt{d}}{\pi},\tag{78}$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}=\frac{c\sqrt{d}}{\pi},\tag{79}$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}=\frac{c\sqrt{d}}{\pi},\tag{80}$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}=\frac{c\sqrt{d}}{\pi},\tag{81}$$

where $a,b,m\in\mathbb{Z}$, $am\neq 0$, $c\in\mathbb{Q}\setminus\{0\}$, and d is a positive squarefree integer. It is known that there are totally 36 such series, see, e.g., S. Cooper [11, Chapter 14].

For a positive integer m, can we find similar series for $(\log m)/\pi$? Motivated by Ramanujan-type series of the forms (78)-(81), the author formulated the following general conjecture.

Conjecture 29 (General Conjecture, 2022-12-08)**.** (i) *If we have an identity (78) with $a,b,m\in\mathbb{Z}$, $am\neq 0$, $c\in\mathbb{Q}\setminus\{0\}$, and $d\in\mathbb{Z}^+$ squarefree, then*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{m^k}(6(ak+b)(H_{2k}-H_k)+a)=\frac{c\sqrt{d}}{\pi}\log|m|,\tag{82}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k}\left(6(ak+b)(H_{2k}-H_k)+a\right)\equiv\left(\frac{-d}{p}\right)(a+b(m^{p-1}-1)) \pmod{p^2} \tag{83}$$

for any odd prime $p\nmid dm$.

(ii) *If we have an identity (79) with $a,b,m\in\mathbb{Z}$, $am\neq 0$, $c\in\mathbb{Q}\setminus\{0\}$, and $d\in\mathbb{Z}^+$ squarefree, then*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}((ak+b)(3H_{3k}+2H_{2k}-5H_k)+a)=\frac{c\sqrt{d}}{\pi}\log|m|, \tag{84}$$

and

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}&((ak+b)(3H_{3k}+2H_{2k}-5H_k)+a)\\ &\equiv\left(\frac{-d}{p}\right)(a+b(m^{p-1}-1)) \pmod{p^2}\end{aligned} \tag{85}$$

for any odd prime $p\nmid dm$.

(iii) *If we have an identity (80) with $a,b,m\in\mathbb{Z}$, $am\neq 0$, $c\in\mathbb{Q}\setminus\{0\}$, and $d\in\mathbb{Z}^+$ squarefree, then*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}(4(ak+b)(H_{4k}-H_k)+a)=\frac{c\sqrt{d}}{\pi}\log|m|, \tag{86}$$

and

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}&(4(ak+b)(H_{4k}-H_k)+a)\\ &\equiv\left(\frac{-d}{p}\right)\big(a+b(m^{p-1}-1)\big) \pmod{p^2}\end{aligned} \tag{87}$$

for any odd prime $p\nmid dm$.

(iv) *If we have an identity (81) with $a,b,m\in\mathbb{Z}$, $am\neq 0$, $c\in\mathbb{Q}\setminus\{0\}$, and $d\in\mathbb{Z}^+$ squarefree, then*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}(3(ak+b)(2H_{6k}-H_{3k}-H_k)+a)=\frac{c\sqrt{d}}{\pi}\log|m|, \tag{88}$$

and

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}&(3(ak+b)(2H_{6k}-H_{3k}-H_k)+a)\\ &\equiv\left(\frac{-d}{p}\right)\big(a+b(m^{p-1}-1)\big) \pmod{p^2}\end{aligned} \tag{89}$$

for any odd prime $p\nmid dm$.

Remark 29. Ramanujan [25] found the irrational series

$$\sum_{k=0}^{\infty}\left(k+\frac{31}{270+48\sqrt{5}}\right)\frac{\binom{2k}{k}^3}{(2^{20}/(\sqrt{5}-1)^8)^k}=\frac{16}{(15+21\sqrt{5})\pi}.$$

In the spirit of part (i) of our general conjecture, we guess that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{(2^{20}/(\sqrt{5}-1)^8)^k}\left(6\left(k+\frac{31}{270+48\sqrt{5}}\right)(H_{2k}-H_k)+1\right)$$
$$=\frac{16}{(15+21\sqrt{5})\pi}\times\log\frac{2^{20}}{(\sqrt{5}-1)^8},$$

which can be easily checked via `Mathematica`. We believe that similar things happen for all irrational Ramanujan-type series.

Example 3.1. In view of the Ramanujan series (cf. [11, Chapter 14] and [25])

$$\sum_{k=0}^{\infty}(5k+1)\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}=\frac{4\sqrt{3}}{\pi}$$

and the known identity

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}=\frac{4\sqrt{\pi}}{3\sqrt{3}\,\Gamma(5/6)^3}$$

(cf. (14.29) of [11, p. 624]), by part (ii) of Conjecture 29 we should have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}(5k+1)(3H_{3k}+2H_{2k}-5H_k)=\frac{4\log 192}{\sqrt{3}\,\pi}-\frac{20\sqrt{\pi}}{3\sqrt{3}\,\Gamma(5/6)^3}.$$

In 2013, Guillera [16, (32)] proved the identity

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{(-64)^k}\left((4k+1)H_k-\frac{2}{3}\right)=-\frac{4\log 2}{\pi}$$

motivated by the Bauer series (8).

