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PERMANENT IDENTITIES, COMBINATORIAL SEQUENCES, AND PERMUTATION STATISTICS

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Abstract. In this paper, we confirm six conjectures on the exact values of some permanents, relating them to the Genocchi numbers of the first and second kinds as well as the Euler numbers. For example, we prove that

$$\mathrm{per}\left[\left\lfloor\frac{2j-k}{n}\right\rfloor\right]_{1\le j,k\le n}=2(2^{n+1}-1)B_{n+1},$$

where $B_0, B_1, B_2, \ldots$ are the Bernoulli numbers. We also show that

$$\mathrm{per}\left[\mathrm{sgn}\left(\cos\pi\frac{i+j}{n+1}\right)\right]_{1\le i,j\le n}=\begin{cases}-\sum_{k=0}^{m}\binom{m}{k}E_{2k+1} & \text{if } n=2m+1,\\ \sum_{k=0}^{m}\binom{m}{k}E_{2k} & \text{if } n=2m,\end{cases}$$

where $\mathrm{sgn}(x)$ is the sign function, and $E_0, E_1, E_2, \ldots$ are the Euler (zigzag) numbers.

In the course of linking the evaluation of these permanents to the aforementioned combinatorial sequences, the classical permutation statistic – the excedance number, together with several kinds of its variants, plays a central role. Our approach features recurrence relations, bijections, as well as certain elementary operations on matrices that preserve their permanents. Moreover, our proof of the second permanent identity leads to a proof of Bala's conjectural continued fraction formula, and an unexpected permutation interpretation for the γ-coefficients of the 2-Eulerian polynomials.

1. Introduction

For a matrix $A = [a_{i,j}]_{1\le i,j\le n}$ over a commutative ring with identity, its determinant and permanent are defined by

$$\det(A)=\det[a_{i,j}]_{1\le i,j\le n}=\sum_{\pi\in\mathfrak{S}_n}\mathrm{sign}(\pi)\prod_{i=1}^{n}a_{i,\pi(i)}$$

and

$$\mathrm{per}(A)=\mathrm{per}[a_{i,j}]_{1\le i,j\le n}=\sum_{\pi\in\mathfrak{S}_n}\prod_{i=1}^{n}a_{i,\pi(i)}$$

respectively, where $\mathfrak{S}_n$ is the symmetric group of all permutations of $[n] := \{1, \ldots, n\}$. Determinants are widely used in mathematics, and permanents are useful in combinatorics.

It is well known that

$$\det[i^{j-1}]_{1\le i,j\le n}=\prod_{1\le i<j\le n}(j-i)=1!2!\cdots(n-1)!$$

as this is of Vandermonde's type. In contrast, Z.-W. Sun [32] proved that

$$\mathrm{per}[i^{j-1}]_{1\le i,j\le n}\equiv 0 \pmod{n}\quad\text{for all } n\ge 3.$$

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Z.-W. Sun [35] evaluated some determinants involving trigonometric functions. For example, he proved that for any odd integer $n>1$ and integers a and b with $\gcd(ab,n)=1$, the following identity holds:

$$\det\left[\tan\pi\frac{aj+bk}{n}\right]_{1\leqslant j,k\leqslant n-1}=\left(\frac{-ab}{n}\right)n^{n-2},$$

where $(\frac{\cdot}{n})$ is the Jacobi symbol.

Motivated by the above work, Sun [33] investigated arithmetic properties of some permanents. For example, he showed that

$$\operatorname{per}\left[\left\lfloor\frac{j+k}{n}\right\rfloor\right]_{1\leqslant j,k\leqslant n}=2^{n-1}+1 \quad\text{and}\quad \operatorname{per}\left[\left\lfloor\frac{j+k-1}{n}\right\rfloor\right]_{1\leqslant j,k\leqslant n}=1, \tag{1.1}$$

where $\lfloor\cdot\rfloor$ is the floor function. Sun [33] also proved that

$$\operatorname{per}\left[\tan\pi\frac{j+k}{p}\right]_{1\leq j,k\leq p-1}\equiv(-1)^{(p+1)/2}2p \pmod{p^2}$$

for any odd prime p.

For any positive integer n, clearly

$$\left\lfloor\frac{2j-k}{n}\right\rfloor\in\{0,\pm1\}\quad\text{for all } j,k=1,\ldots,n.$$

Inspired by this as well as his preprint [33], Sun [34] posed the following novel conjecture involving the Bernoulli numbers $B_0,B_1,\ldots$ given by

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}\quad(|x|<2\pi).$$

Conjecture 1.1 (Z.-W. Sun)**.** For any positive integer n, we have

$$\operatorname{per}\left[\left\lfloor\frac{2j-k}{n}\right\rfloor\right]_{1\leq j,k\leq n}=2(2^{n+1}-1)B_{n+1}. \tag{1.2}$$

This conjecture has aroused quite some interests and triggered further conjectures over the online forum `MathOverflow` concerning the evaluations of various permanents. We aim to prove six of the conjectures (including the above one) posted there.

Three combinatorial sequences play major roles in the current work. They are the Genocchi numbers, the median Genocchi numbers, and the Euler numbers. We collect their definitions here and state the remaining five conjectures afterwards.

The Genocchi numbers (of the first kind) $G_1,G_2,\ldots$ are given by

$$\frac{2x}{e^x+1}=\sum_{n=1}^{\infty}G_n\frac{x^n}{n!}\quad(|x|<\pi).$$

It is known that $G_n=2(1-2^n)B_n$ for any positive integer n (cf. [29, A036968]); in particular, $G_{2n+1}=2(1-2^{2n+1})B_{2n+1}=0$ and

$$(-1)^nG_{2n}=2(2^{2n}-1)(-1)^{n-1}B_{2n}>0$$

for all $n=1,2,3,\ldots$. Note that

$$G_1=1,\ G_2=-1,\ G_4=1,\ G_6=-3,\ G_8=17,\ G_{10}=-155,\ G_{12}=2073.$$

The median Genocchi numbers (or Genocchi numbers of the second kind, cf. [29, A005439]) $H_1, H_3, H_5, \ldots$ can be defined by their relation with G_{2n} ($n \geq 1$):

$$H_{2n-1} = (-1)^n \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} G_{2n-2j} \quad \text{for all } n = 1, 2, 3, \ldots.$$

For example,

$$H_1 = 1,\ H_3 = 2,\ H_5 = 8,\ H_7 = 56,\ H_9 = 608,\ H_{11} = 9440.$$

The *Euler numbers* $(E_n)_{n\geq 0}$ are defined as the coefficients of the Taylor expansion

$$\sec(x) + \tan(x) = \sum_{n\geq 0} E_n \frac{x^n}{n!} = 1 + 1\frac{x}{1!} + 1\frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 16\frac{x^5}{5!} + 61\frac{x^6}{6!} + 272\frac{x^7}{7!} + \cdots.$$

It was André [1] in 1879 who first discovered the interpretation of E_n as the number of *alternating (down-up) permutations* of length n. The Euler numbers E_{2n} of even indices are called *secant numbers*, while those E_{2n-1} with odd indices are called *tangent numbers*.

Motivated by Conjecture 1.1, P. Luschny [29, A005439] made the following similar conjecture involving median Genocchi numbers.

Conjecture 1.2 (P. Luschny)**.** Let n be any positive integer and define

$$M_{2n} := \left[\left\lfloor \frac{2j-k-1}{2n} \right\rfloor\right]_{1\leq j,k\leq 2n}.$$

Then we have

$$\mathrm{per}(M_{2n}) = (-1)^n H_{2n-1}. \tag{1.3}$$

Recall that the sign function is given by

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

The following third, fourth, and fifth conjectures (involving the sign function) were raised by Deyi Chen [9, 10].

Conjecture 1.3 (D. Chen)**.** Let n be any positive integer, and set

$$A_{2n} := \left[\mathrm{sgn}\left(\tan \pi \frac{i+j}{2n+1}\right)\right]_{1\leq i,j\leq 2n}.$$

Then we have

$$\mathrm{per}(A_{2n}) = \mathrm{per}(A_{2n}^{-1}) = (-1)^n E_{2n}. \tag{1.4}$$

Conjecture 1.4 (D. Chen)**.** Let n be a positive integer, and set

$$P_n := \left[\mathrm{sgn}\left(\sin \pi \frac{i+j}{n+1}\right)\right]_{1\leq i,j\leq n}.$$

Then

$$\mathrm{per}(P_{2n}) = \mathrm{per}(P_{2n}^{-1}) = (-1)^n E_{2n}. \tag{1.5}$$

Conjecture 1.5 (D. Chen)**.** Let n be a positive integer, and set

$$Q_n := \left[\operatorname{sgn}\left(\sin \pi \frac{i+2j}{n+1}\right)\right]_{1\le i,j\le n}.$$

Then

$$\operatorname{per}(Q_n) = (-1)^n E_n. \tag{1.6}$$

In view of the above conjectures relating permanents involving "tan" and "sin" directly to (signed) Euler numbers, it seems natural to consider the trigonometric function "cos" instead, and compute the corresponding permanents. This consideration leads to the most recent conjecture of Deyi Chen [11].

Conjecture 1.6 (D. Chen)**.** Let n be a positive integer, and define

$$R_n := \left[\operatorname{sgn}\left(\cos \pi \frac{i+j}{n+1}\right)\right]_{1\le i,j\le n}.$$

Then

$$\operatorname{per}(R_n) = \begin{cases} -\sum_{k=0}^{m} \binom{m}{k} E_{2k+1} & \text{if } n = 2m+1, \\ \sum_{k=0}^{m} \binom{m}{k} E_{2k} & \text{if } n = 2m. \end{cases} \tag{1.7}$$

We are going to prove Conjectures 1.1 and 1.2 in Section 2, where four permutation interpretations of Kreweras' triangle are derived as byproduct. Relying on the classical sign-balance results for the excedance polynomials over permutations and derangements, as well as certain elementary action on matrices, we will prove Conjectures 1.3–1.5 in Section 3. The proof of Conjecture 1.6 is a combination of the Foata–Strehl action [18] and the bivariate generating functions of two types of Poupard numbers studied by Foata and Han [16]. This will be done in Section 4, where a conjecture posted to [29, A005799] by Peter Bala is also confirmed. Moreover, we will give in the last section a new permutation interpretation for the γ-coefficients of the descent polynomials on the multiset $\{1,1,2,2,\dots,n,n\}$, and conclude our paper by posing some related open problems for further research.

2. Proofs of Conjectures 1.1-1.2 and some relevant results

2.1. **Proof of Conjecture 1.1.** We begin with some initial observations and analysis on Conjecture 1.1. Let

$$L_n := \left[\left\lfloor \frac{2j-k}{n} \right\rfloor\right]_{1\le j,k\le n},$$

which is a matrix with entries in $\{0,\pm 1\}$. We observe the following sign patterns (whether an entry is 1, 0 or -1) of the matrix L_n.