Conjecture 30 (2022-10-16)**.** *We have*

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}H_{2k}^{(3)}=\frac{15\zeta(3)}{4\pi}-2G. \tag{90}$$

Remark 30. For any $m,n\in\mathbb{Z}^+$ and odd prime p not dividing $2^{2m-1}-1$, we have $H_{p-1}^{(2m-1)}\equiv 0 \pmod p$ since $\sum_{j=1}^{p-1}1/(2j)^{2m-1}\equiv\sum_{k=1}^{p-1}1/k^{2m-1} \pmod p$, thus

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}(4k+1)\binom{(p-1)/2}{k}^n H_{2k}^{(2m-1)}\\ =&\sum_{k=0}^{(p-1)/2}\left(4\left(\frac{p-1}{2}-k\right)+1\right)\binom{(p-1)/2}{k}^n H_{p-1-2k}^{(2m-1)}\\ \equiv&-\sum_{k=0}^{(p-1)/2}(4k+1)\binom{(p-1)/2}{k}^n H_{2k}^{(2m-1)} \pmod{p}\end{aligned}$$

and hence

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^n}{(-4)^{kn}}H_{2k}^{(2m-1)}\equiv 0 \pmod{p}. \tag{91}$$

(Note that $\binom{-1/2}{k}=\binom{2k}{k}/(-4)^k$ for any $k\in\mathbb{N}$.)

Conjecture 31 (2022-12-04)**.** (i) *We have*

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k}H_k=\frac{4}{3}\cdot\frac{\sqrt[3]{2}\sqrt{\pi}}{\Gamma(5/6)^3}-\frac{8\log 2}{\pi} \tag{92}$$

and

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k}H_{2k}=\frac{2}{3}\cdot\frac{\sqrt[3]{2}\sqrt{\pi}}{\Gamma(5/6)^3}-\frac{8\log 2}{3\pi}. \tag{93}$$

(ii) *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{256^k}((6k+1)(3H_{2k}-H_k)-1)\equiv(-1)^{(p+1)/2} \pmod{p^4}. \tag{94}$$

If $p>3$, then

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{256^k}((6k+1)(H_{2k}-H_k)+1)\\ \equiv&\left(\frac{-1}{p}\right)\left(1+\frac{4}{3}p\,q_p(2)-\frac{2}{3}p^2q_p(2)^2\right) \pmod{p^3}.\end{aligned} \tag{95}$$

Remark 31. It is known (cf. (14.27) of [11, p. 623]) that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{256^k}=\frac{2}{3}\cdot\frac{\sqrt[3]{2}\sqrt{\pi}}{\Gamma(5/6)^3}.$$

In view of this, (92) and (93) together implies the identities

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{256^k}((6k+1)(3H_{2k}-H_k)-1)=0$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{256^k}\left((6k+1)(H_{2k}-H_k)+1\right)=\frac{16\log 2}{3\pi}.$$

The last identity is also implied by Conjecture 29(i).

Conjecture 32 (2022-10-12)**.** *Let $p>3$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{256^k}\left(H_{2k}^{(2)}-\frac{5}{16}H_k^{(2)}\right)\equiv\frac{7}{24}\left(\frac{-1}{p}\right)p^2B_{p-3}\pmod{p^3},\tag{96}$$

and

$$\sum_{k=0}^{p-1}(6k+1)\frac{\binom{2k}{k}^3}{256^k}\left(H_{2k}^{(2)}-\frac{5}{16}H_k^{(2)}\right)\equiv -pE_{p-3}\pmod{p^2}.\tag{97}$$

(ii) *If $p\equiv 2\pmod 3$, then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{256^k}\left(H_{2k}^{(2)}-\frac{5}{16}H_k^{(2)}\right)\equiv 0\pmod{p}.\tag{98}$$

Remark 32. For any prime $p>3$, Guo and Lian [19] proved that the two sides of (96) are congruent modulo p^2.

Conjecture 33 (2022-10-11)**.** (i) *We have the identity*

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k}\left(H_{2k}^{(3)}-\frac{7}{64}H_k^{(3)}\right)=\frac{25\zeta(3)}{8\pi}-G.\tag{99}$$

(ii) *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{256^k}\left(H_{2k}^{(3)}-\frac{7}{64}H_k^{(3)}\right)\equiv -\frac{1}{2}E_{p-3}\pmod{p}\tag{100}$$

and

$$\sum_{k=0}^{p-1}(6k+1)\frac{\binom{2k}{k}^3}{256^k}\left(H_{2k}^{(3)}-\frac{7}{64}H_k^{(3)}\right)\equiv -\frac{3}{2}E_{p-3}\pmod{p}.\tag{101}$$

Remark 33. For any $k\in\mathbb{Z}^+$, it is easy to see that

$$H_{2k}^{(3)}-\frac{7}{64}H_k^{(3)}=\sum_{j=1}^{k}\left(\frac{1}{(2j-1)^3}+\frac{1}{(4j)^3}\right).$$

Conjecture 34 (2022-10-12)**.** *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\left(H_{2k}^{(2)}-\frac{5}{16}H_k^{(2)}\right)\equiv\left(\frac{2}{p}\right)\frac{p}{4}E_{p-3}\pmod{p^2}\tag{102}$$

and

$$\sum_{k=0}^{p-1}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\left(H_{2k}^{(2)}-\frac{5}{16}H_k^{(2)}\right)\equiv\frac{p}{16}E_{p-3}\left(\frac14\right)\pmod{p^2}. \tag{103}$$

Remark 34. Note that (102) is stronger than (7).

Conjecture 35 (2022-10-16)**.** (i) *We have the identity*

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\left(H_{2k}^{(3)}-\frac{7}{64}H_k^{(3)}\right)=\frac{57}{16}\cdot\frac{\zeta(3)}{\sqrt{2}\,\pi}-L, \tag{104}$$

where

$$L:=L\left(2,\left(\frac{-8}{\cdot}\right)\right)=\sum_{n=1}^{\infty}\frac{\left(\frac{-8}{n}\right)}{n^2}=\sum_{k=0}^{\infty}\frac{(-1)^{k(k-1)/2}}{(2k+1)^2}$$

with $(\frac{-8}{\cdot})$ *the Kronecker symbol.*

(ii) *Let* p *be an odd prime. Then*

$$\sum_{k=0}^{p-1}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\left(H_{2k}^{(3)}-\frac{7}{64}H_k^{(3)}\right)\equiv 0\pmod{p}. \tag{105}$$

Remark 35. For a general odd prime p, we are unable to find a closed form for the left-hand side of the congruence (105) modulo p^2.

Conjecture 36 (2022-12-09)**.** *For any prime* $p>3$*, we have*

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{4096^k}\left((42k+5)(H_{2k}-H_k)+7\right)\\ &\equiv\left(\frac{-1}{p}\right)\left(7+10p\,q_p(2)-5p^2q_p(2)^2\right)\pmod{p^3}.\end{aligned} \tag{106}$$

Remark 36. In view of the Ramanujan series (cf. [25])

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi},$$

by part (i) of Conjecture 29 we should have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{4096^k}\left((42k+5)(H_{2k}-H_k)+7\right)=\frac{32\log 2}{\pi},$$

and

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{4096^k}\left(6(42k+5)(H_{2k}-H_k)+42\right)\\ &\equiv\left(\frac{-1}{p}\right)\left(42+5p\,q_p(2^{12})\right)\equiv\left(\frac{-1}{p}\right)\left(42+60p\,q_p(2)\right)\pmod{p^2}\end{aligned}$$

for any odd prime p.

Conjecture 37 (2022-10-11)**.** (i) *We have the identity*

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\left(H_{2k}^{(2)}-\frac{25}{92}H_k^{(2)}\right)=\frac{2\pi}{69}. \tag{107}$$

(ii) *Let $p>3$ be a prime. If $p\neq 23$, then*

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\left(H_{2k}^{(2)}-\frac{25}{92}H_k^{(2)}\right)\\ \equiv&\left(\frac{-1}{p}\right)\frac{3}{20}\left(p^4B_{p-5}-5H_{p-1}\right)\pmod{p^5}.\end{aligned} \tag{108}$$

Also,

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\left(H_{2k}^{(2)}-\frac{25}{92}H_k^{(2)}\right)\equiv -pE_{p-3}\pmod{p^2}. \tag{109}$$

Remark 37. It is interesting to compare this conjecture with Remark 36.

Conjecture 38 (2022-10-12)**.** (i) *We have*

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\left(H_{2k}^{(3)}-\frac{43}{352}H_k^{(3)}\right)=\frac{555}{77}\cdot\frac{\zeta(3)}{\pi}-\frac{32}{11}G. \tag{110}$$

(ii) *For any prime $p>7$, we have*

$$\sum_{k=0}^{(p-1)/2}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\left(11H_{2k}^{(3)}-\frac{43}{32}H_k^{(3)}\right)\equiv -16E_{p-3}\pmod{p} \tag{111}$$

and

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\left(11H_{2k}^{(3)}-\frac{43}{32}H_k^{(3)}\right)\equiv -27E_{p-3}\pmod{p}. \tag{112}$$

Remark 38. Conjecture 38 looks quite challenging.

Conjecture 39 (2022-12-05)**.** (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}\left((6k+1)(H_{2k}-2H_k)+3\right)=\frac{9\sqrt{3}\log 3}{2\pi} \tag{113}$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}(6k+1)(3H_{3k}-H_k)=\frac{9\sqrt{3}\log 2}{\pi}. \tag{114}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}\left((6k+1)(H_{2k}-2H_k)+3\right)\equiv\left(\frac{p}{3}\right)\frac{3^p+3}{2}\pmod{p^2} \tag{115}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}(6k+1)(3H_{3k}-H_k)\equiv 3\left(\frac{p}{3}\right)p\,q_p(2)\pmod{p^2}. \tag{116}$$

Remark 39. This is motivated by the Ramanujan series (cf. [11, Chapter 14] and [25])

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}=\frac{3\sqrt{3}}{\pi}.$$

Conjecture 40 (2022-12-04)**.** (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k}\left((20k+3)(H_{2k}-3H_k)+12\right)=\frac{56\log 2}{\pi}. \tag{117}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k}\left((20k+3)(H_{2k}-3H_k)+12\right)\equiv\left(\frac{-1}{p}\right)(12+21p\,q_p(2))\pmod{p^2}. \tag{118}$$

Remark 40. This is motivated by the Ramanujan series (cf. [11, Chapter 14] and [25])

$$\sum_{k=0}^{\infty}(20k+3)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k}=\frac{8}{\pi}.$$

Conjecture 41 (2023-06-16)**.** (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k}\left((20k+3)\left(H_{4k}^{(2)}-\frac{H_{2k}^{(2)}}{4}-\frac{H_k^{(2)}}{16}\right)+\frac{1}{4k+1}\right)=\frac{\pi}{6}. \tag{119}$$

(ii) *For any prime $p>3$, we have*

$$\begin{aligned}p\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k}&\left((20k+3)\left(H_{4k}^{(2)}-\frac{H_{2k}^{(2)}}{4}-\frac{H_k^{(2)}}{16}\right)+\frac{1}{4k+1}\right)\\&\equiv\left(\frac{-1}{p}\right)+\frac{10}{3}p^2E_{p-3}\pmod{p^3}.\end{aligned} \tag{120}$$

Remark 41. The identity (119) looks curious and challenging.

Conjecture 42 (2022-12-04)**.** (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\left((8k+1)(3H_{2k}-4H_k)+6\right)=\frac{16\sqrt{3}\log 2}{\pi}. \tag{121}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\left((8k+1)(3H_{2k}-4H_k)+6\right)\equiv\left(\frac{p}{3}\right)(6+8p\,q_p(2))\pmod{p^2}. \tag{122}$$

Remark 42. This is motivated by the Ramanujan series (9).