Fact 2.1. (1) *For even $n = 2m \ge 2$, we have $\operatorname{per}(L_n) = B_{n+1} = 0$, since the m-th row of L_n contains only zeros.*

(2) *For odd $n = 2m+1$, the first m rows of L_n begin with 0s and end with -1s, the next m rows (the $(m+1)$-th row to the $2m$-th row) begin with 1s and end with 0s, while the last row contains only 1s. In particular, we have $\operatorname{sgn}(\operatorname{per}(L_n)) = (-1)^m = \operatorname{sgn}(B_{n+1}) = -\operatorname{sgn}(G_{2m+2})$.*

(3) *For odd $n = 2m+1$, the m-th row of L_n begins with $2m$ consecutive 0s and ends with one -1, while the $(m+1)$-th row of L_n begins with one 1 and continues with $2m$ consecutive 0s.*

In view of Fact 2.1(1)-(2), it suffices to show that

$$|\operatorname{per}(L_{2m+1})| = (-1)^{m+1} G_{2m+2} \quad \text{for every } m = 0, 1, 2, \dots. \tag{2.1}$$

Now Fact 2.1(3) tells us that when we expand $\mathrm{per}(L_{2m+1})$ to get non-zero terms, the choices for the m-th and $(m+1)$-th rows are unique, namely, we are forced to choose the last entry -1 for the m-th row and the first entry 1 for the $(m+1)$-th row. Therefore, we shall delete these two rows, as well as the first and last columns of L_{2m+1}, and extract -1 from each of the first $m-1$ rows to consider the matrix $\tilde{L}_{2m-1} = (\ell_{i,j})_{1\le i,j\le 2m-1}$, where

$$\ell_{i,j} = \begin{cases} 1 & \text{if } 1 \le i \le m-1 \ \&\ 2i \le j \le 2m-1, \\ & \text{or } m \le i \le 2m-2 \ \&\ 1 \le j \le 2(i-m)+2, \text{ or } i = 2m-1, \\ 0 & \text{otherwise,} \end{cases}$$

and then observe that

$$\mathrm{per}(\tilde{L}_{2m-1}) = |\mathrm{per}(L_{2m+1})|. \tag{2.2}$$

For example, the first four matrices are $\tilde{L}_1 = [1]$, and

$$\tilde{L}_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ \tilde{L}_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},\ \tilde{L}_7 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

In order to calculate $\mathrm{per}(\tilde{L}_{2m-1})$, we expand $\tilde{L}_{2m-1}$ along the bottom row, and denote the $(2m-1,i)$-minor by $\tilde{L}_{2m-1,i}$ $(1 \le i \le 2m-1)$. That is, $\tilde{L}_{2m-1,i}$ is the permanent of the submatrix obtained from deleting the $(2m-1)$-th row and the i-th column of $\tilde{L}_{2m-1}$. The following recurrence relation fully characterizes these minors, and it is vital to our proof of Conjecture 1.1.

Lemma 2.2. *For integers $m \geqslant k \geqslant 1$, we have*

$$\tilde{L}_{2m-1,2k-1} = \tilde{L}_{2m-1,2k-2} + \sum_{i=2k-2}^{2m-3} \tilde{L}_{2m-3,i} \tag{2.3}$$

and

$$\tilde{L}_{2m-1,2k} = \tilde{L}_{2m-1,2k-1} - \sum_{i=1}^{2k-2} \tilde{L}_{2m-3,i}, \tag{2.4}$$

where we set $\tilde{L}_{2m-1,0} = \tilde{L}_{2m-1,2m} = 0$.

Proof. If we delete the m-th row and the $(2m-1)$-th row, and the first two columns of $\tilde{L}_{2m-1}$, we get a submatrix that becomes $\tilde{L}_{2m-3}$ once we flip it upside down, then left to right. This shows the $k=1$ case of (2.3).

Next, for $k \ge 2$, comparing the $(2k-1)$-th column of $\tilde{L}_{2m-1}$ with the $(2k-2)$-th column, we see that the only discrepancy is $\ell_{m+k-2,2k-2} = 1$ while $\ell_{m+k-2,2k-1} = 0$. This means the difference of the two minors, $\tilde{L}_{2m-1,2k-1} - \tilde{L}_{2m-1,2k-2}$, is given by the permanent of the submatrix obtained from deleting the $(m+k-2)$-th, $(2m-1)$-th rows, and the $(2k-2)$-th, $(2k-1)$-th columns of $\tilde{L}_{2m-1}$. Expanding this permanent along the $(k-1)$-th row, we see that it coincides with the summation

$$\sum_{i=2k-2}^{2m-3} \tilde{L}_{2m-3,i}.$$

This proves (2.3). The equality (2.4) can be proved similarly, and we omit the details. □

The signless Genocchi numbers $(-1)^n G_{2n}$ possess many interesting combinatorial and arithmetic properties. The first combinatorial interpretation of Genocchi numbers was found by Dumont [12] in 1974, which asserts that $|\mathcal{D}_{2n-1}| = (-1)^n G_{2n}$ with

$$\mathcal{D}_{2n-1} := \{\sigma \in \mathfrak{S}_{2n-1} : \forall\, i \in [2n-2],\ \sigma(i) > \sigma(i+1) \text{ if and only if } \sigma(i) \text{ is even}\}.$$

Since then, many other interpretations of Genocchi numbers have been found in the literature; see [4, 7, 14, 20, 22, 24] and the references therein.