Conjecture 43 (2022-10-15)**.** *For any prime $p > 3$, we have*

$$\sum_{k=0}^{(p-1)/2}(8k+1)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\left(H_{2k}^{(2)}-\frac{5}{18}H_k^{(2)}\right)\equiv\frac{p}{36}B_{p-2}\left(\frac{1}{3}\right)\pmod{p^2},\tag{123}$$

$$\sum_{k=0}^{p-1}(8k+1)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\left(H_{2k}^{(2)}-\frac{5}{18}H_k^{(2)}\right)\equiv-\frac{5}{24}pB_{p-2}\left(\frac{1}{3}\right)\pmod{p^2}.\tag{124}$$

Remark 43. The congruence (123) is motivated by (11).

Conjecture 44 (2022-10-19)**.** (i) *For any prime $p > 3$, we have*

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^4}{256^k}\left(H_{2k}^{(2)}-\frac{1}{2}H_k^{(2)}\right)\\ \equiv&\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^4}{256^k}\left(H_{2k}^{(2)}-\frac{1}{2}H_k^{(2)}\right)\equiv\frac{7}{6}p^2B_{p-3}\pmod{p^3}.\end{aligned}\tag{125}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^4}{256^k}H_{2k}^{(3)}\equiv\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^4}{256^k}H_{2k}^{(3)}\equiv\frac{3}{2}pB_{p-3}\pmod{p^2}.\tag{126}$$

Remark 44. Guo and Lian [19, (1.7)] proved that for any prime $p > 3$ we have

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^4}{256^k}\left(H_{2k}^{(2)}-\frac{1}{2}H_k^{(2)}\right)\equiv 0\pmod{p^2}.$$

4. Series and congruences with summands containing at least five binomial coefficients

The following two conjectures are motivated by the identity

$$\sum_{k=1}^{\infty}\frac{(-1)^k(205k^2-160k+32)}{k^5\binom{2k}{k}^5}=-2\zeta(3)$$

established by T. Amdeberhan and D. Zeilberger [2] in 1997 via the WZ method.

Conjecture 45 (2022-12-09)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^5\binom{2k}{k}^5}\left((205k^2-160k+32)(H_{2k-1}-H_{k-1})-41k+16\right)=\frac{\pi^4}{60}.\tag{127}$$

(ii) *For any prime $p>5$, we have*

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^5\big((205k^2+160k+32)(H_{2k}-H_k)+41k+16\big)\equiv 16p+64p^2H_{p-1}\pmod{p^6}.\tag{128}$$

Conjecture 46 (2022-12-09). (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-1)^k((205k^2-160k+32)(4H^{(2)}_{2k-1}-12H^{(2)}_{k-1})-43)}{k^5\binom{2k}{k}^5}=-8\zeta(5).\tag{129}$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k((205k^2-160k+32)(4H^{(2)}_{2k-1}-12H^{(2)}_{k-1})-43)}{k^5\binom{2k}{k}^5}\equiv -200B_{p-5}\pmod{p},\tag{130}$$

and

$$\sum_{k=1}^{p-1}(-1)^k\binom{2k}{k}^5((205k^2+160k+32)(4H^{(2)}_{2k}-12H^{(2)}_{k})+43)\equiv 256pH_{p-1}\pmod{p^5}\tag{131}$$

if $p>5$.

The following two conjectures are motivated by the identity

$$\sum_{k=1}^{\infty}\frac{(10k^2-6k+1)(-256)^k}{k^5\binom{2k}{k}^5}=-28\zeta(3)$$

(cf. [14, Identity 8]).

Conjecture 47 (2022-12-09). (i) *We have*

$$\sum_{k=1}^{\infty}\frac{(-256)^k}{k^5\binom{2k}{k}^5}\big((10k^2-6k+1)(2H_{2k-1}-H_{k-1})-3k+1\big)=-\frac{\pi^4}{2}.\tag{132}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^5}{(-256)^k}\big((10k^2+6k+1)(2H_{2k}-H_k)+3k+1\big)\equiv p+\frac{14}{3}p^4B_{p-3}\pmod{p^5}.\tag{133}$$

Remark 45. For any prime $p>3$, the author [37, Conjecture 31(ii)] conjectured the congruence

$$\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^5}{(-256)^k}(10k^2+6k+1)\equiv-\frac{7}{2}p^5B_{p-3}\pmod{p^6}$$

which implies that

$$\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^5}{(-256)^k}\big((10k^2+6k+1)(2H_{2k}-H_k)+3k+1\big)$$
$$\equiv\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}^5}{(-256)^k}(10k^2+6k+1)\cdot\frac{2}{p}\equiv -7p^4B_{p-3}\pmod{p^5}.$$

Conjecture 48 (2022-12-09)**.** (i) *We have*

$$\sum_{k=1}^\infty\frac{(-256)^k((10k^2-6k+1)(4H^{(2)}_{2k-1}-3H^{(2)}_{k-1})-2)}{k^5\binom{2k}{k}^5}=-124\zeta(5).\tag{134}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=1}^{(p-1)/2}\frac{(-256)^k}{k^5\binom{2k}{k}^5}((10k^2-6k+1)(4H^{(2)}_{2k-1}-3H^{(2)}_{k-1})-2)\equiv -124B_{p-5}\pmod{p}\tag{135}$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^5}{(-256)^k}((10k^2+6k+1)(4H^{(2)}_{2k}-3H^{(2)}_{k})+2)\equiv\frac{28}{3}p^3B_{p-3}\pmod{p^4}.\tag{136}$$

The following two conjectures are motivated by the identity

$$\sum_{k=1}^\infty\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3)$$

conjectured by the author [30] and confirmed recently by Au [5].

Conjecture 49 (2022-12-09)**.** (i) *We have*

$$\sum_{k=1}^\infty\frac{(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}\big((28k^2-18k+3)(4H_{2k-1}-3H_{k-1})-20k+6\big)=-\frac{\pi^4}{2}.\tag{137}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^4\binom{3k}{k}}{(-64)^k}\big((28k^2+18k+3)(4H_{2k}-3H_k)+20k+6\big)\equiv 6p-14p^4B_{p-3}\pmod{p^5}.\tag{138}$$

Conjecture 50 (2022-12-09)**.** (i) *We have*

$$\sum_{k=1}^\infty\frac{(-64)^k((28k^2-18k+3)(2H^{(2)}_{2k-1}-3H^{(2)}_{k-1})-2)}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-31\zeta(5).\tag{139}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^4\binom{3k}{k}}{(-64)^k}\left((28k^2+18k+3)(2H_{2k}^{(2)}-3H_k^{(2)})+2\right) \equiv -7p^3B_{p-3} \pmod{p^4}. \tag{140}$$

The following two conjectures are motivated by the identity

$$\sum_{k=0}^{\infty}(20k^2+8k+1)\frac{\binom{2k}{k}^5}{(-4096)^k}=\frac{8}{\pi^2}$$

(cf. [14, Identity 8]).

Conjecture 51 (2022-12-09)**.** (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^5}{(-4096)^k}((20k^2+8k+1)H_k-6k-1)=-\frac{16\log 2}{\pi^2} \tag{141}$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^5}{(-4096)^k}(5(20k^2+8k+1)H_{2k}-10k-1)=-\frac{32\log 2}{\pi^2}. \tag{142}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^5}{(-4096)^k}((20k^2+8k+1)H_k-6k-1) \equiv -p-2p^2q_p(2)+p^3q_p(2)^2-\frac{2}{3}p^4q_p(2)^3 \pmod{p^5}, \tag{143}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^5}{(-4096)^k}(5(20k^2+8k+1)H_k-10k-1) \equiv -p-4p^2q_p(2)+2p^3q_p(2)^2 \pmod{p^4}. \tag{144}$$

Conjecture 52 (2022-12-09)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}^5}{(-4096)^k}((20k^2+8k+1)(8H_{2k}^{(2)}-3H_k^{(2)})+4)=-\frac{4}{3}. \tag{145}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^5}{(-4096)^k}((20k^2+8k+1)(8H_{2k}^{(2)}-3H_k^{(2)})+4)\equiv\frac{14}{3}p^3B_{p-3} \pmod{p^4} \tag{146}$$

and

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^5}{(-4096)^k}((20k^2+8k+1)(8H_{2k}^{(2)}-3H_k^{(2)})+4)\equiv-\frac{28}{3}p^3B_{p-3}\pmod{p^4}. \tag{147}$$

The following two conjectures are motivated by the identity

$$\sum_{k=0}^{\infty}(820k^2+180k+13)\frac{\binom{2k}{k}^5}{(-2^{20})^k}=\frac{128}{\pi^2}$$

(cf. [10, Identity 9]).

Conjecture 53 (2022-12-09)**.** (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^5}{(-2^{20})^k}\big((820k^2+180k+13)(H_{2k}-H_k)+164k+18\big)=\frac{256\log 2}{\pi^2}. \tag{148}$$

(ii) *For any odd prime p, we have*

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^5}{(-2^{20})^k}\big((820k^2+180k+13)(H_{2k}-H_k)+164k+18\big)\\&\quad\equiv\ 18p+26p^2q_p(2)-13p^3q_p(2)^2\pmod{p^4}.\end{aligned} \tag{149}$$

Conjecture 54 (2022-12-09)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}^5}{(-2^{20})^k}((820k^2+180k+13)(11H_{2k}^{(2)}-3H_k^{(2)})+43)=-\frac{1}{3}. \tag{150}$$

(ii) *Let $p>3$ be a prime. Then*

$$\begin{aligned}&\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^5}{(-2^{20})^k}((820k^2+180k+13)(11H_{2k}^{(2)}-3H_k^{(2)})+43)\\&\qquad\equiv-\frac{77}{6}p^3B_{p-3}\pmod{p^4},\end{aligned} \tag{151}$$

and

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^5}{(-2^{20})^k}((820k^2+180k+13)(11H_{2k}^{(2)}-3H_k^{(2)})+43)\\&\qquad\equiv-\frac{11}{4}pH_{p-1}\pmod{p^5}\end{aligned} \tag{152}$$

if $p>5$.

The following two conjectures are motivated by the known identity

$$\sum_{k=0}^{\infty}(74k^2+27k+3)\frac{\binom{2k}{k}^4\binom{3k}{k}}{4096^k}=\frac{48}{\pi^2}$$

(cf. [15]).

Conjecture 55 (2022-12-09). (i) *We have*

$$\sum_{k=0}^\infty\frac{\binom{2k}k^4\binom{3k}k}{4096^k}\big((74k^2+27k+3)H_{2k}-17k-3\big)=0\tag{153}$$

and

$$\begin{aligned}&\sum_{k=0}^\infty\frac{\binom{2k}k^4\binom{3k}k}{4096^k}((74k^2+27k+3)(51H_{3k}+250H_{2k}-153H_k)+15)\\&\qquad=\frac{9792\log 2}{\pi^2}.\end{aligned}\tag{154}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}k^4\binom{3k}k}{4096^k}((74k^2+27k+3)H_{2k}-17k-3)\equiv-3p+7p^4B_{p-3}\pmod{p^5},\tag{155}$$

and

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{\binom{2k}k^4\binom{3k}k}{4096^k}((74k^2+27k+3)(51H_{3k}+250H_{2k}-153H_k)+15)\\&\quad\equiv\ 15p+612p^2q_p(2)-306p^3q_p(2)^2\pmod{p^4}.\end{aligned}\tag{156}$$