Our final step for the proof of Conjecture 1.1 is to realize that the recurrences (2.3) and (2.4) are precisely the recurrences for Kreweras' triangle [22]:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 3 & 3 & 5 & 3 & 3 & & \\
 & 17 & 17 & 31 & 25 & 31 & 17 & 17 & \\
155 & 155 & 293 & 259 & 349 & 259 & 293 & 155 & 155 \\
 & & & & \vdots & & & &
\end{array}$$

This is a well-known triangle that refines the Genocchi numbers. Its entry in the m-th row and the k-th column is given by

$$K_{2m-1,k} := |\{\sigma \in \mathcal{D}_{2m+1} : \sigma(1) = k+1\}|.$$

For instance, $K_{5,2} = 3$ counts the three qualified permutations in $\mathcal{D}_7$ that begin with letter 3:

$$3421657, \quad 3564217, \quad 3642157.$$

Kreweras [22] proved that the triangle $K_{2m-1,k}$ $(1 \le k \le 2m-1)$ shares the same recurrence relation as $\tilde{L}_{2m-1,k}$ in Lemma 2.2. This leads to the following refinement of Conjecture 1.1.

Theorem 2.3. *Let m be a positive integer. For each $k = 1, 2, \ldots, 2m-1$, we have $\tilde{L}_{2m-1,k} = K_{2m-1,k}$. In particular, $\operatorname{per}(\tilde{L}_{2m-1}) = (-1)^{m+1} G_{2m+2}$, and hence Conjecture 1.1 holds.*

2.2. **Four permutation interpretations of Kreweras' triangle.** It is interesting to point out that Theorem 2.3 is equivalent to the following new permutation interpretation of Kreweras' triangle.

Corollary 2.4. *The Kreweras number $K_{2m-1,k}$ with $1 \le k \le 2m-1$ enumerates permutations $\sigma \in \mathfrak{S}_{2m-1}$ satisfying $\sigma^{-1}(2m-1) = k$ and $\sigma(i) \notin [\lfloor \frac{i}{2} \rfloor + 1, \lceil \frac{i}{2} \rceil + m - 2]$ for each $i \in [2m-1]$.*

Let $\gamma_{n,i}$ be the row vector with the first i entries being 0 and the remaining $n-i$ entries being 1. Let $\bar{\gamma}_{n,i} := \mathbf{1}_n^T - \gamma_{n,i}$ be the complement of $\gamma_{n,i}$, where $\mathbf{1}_n$ is the n-dimensional column vector with all entries equal to 1. For brevity, we write $\gamma_{n,i}$ as γ_i when n is fixed. Observe that

$$\tilde{L}_{2m-1} = (\gamma_1, \gamma_3, \ldots, \gamma_{2m-3}, \bar{\gamma}_2, \bar{\gamma}_4, \ldots, \bar{\gamma}_{2m-2}, \mathbf{1}_{2m-1}^T)^T.$$

Rearrange the rows of $\tilde{L}_{2m-1}$ to form another matrix

$$L^*_{2m-1} = (\bar{\gamma}_2, \bar{\gamma}_4, \ldots, \bar{\gamma}_{2m-2}, \mathbf{1}_{2m-1}^T, \gamma_1, \gamma_3, \ldots, \gamma_{2m-3})^T$$

with the same permanent. For example,

$$L_3^* = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \; L_5^* = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \; L_7^* = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Thus, Theorem 2.3 implies the following alternative permutation interpretation of Kreweras' triangle.

Corollary 2.5. *The Kreweras number $K_{2m-1,k}$ enumerates permutations $\sigma \in \mathfrak{S}_{2m-1}$ satisfying $\sigma^{-1}(m) = k$ and $\lceil \frac{i}{2} \rceil \le \sigma(i) \le m + \lfloor \frac{i}{2} \rfloor$ for each $i \in [2m-1]$.*

We may also rearrange the rows of $\tilde{L}_{2m-1}$ as

$$L_{2m-1}^{\star} = (\gamma_1, \bar{\gamma}_2, \gamma_3, \bar{\gamma}_4, \ldots, \gamma_{2m-3}, \bar{\gamma}_{2m-2}, \mathbf{1}_{2m-1}^T)^T.$$

For example,

$$L_3^{\star} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \; L_5^{\star} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \; L_7^{\star} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

This rearrangement, together with Theorem 2.3, yields the following third interpretation of Kreweras' triangle.

Corollary 2.6. *The Kreweras number $K_{2m-1,k}$ enumerates permutations $\sigma \in \mathfrak{S}_{2m-1}$ for which $\sigma(2m-1) = k$, and $\sigma(i) > i$ if and only if $i \in \{2n-1 : \; 1 \leqslant n < m\}$.*

A permutation $\sigma \in \mathfrak{S}_{2m}$ is called a *Dumont permutation of the second kind* [7] if $\sigma(2i) < 2i$ and $\sigma(2i-1) \ge 2i-1$ for every $i \in [m]$. It is clear that the permanent of the $2m \times 2m$ matrix

$$L_{2m}^{\star} = (\mathbf{1}_{2m}^T, \bar{\gamma}_1, \gamma_2, \bar{\gamma}_3, \gamma_4, \ldots, \bar{\gamma}_{2m-3}, \gamma_{2m-2}, \bar{\gamma}_{2m-1})^T$$

enumerates Dumont permutations of the second kind in $\mathfrak{S}_{2m}$. For example,

$$L_4^{\star} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \; L_6^{\star} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \; L_8^{\star} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Observe that $\bar{\gamma}_1 = (1, 0, 0, \ldots, 0)$. Deleting the second row and the first column from $L_{2m}^{\star}$ yields the $(2m-1) \times (2m-1)$-matrix $(\mathbf{1}_{2m-1}^T, \gamma_1, \bar{\gamma}_2, \gamma_3, \bar{\gamma}_4, \ldots, \gamma_{2m-3}, \bar{\gamma}_{2m-2})^T$, which is clearly a row rearrangement of $L_{2m-1}^{\star}$. This, together with Corollary 2.6, implies the following interpretation of Kreweras' triangle in terms of Dumont permutations of the second kind.