Conjecture 56 (2022-12-09). (i) *We have*

$$\sum_{k=1}^\infty\frac{\binom{2k}k^4\binom{3k}k}{4096^k}((74k^2+27k+3)(92H_{2k}^{(2)}-33H_k^{(2)})+112)=160.\tag{157}$$

(ii) *For any odd prime p, we have*

$$\begin{aligned}&\sum_{k=1}^{p-1}\frac{\binom{2k}k^4\binom{3k}k}{4096^k}\Big((74k^2+27k+3)(92H_{2k}^{(2)}-33H_k^{(2)})+112\Big)\\&\qquad\equiv 644p^3B_{p-3}\pmod{p^4}.\end{aligned}\tag{158}$$

The following two conjectures are motivated by the identity

$$\sum_{k=0}^\infty(120k^2+34k+3)\frac{\binom{2k}k^4\binom{4k}{2k}}{2^{16k}}=\frac{32}{\pi^2}$$

(cf. [14, Identity 10]).

Conjecture 57 (2022-12-09). (i) *We have*

$$\sum_{k=0}^\infty\frac{\binom{2k}k^4\binom{4k}{2k}}{2^{16k}}(2(120k^2+34k+3)H_{4k}-16k-1)=0\tag{159}$$

and

$$\sum_{k=0}^\infty\frac{\binom{2k}k^4\binom{4k}{2k}}{2^{16k}}((120k^2+34k+3)(H_{2k}-2H_k)+68k+9)=\frac{128\log 2}{\pi^2}.\tag{160}$$

(ii) *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^4\binom{4k}{2k}}{2^{16k}}((120k^2+34k+3)(H_{2k}-2H_k)+68k+9)\equiv 9p+12p^2q_p(2)-6p^3q_p(2)^2\pmod{p^4},\tag{161}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^4\binom{4k}{2k}}{2^{16k}}(2(120k^2+34k+3)H_{4k}-16k-1)\equiv -p+\frac{77}{6}p^4B_{p-3}\pmod{p^5}\tag{162}$$

if $p>3$.

Conjecture 58 (2022-12-09)**.** (i) *We have*

$$\sum_{k=1}^{\infty}\frac{\binom{2k}{k}^4\binom{4k}{2k}}{2^{16k}}((120k^2+34k+3)(23H_{2k}^{(2)}-7H_k^{(2)})+24)=\frac{16}{3}.\tag{163}$$

(ii) *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^4\binom{4k}{2k}}{2^{16k}}((120k^2+34k+3)(23H_{2k}^{(2)}-7H_k^{(2)})+24)\equiv\frac{161}{2}p^3B_{p-3}\pmod{p^4},\tag{164}$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^4\binom{4k}{2k}}{2^{16k}}((120k^2+34k+3)(23H_{2k}^{(2)}-7H_k^{(2)})+24)\equiv -23pH_{p-1}\pmod{p^5}\tag{165}$$

if $p\neq 5$.

Conjecture 59 (2022-12-09)**.** (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3\binom{3k}{k}\binom{4k}{2k}}{(-24^4)^k}\big((252k^2+63k+5)(4H_{4k}+3H_{3k}-7H_k)+504k+63\big)=\frac{192\log 24}{\pi^2}.\tag{166}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3\binom{3k}{k}\binom{4k}{2k}}{(-24^4)^k}\big((252k^2+63k+5)(4H_{4k}+3H_{3k}-7H_k)+504k+63\big)\equiv 63p+5p^2q_p(24^4)-\frac{5}{2}p^3q_p(24^4)^2\pmod{p^4}.\tag{167}$$

Remark 46. Conjecture 59 is motivated by the identity

$$\sum_{k=0}^{\infty}(252k^2+63k+5)\frac{\binom{2k}{k}^3\binom{3k}{k}\binom{4k}{2k}}{(-24^4)^k}=\frac{48}{\pi^2}$$

(cf. [10]).

Conjecture 60 (2023-01-16). (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{10^{6k}}\big(3(532k^2+126k+9)(H_{6k}-H_k)+532k+63\big)=\frac{1125\log 10}{4\pi^2}. \tag{168}$$

(ii) *For any odd prime $p\neq 5$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{10^{6k}}\big(3(532k^2+126k+9)(H_{6k}-H_k)+532k+63\big)\equiv 63p+\frac{9}{2}p^2q_p(10^6)-\frac{9}{4}p^3q_p(10^6)^2 \pmod{p^4}. \tag{169}$$

Remark 47. Conjecture 60 is motivated by the identity

$$\sum_{k=0}^{\infty}(532k^2+126k+9)\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{10^{6k}}=\frac{375}{4\pi^2}$$

(cf. [10]).

Conjecture 61 (2023-01-17). (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{(-2^{18})^k}\big(6(1930k^2+549k+45)(H_{6k}-H_k)+3860k+549\big)=\frac{6912\log 2}{\pi^2}. \tag{170}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{(-2^{18})^k}\big(6(1930k^2+549k+45)(H_{6k}-H_k)+3860k+549\big)\equiv 549p+45p^2q_p(2^{18})-\frac{45}{2}p^3q_p(2^{18})^2 \pmod{p^4}. \tag{171}$$

Remark 48. Conjecture 61 is motivated by the identity

$$\sum_{k=0}^{\infty}(1930k^2+549k+45)\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{(-2^{18})^k}=\frac{384}{\pi^2}$$

(cf. [10]).

Conjecture 62 (2023-01-17). (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{(-2^{18}3^65^3)^k}\big(2(5418k^2+693k+29)(H_{6k}-H_k)+3612k+231\big)$$
$$=\frac{128\sqrt5}{\pi^2}\log(2^63^25). \tag{172}$$

(ii) *For any prime $p>5$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{(-2^{18}3^65^3)^k}\big(2(5418k^2+693k+29)(H_{6k}-H_k)+3612k+231\big)$$
$$\equiv\left(\frac{5}{p}\right)\left(231p+\frac{29}{3}p^2q_p(2^{18}3^65^3)-\frac{29}{6}p^3q_p(2^{18}3^65^3)^2\right)\pmod{p^4}. \tag{173}$$

Remark 49. Conjecture 62 is motivated by the identity

$$\sum_{k=0}^{\infty}(5418k^2+693k+29)\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{(-2^{18}3^65^3)^k}=\frac{128\sqrt5}{\pi^2}$$

(cf. [10]).

Conjecture 63 (2023-01-17). *For $k\in\mathbb{N}$, set*

$$H(k):=6H_{6k}+4H_{4k}-3H_{3k}-2H_{2k}-5H_k.$$

(i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}\binom{4k}{2k}\binom{6k}{3k}}{(-2^{22}3^3)^k}\big((1640k^2+278k+15)H(k)+3280k+278\big)$$
$$=\frac{256}{\sqrt3\pi^2}\log(2^{22}3^3). \tag{174}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}\binom{4k}{2k}\binom{6k}{3k}}{(-2^{22}3^3)^k}\big((1640k^2+278k+15)H(k)+3280k+278\big)$$
$$\equiv\left(\frac{3}{p}\right)\left(278p+15p^2q_p(2^{22}3^3)-\frac{15}{2}p^3q_p(2^{22}3^3)^2\right)\pmod{p^4}. \tag{175}$$

Remark 50. Conjecture 63 is motivated by the identity

$$\sum_{k=0}^{\infty}(1640k^2+278k+15)\frac{\binom{2k}{k}^2\binom{3k}{k}\binom{4k}{2k}\binom{6k}{3k}}{(-2^{22}3^3)^k}=\frac{256}{\sqrt3\pi^2}$$

(cf. [10]).

Conjecture 64 (2023-01-17). *For $k \in \mathbb{N}$, set*

$$\mathcal{H}(k) := 4H_{8k} - 2H_{4k} + H_{2k} - 3H_k.$$

(i) *We have*

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3\binom{4k}{2k}\binom{8k}{4k}}{(2^{18}7^4)^k}\big((1920k^2+304k+15)\mathcal{H}(k)+1920k+152\big) = \frac{56\sqrt{7}}{\pi^2}(9\log 2+2\log 7). \tag{176}$$

(ii) *For any odd prime $p \neq 7$, we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3\binom{4k}{2k}\binom{8k}{4k}}{(2^{18}7^4)^k}\big((1920k^2+304k+15)\mathcal{H}(k)+1920k+152\big) \equiv \left(\frac{7}{p}\right)\left(152p+\frac{15}{2}p^2q_p(2^{18}7^4)-\frac{15}{4}p^3q_p(2^{18}7^4)^2\right) \pmod{p^4}. \tag{177}$$

Remark 51. Conjecture 64 is motivated by the identity

$$\sum_{k=0}^{\infty}(1920k^2+304k+15)\frac{\binom{2k}{k}^3\binom{4k}{2k}\binom{8k}{4k}}{(2^{18}7^4)^k} = \frac{56\sqrt{7}}{\pi^2}$$

(cf. [10]).

Conjecture 65 (2022-12-09). (i) *We have*

$$\sum_{k=1}^{\infty}\frac{256^k}{k^7\binom{2k}{k}^7}\left((21k^3-22k^2+8k-1)(4H^{(2)}_{2k-1}-5H^{(2)}_{k-1})-6k+2\right) = \frac{\pi^6}{24}. \tag{178}$$

(ii) *For any odd prime p, we have*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^7}{256^k}\left((21k^3+22k^2+8k+1)(4H^{(2)}_{2k}-5H^{(2)}_{k})+6k+2\right) \equiv 2p \pmod{p^5}. \tag{179}$$

Remark 52. This is motivated by the identity

$$\sum_{k=1}^{\infty}\frac{(21k^3-22k^2+8k-1)256^k}{k^7\binom{2k}{k}^7} = \frac{\pi^4}{8}$$

conjectured by Guillera [13].

The following three conjectures are motivated by the identity

$$\sum_{k=0}^{\infty}(168k^3+76k^2+14k+1)\frac{\binom{2k}{k}^7}{2^{20k}} = \frac{32}{\pi^3} \tag{180}$$

conjectured by B. Gourevich (cf. [10]).

Conjecture 66 (2022-12-09)**.** (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^7}{2^{20k}}\big(7(168k^3+76k^2+14k+1)(H_{2k}-H_k)+252k^2+76k+7\big)$$
$$=\frac{320\log 2}{\pi^3}. \tag{181}$$

(ii) *For any prime $p>5$, we have*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^7}{2^{20k}}\big(7(168k^3+76k^2+14k+1)(H_{2k}-H_k)+252k^2+76k+7\big)$$
$$\equiv\left(\frac{-1}{p}\right)\left(7p^2+10p^3q_p(2)-5p^4q_p(2)^2+\frac{10}{3}p^5q_p(2)^3-\frac{5}{2}p^6q_p(2)^4\right)$$
$$\pmod{p^7}. \tag{182}$$

Conjecture 67 (2022-12-09)**.** (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^7}{2^{20k}}\left((168k^3+76k^2+14k+1)(16H_{2k}^{(2)}-5H_k^{(2)})+8(6k+1)\right)=\frac{80}{3\pi}. \tag{183}$$