Corollary 2.7. *Let $\mathcal{D}^2_{2m}$ be the set of Dumont permutations of the second kind with length $2m$. Then the Kreweras number $K_{2m-1,k}$ enumerates $\sigma \in \mathcal{D}^2_{2m}$ satisfying $\sigma(1) = k+1$.*

Remark 2.8. A variation of Foata's first fundamental transformation[1] (cf. [30, Prop. 1.3.1]) establishes a one-to-one correspondence between $\{\sigma \in \mathcal{D}^2_{2m} : \sigma(1) = k+1\}$ and $\{\sigma \in \mathcal{D}_{2m+1} : \sigma(1) = k+1\}$, which provides an alternative approach to Corollary 2.7. In fact, we first write $\sigma \in \mathcal{D}^2_{2m}$ as a product of disjoint cycles meeting the following requirements:

(a) each cycle is written with the last element being the smallest one in the cycle;
(b) the cycles are arranged in increasing order of their smallest elements.

Now, define $\hat{\sigma}$ as the word (or permutation) obtained by first erasing the parentheses in the above form of σ and then adding the letter $2m+1$ to the end. Then $\sigma \mapsto \hat{\sigma}$ is the desired bijection between $\mathcal{D}^2_{2m}$ and $\mathcal{D}_{2m+1}$ such that $\sigma(1) = \hat{\sigma}(1)$.

2.3. **Proof of Conjecture 1.2.** We first make some observations similar to Fact 2.1 by determining the sign, rearranging the columns/rows, and deleting the two rows which contain a unique 1 or -1, etc.. This leads us to consider the matrix $\tilde{M}_{2n} = [m_{i,j}]_{1\le i,j\le 2n}$ (induced from M_{2n+2}), where

$$m_{i,j} = \begin{cases} 1 & \text{if } \lceil \frac{i}{2} \rceil \le j \le n + \lceil \frac{i}{2} \rceil, \\ 0 & \text{otherwise.} \end{cases}$$

The first three matrices of this type are given by

$$\tilde{M}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; \tilde{M}_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \; \tilde{M}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Conjecture 1.2 has the following equivalent version:

$$\mathrm{per}(\tilde{M}_{2n}) = H_{2n+1} \quad \text{for all } n \ge 1. \tag{2.5}$$

A salient feature of the matrix $\tilde{M}_{2n}$ is that the $(2i-1)$-th and $2i$-th rows are identical, for each $1 \le i \le n$. Therefore, when we expand the matrix to compute its permanent, our choices for the $(2i-1)$-th and $2i$-th rows can always be swapped. This is in agreement with the fact that H_{2n+1} is divisible by 2^n. Actually, the integers $h_n := H_{2n+1}/2^n$ $(n \ge 1)$ are usually referred to as the *normalized median Genocchi numbers*.

The permanent $\mathrm{per}(\tilde{M}_{2n})$ can be interpreted as a sum over certain subset of permutations in $\mathfrak{S}_{2n}$. Recall that

$$\mathrm{DES}(\pi) := \{i \in [n-1] : \pi(i) > \pi(i+1)\}$$

is the *descent set* of the permutation $\pi \in \mathfrak{S}_n$. Let $\tilde{\mathfrak{S}}_{2n}$ be the set of permutations $\pi = \pi(1)\cdots\pi(2n)$ in $\mathfrak{S}_{2n}$ for which $\{1, 3, 5, \ldots, 2n-1\} \subseteq \mathrm{DES}(\pi)$ and $\lceil \frac{i}{2} \rceil \le \pi(i) \le n + \lceil \frac{i}{2} \rceil$. By the previous discussions, $2^n \cdot |\tilde{\mathfrak{S}}_{2n}| = \mathrm{per}(\tilde{M}_{2n})$. Hence it remains to show that

$$|\tilde{\mathfrak{S}}_{2n}| = h_n. \tag{2.6}$$

Indeed, there is a well-known combinatorial model in the literature called the *Dellac configuration* [4, 21], which is known [14] to be enumerated by the normalized median Genocchi numbers h_n.

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[1] The original form of Foata's first fundamental transformation will be used in Section 4.1.

Definition 2.9. A Dellac configuration of size n is a tableau of width n and height $2n$ that contains $2n$ dots between the line $y = x$ and $y = n + x$, such that each row contains exactly one dot and each column contains exactly two dots.