(ii) *For any prime $p>5$, we have*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^7}{2^{20k}}\left((168k^3+76k^2+14k+1)(16H_{2k}^{(2)}-5H_k^{(2)})+8(6k+1)\right)$$
$$\equiv\left(\frac{-1}{p}\right)8p \pmod{p^6}. \tag{184}$$

Conjecture 68 (2023-06-19)**.** (i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^7}{2^{20k}}\left((168k^3+76k^2+14k+1)(128H_{2k}^{(4)}-7H_k^{(4)})+\frac{64}{2k+1}\right)=\frac{976}{45}\pi. \tag{185}$$

(ii) *For any odd prime $p\neq 5$, we have*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^7}{2^{20k}}\left((168k^3+76k^2+14k+1)\left(128H_{2k}^{(4)}-7H_k^{(4)}\right)+\frac{64}{2k+1}\right)$$
$$\equiv -256q_p(2) \pmod{p}. \tag{186}$$

Conjecture 69 (2023-06-19)**.** *Set*

$$P(x)=4528x^4+3180x^3+972x^2+147x+9.$$

(i) *We have*

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^7\binom{3k}{k}\binom{4k}{2k}}{(-2^{24})^k}\left(\left(H_{4k}+\frac{H_{2k}}{2}\right)P(k)-484k^3-108k^2+9k+3\right)=0, \tag{187}$$

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^7\binom{3k}{k}\binom{4k}{2k}}{(-2^{24})^k}\left(\left(11H_{2k}^{(2)}-4H_k^{(2)}\right)P(k)+1780k^2+633k+63\right)=\frac{512}{\pi^2}, \tag{188}$$

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^7\binom{3k}{k}\binom{4k}{2k}}{(-2^{24})^k}\left(\left(9H_{2k}^{(3)}-2H_k^{(3)}\right)P(k)-\frac{916k^2+65k-9}{2k+1}\right)=0, \tag{189}$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^7\binom{3k}{k}\binom{4k}{2k}}{(-2^{24})^k}\left(\left(49H_{2k}^{(4)}-2H_k^{(4)}\right)P(k)+\frac{5996k^2+3071k+329}{(2k+1)^2}\right)=\frac{4096}{15}. \tag{190}$$

(ii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^7\binom{3k}{k}\binom{4k}{2k}}{(-2^{24})^k}\left(\left(H_{4k}+\frac{H_{2k}}{2}\right)P(k)-484k^3-108k^2+9k+3\right)\equiv 3p^3-\frac{1953}{10}p^8B_{p-5}\pmod{p^9}, \tag{191}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^7\binom{3k}{k}\binom{4k}{2k}}{(-2^{24})^k}\left(\left(11H_{2k}^{(2)}-4H_k^{(2)}\right)P(k)+1780k^2+633k+63\right)\equiv 63p^2-\frac{9207}{10}p^7B_{p-5}\pmod{p^8}, \tag{192}$$

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^7\binom{3k}{k}\binom{4k}{2k}}{(-2^{24})^k}\left(\left(9H_{2k}^{(3)}-2H_k^{(3)}\right)P(k)-\frac{916k^2+65k-9}{2k+1}\right)\equiv 9p-\frac{81}{2}p^2H_{p-1}-\frac{243}{8}p^6B_{p-5}\pmod{p^7}, \tag{193}$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^7\binom{3k}{k}\binom{4k}{2k}}{(-2^{24})^k}\left(\left(49H_{2k}^{(4)}-2H_k^{(4)}\right)P(k)+\frac{5996k^2+3071k+329}{(2k+1)^2}\right)\equiv -\frac{441}{2}\left(pH_{p-1}+\frac{3}{4}p^5B_{p-5}\right)\pmod{p^6}. \tag{194}$$

Remark 53. This is motivated by the conjectural identity

$$\sum_{k=0}^{\infty}P(k)\frac{\binom{2k}{k}^7\binom{3k}{k}\binom{4k}{2k}}{(-2^{24})^k}=\frac{768}{\pi^4}$$

(cf. [10]).

Conjecture 70 (2023-06-19). *Set*

$$Q(x) = 43680k^4 + 20632k^3 + 4340k^2 + 466k + 21$$

and

$$R(x) = 87360x^3 + 30948x^2 + 4340x + 233.$$

(i) *We have*

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^8\binom{4k}{2k}}{2^{32k}} ((2(H_{4k} + 3H_{2k} - 4H_k)Q(k) + R(k)) = \frac{2^{15}\log 2}{\pi^4} \tag{195}$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^8\binom{4k}{2k}}{2^{32k}} \left(\left(7H_{2k}^{(2)} - 2H_k^{(2)}\right) Q(k) + 3624k^2 + 926k + 69\right) = \frac{2048}{3\pi^2}. \tag{196}$$

(ii) *Let p be an odd prime. If $p \neq 5$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^8\binom{4k}{2k}}{2^{32k}} ((2(H_{4k} + 3H_{2k} - 4H_k)Q(k) + R(k)) \equiv 233p^3 + 336p^4q_p(2) - 168p^5q_p(2)^2 + 112p^6q_p(2)^3 - 84p^7q_p(2)^4 \pmod{p^8}. \tag{197}$$

Provided $p > 3$, we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^8\binom{4k}{2k}}{2^{32k}} \left(\left(7H_{2k}^{(2)} - 2H_k^{(2)}\right) Q(k) + 3624k^2 + 926k + 69\right) \equiv 69p^2 + \frac{1953}{20}p^7B_{p-5} \pmod{p^8}. \tag{198}$$

Remark 54. This is motivated by the conjectural identity

$$\sum_{k=0}^{\infty} Q(k)\frac{\binom{2k}{k}^8\binom{4k}{2k}}{2^{32k}} = \frac{2048}{\pi^4}$$

(cf. [10]).

Acknowledgments. The work was supported by the Natural Science Foundation of China (grant no. 12371004).

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