Proof of Conjecture 1.2. Let DC_n be the set of Dellac configurations of size n. Now given a configuration $C \in \mathrm{DC}_n$, we read the y-coordinates of the dots in C, from the leftmost column to the rightmost and top-down inside each column. It is easy to check that this list of coordinates forms precisely a permutation in $\tilde{\mathfrak{S}}_{2n}$. This gives a bijection between DC_n and $\tilde{\mathfrak{S}}_{2n}$, and thus (2.6) holds true. □

3. Proofs of Conjectures 1.3∼1.5

A permutation $\pi \in \mathfrak{S}_n$ is called a *derangement* if $\pi(i) \neq i$ for all $i \in [n]$. For convenience, we set

$$\mathfrak{D}_n := \{\pi \in \mathfrak{S}_n : \pi(i) \neq i \text{ for all } i \in [n]\}.$$

Let us recall the following two classical results:

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{exc}(\pi)} = \begin{cases} (-1)^m E_{2m+1} & \text{if } n = 2m+1, \\ 0 & \text{if } n = 2m, \end{cases} \tag{3.1}$$

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{exc}(\pi)} = \begin{cases} 0 & \text{if } n = 2m+1, \\ (-1)^m E_{2m} & \text{if } n = 2m, \end{cases} \tag{3.2}$$

where $\mathrm{exc}(\pi)$ is the number of *excedances* of π, i.e.,

$$\mathrm{exc}(\pi) := |\{i \in [n] : \pi(i) > i\}|.$$

Note that (3.1) and (3.2) are due to Euler [13] and Roselle [28] respectively, and a joint combinatorial proof of them can be found in [17, Chap. 5] (see also [25]). We emphasize that the proofs of Conjectures 1.3∼1.5 hinge on linking these conjectures to the above two identities. In particular, the proof of Conjecture 1.3 is more involved and it requires block matrix decomposition and a variant of excedance that we denote as "exph". This proof is thus given after the proofs of Conjectures 1.4 and 1.5.

3.1. **Proof of Conjecture 1.4.** In view of the three matrices

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ P_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix}, \ P_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 \end{bmatrix},$$

it is not hard to see the connection between the pattern of signs of P_{2n} and the permutation statistic $\mathrm{exc}(\pi)$. More precisely, if we interpret each -1 entry of P_{2n} sitting at the i-th row (counting from top to bottom) and the j-th column (counting from right to left) as an excedance $i = \pi(j) > j$, then

$$\mathrm{per}(P_{2n}) = \sum_{\pi \in \mathfrak{D}_{2n}} (-1)^{\mathrm{exc}(\pi)}. \tag{3.3}$$

Applying (3.2) we get $\mathrm{per}(P_{2n}) = (-1)^n E_{2n}$.

A straightforward computation checks that the inverse matrix of P_{2n} is $[x_{i,j}]_{1\le i,j\le 2n}$, where

$$x_{i,j} = \begin{cases} 0 & \text{if } i+j = 2n+1, \\ 1 & \text{if } i+j < 2n+1 \text{ and } i \equiv j \pmod 2, \\ & \text{or } i+j > 2n+1 \text{ and } i \not\equiv j \pmod 2, \\ -1 & \text{otherwise.} \end{cases}$$

The first three inverse matrices are

$$P_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; P_4^{-1} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}, \; P_6^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

For each permutation $\pi \in \mathfrak{S}_n$, an index $i \in [n]$ is called an *excedance of type P*, if either $\pi(i) > i$ and $\pi(i) - i$ is odd, or $\pi(i) < i$ and $\pi(i) - i$ is even. Let $\mathrm{exc}_P(\pi)$ denote the number of type P excedances of π. Note that exc_P is not an Eulerian statistic, i.e., its distribution over $\mathfrak{S}_n$ is different from that of exc. Nonetheless, exc_P and exc have the same sign-balance over both $\mathfrak{S}_n$ and $\mathfrak{D}_n$.

Theorem 3.1. *For each $n \ge 1$, we have*

$$\sum_{\pi\in\mathfrak{S}_n} (-1)^{\mathrm{exc}(\pi)} = \sum_{\pi\in\mathfrak{S}_n} (-1)^{\mathrm{exc}_P(\pi)} \text{ and } \sum_{\pi\in\mathfrak{D}_n} (-1)^{\mathrm{exc}(\pi)} = \sum_{\pi\in\mathfrak{D}_n} (-1)^{\mathrm{exc}_P(\pi)}. \tag{3.4}$$

Proof. For our convenience, we shall consider inverse excedence ($\mathrm{iexc}(\pi) := \mathrm{exc}(\pi^{-1})$) instead of excedance. Note that exc and iexc are equidistributed over both $\mathfrak{S}_n$ and $\mathfrak{D}_n$.

The idea is to begin with the cycle decomposition of certain permutation $\pi \in \mathfrak{S}_{n-1}$, and insert n into π, say between i and $\pi(i)$, to get a new permutation $\pi' \in \mathfrak{S}_n$.

i	$\pi(i)$	$\pi(i) - i$	n	iexc	exc_P
even	even	> 0	even/odd	$+1$	$+1$
even	even	< 0	even/odd	$+0$	$+0$
even	odd	> 0	even	$+1$	-1
even	odd	> 0	odd	$+1$	$+1$
even	odd	< 0	even	$+0$	$+0$
even	odd	< 0	odd	$+0$	$+2$
odd	even	> 0	even	$+1$	$+1$
odd	even	> 0	odd	$+1$	-1
odd	even	< 0	even	$+0$	$+2$
odd	even	< 0	odd	$+0$	$+0$
odd	odd	> 0	even/odd	$+1$	$+1$
odd	odd	< 0	even/odd	$+0$	$+0$
even/odd	even/odd	$= 0$	even/odd	$+1$	$+1$

TABLE 1. Various cases of inserting n between i and $\pi(i)$

The rest of the proof is to verify, in a case-by-case fashion, that the parity changes of both iexc and exc_P are the same, when we go from π to π'. When n is a fixed point for π', we see neither iexc nor exc_P changes in this case. We collect the remaining cases in the above table so that the readers can easily check for themselves. The bottom case in the table means that n joins a fixed point $\pi(i) = i$ of π to form a 2-cycle $(i\ n)$ of π'. □

By the definitions of the inverse matrix P_{2n}^{-1} and the new statistic exc_P, we have

$$\mathrm{per}(P_{2n}^{-1}) = \sum_{\pi \in \mathfrak{D}_{2n}} (-1)^{\mathrm{exc}_P(\pi)}.$$

This result combined with (3.2) and the second identity in (3.4) proves the equality $\mathrm{per}(P_{2n}^{-1}) = (-1)^n E_{2n}$.

There is another elementary proof of the identity

$$\mathrm{per}(P_{2n}) = \mathrm{per}(P_{2n}^{-1}) \tag{3.5}$$

that we are going to provide below. The following elementary transformations concerning evaluations of permanents are clear.

Lemma 3.2. *Let $A = [a_{i,j}]_{1 \le i,j \le n}$ be an $n \times n$ matrix over a commutative ring R with identity.*

(1) *For any $1 \le k \le n$, the matrix obtained from A by multiplying each entry in the k-th column (or row) by $c \in R$ has permanent that equals $c \cdot \mathrm{per}(A)$.*
(2) *Exchanging any two columns (or rows) of A preserves the permanent.*

Definition 3.3 (Hadamard product)**.** For two matrices $A = [a_{i,j}]_{1 \le i,j \le n}$ and $B = [b_{i,j}]_{1 \le i,j \le n}$ over a ring, the Hadamard product of A and B is $A \circ B = [c_{i,j}]_{1 \le i,j \le n}$ with $c_{i,j} = a_{i,j} b_{i,j}$.

Definition 3.4 (An action on matrix)**.** Let $A = [a_{i,j}]_{1 \le i,j \le n}$ be an $n \times n$ matrix over $\mathbb{R}$. For $1 \le k, l \le n$, define $\phi_{k,l}(A)$ to be the matrix obtained from A by multiplying the k-th row and the l-th column by -1. Note that the (k,l)-entry of $\phi_{k,l}(A)$ remains $a_{k,l}$, since it multiplies -1 twice. By Lemma 3.2 (1), we have

$$\mathrm{per}(\phi_{k,l}(A)) = \mathrm{per}(A). \tag{3.6}$$

The usefulness of $\phi_{k,l}$ is demonstrated by the next lemma, and we shall derive further properties of this action in the last section; see Proposition 5.1.

Lemma 3.5. *Let $A = [a_{i,j}]_{1 \le i,j \le n}$ be an $n \times n$ matrix over $\mathbb{R}$. Then*

$$\Bigl(\prod_{k+l \le n} \phi_{k,l}\Bigr)(A) = A \circ H_n,$$

where $H_n = [h_{i,j}]_{1 \le i,j \le n}$ with $h_{i,j} = (-1)^{i+j}$. Consequently, $\mathrm{per}(A) = \mathrm{per}(A \circ H_n)$.

Proof. The number of times that the entry $a_{i,j}$ changes its sign under $\prod_{k+l \le n} \phi_{k,l}$ equals $n - i + n - j = 2n - (i + j)$, which has the same parity as $i + j$. This proves the first statement in Lemma 3.5. The second statement follows from the first one and the equality (3.6). □

In view of the relation

$$P_{2n}^{-1} = P_{2n} \circ H_{2n},$$

Lemma 3.5 provides an alternative approach to (3.5).

3.2. **Proof of Conjecture 1.5.** The first few terms of Q_n are

$$Q_1 = \begin{bmatrix} -1 \end{bmatrix}, \; Q_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \; Q_3 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \; Q_4 = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 \\ -1 & -1 & 0 & 1 \end{bmatrix},$$

$$Q_5 = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & 0 \\ 1 & -1 & -1 & -1 & 1 \\ 0 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \end{bmatrix}, \quad Q_6 = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 0 \\ 1 & 0 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 \end{bmatrix},$$

$$Q_7 = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix}, \quad Q_8 = \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

For $0 \le i \le n-1$, let $\alpha_{n,i}$ denote the column vector of dimension n whose $(n-i)$-th entry is 0 and all entries above (resp. below) this zero entry are 1 (resp. -1). For example, $\alpha_{6,2} = (1,1,1,0,-1,-1)^T$. For the sake of simplicity, we write $\alpha_{n,i}$ as α_i when n is fixed. The matrix Q_n has the following structure.

Lemma 3.6. *We have*

$$Q_{2n} = (\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}, -\alpha_0, -\alpha_2, \ldots, -\alpha_{2n-2})$$

and

$$Q_{2n+1} = (\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}, -\mathbf{1}_{2n+1}, -\alpha_1, -\alpha_3, \ldots, -\alpha_{2n-1}).$$

The following lemma is clear from the definition of permanents.

Lemma 3.7. *For a matrix with two identical columns having one zero at the same position, replacing the zero by* 1 *and the other zeros by* -1 *does not change its permanent.*

Proof of Conjecture 1.5. In view of (1.5),

$$\mathrm{per}(P_{2n}) = (-1)^n E_{2n}. \tag{3.7}$$

By Lemmas 3.2 and 3.6, we have

$$\begin{aligned} \mathrm{per}(Q_{2n}) &= (-1)^n \mathrm{per}(\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}, \alpha_0, \alpha_2, \ldots, \alpha_{2n-2}) \\ &= (-1)^n \mathrm{per}(\alpha_0, \alpha_1, \ldots, \alpha_{2n-1}) = (-1)^n \mathrm{per}(P_{2n}). \end{aligned}$$

Combining this with (3.7), we see that Conjecture 1.5 holds for even n.

It remains to deal with Conjecture 1.5 for odd n. Let β_i be the column vector obtained by replacing the only zero in α_i by 1. By Lemmas 3.2 and 3.7, we have

$$\begin{aligned} \mathrm{per}(Q_{2n+1}) &= (-1)^n \mathrm{per}(\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}, -\mathbf{1}_{2n+1}, \alpha_1, \alpha_3, \ldots, \alpha_{2n-1}) \\ &= (-1)^n \mathrm{per}(\alpha_1, \alpha_1, \alpha_3, \alpha_3, \ldots, \alpha_{2n-1}, \alpha_{2n-1}, -\mathbf{1}_{2n+1}) \\ &= (-1)^n \mathrm{per}(\beta_1, \beta_2, \ldots, \beta_{2n-1}, \beta_{2n}, -\mathbf{1}_{2n+1}) \end{aligned}$$

$$= (-1)^n \sum_{\sigma\in\mathfrak{S}_{2n+1}} (-1)^{\mathrm{wexc}(\sigma)},$$

where $\mathrm{wexc}(\sigma) := |\{i \in [2n+1] : \sigma(i) \geq i\}|$ is the number of weak excedances of σ. In view of the permutation interpretation (3.1) of tangent numbers, we have

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_{2n+1}} (-1)^{\mathrm{wexc}(\sigma)} &= \sum_{\sigma\in\mathfrak{S}_{2n+1}} (-1)^{2n+1-|\{i\in[2n+1]:\,\sigma(i)<i\}|}\\ &= -\sum_{\sigma\in\mathfrak{S}_{2n+1}} (-1)^{|\{i\in[2n+1]:\,\sigma(i)<i\}|}\\ &= -\sum_{\sigma\in\mathfrak{S}_{2n+1}} (-1)^{\mathrm{exc}(\sigma^{-1})}\\ &= -\sum_{\sigma\in\mathfrak{S}_{2n+1}} (-1)^{\mathrm{exc}(\sigma)} = (-1)^{n+1}E_{2n+1},\end{aligned}$$

which proves the odd case of Conjecture 1.5. □

3.3. **Proof of Conjecture 1.3.** We will make use of the multiplication of block matrices to facilitate our computation. For every $n \geq 1$, we introduce two $n \times n$ matrices:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad\text{and}\quad J_n = \begin{bmatrix} 0 & \cdots & 0 & 1\\ \vdots & \iddots & \iddots & 0\\ 0 & \iddots & \iddots & \vdots\\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Recall the matrix $P_n = [\mathrm{sgn}(\sin(\frac{i+j}{n+1}\pi))]_{1\leq i,j\leq n}$ defined in Conjecture 1.4. It is easy to verify that

$$J_n^2 = I_n \quad\text{and}\quad J_nP_n = -P_nJ_n. \tag{3.8}$$

We derive the following block decomposition of the matrix A_{2n} from its definition:

$$A_{2n} = \begin{bmatrix} P_n - J_n & -P_n\\ -P_n & P_n + J_n \end{bmatrix}.$$

With the aid of (3.8), it is fairly easy to check that the inverse matrix of A_{2n} has the following block decomposition:

$$A_{2n}^{-1} = \begin{bmatrix} P_n - J_n & P_n\\ P_n & P_n + J_n \end{bmatrix}.$$

In other words, $A_{2n}^{-1} = [\tilde{a}_{i,j}]_{1\leq i,j\leq 2n}$, where

$$\tilde{a}_{i,j} := \begin{cases} 0 & \text{if } i+j = 2n+1,\\ -1 & \text{if } i+j \geq n+1 \text{ and } \max(i,j) \leq n,\\ & \text{or if } i+j \geq 2n+2 \text{ and } i \leq n,\\ & \text{or if } i+j \geq 2n+2 \text{ and } j \leq n,\\ & \text{or if } i+j \geq 3n+2,\\ 1 & \text{otherwise.} \end{cases}$$

Comprehending the sign pattern of A_{2n}^{-1}, we are naturally led to define the following variant of excedance, that we call the number of *excedances induced by* ϕ (a bijection to be defined later), and denoted as $\mathrm{exph}(\pi)$ for each permutation $\pi \in \mathfrak{S}_{2n}$. Each $i \in [2n]$ is said to be an excedance of π induced by ϕ, if it falls in one of the following situations.

I. $1 \le i < \pi(i) \le n$;
II. $n+1 \le i \le 2n$ and $i-n \le \pi(i) \le n$;
III. $n+1 \le i < \pi(i) \le 2n$;
IV. $1 \le i < \pi(i) - n$ and $n+1 \le \pi(i) \le 2n$.

Now we can interprete $\mathrm{per}(A_{2n}^{-1})$ as the signed sum over all derangements of $[2n]$. Namely, each $\pi \in \mathfrak{D}_{2n}$ corresponds to one term in the expansion of $\mathrm{per}(A_{2n}^{-1})$, and each pair $(i, \pi(i))$ corresponds to the entry in the $\pi(i)$-th (from top to bottom) row and the i-th (from right to left) column of A_{2n}^{-1}. This entry is -1 if and only if i is an excedance of π introduced by ϕ; otherwise it is 1. In other words, we have

$$\mathrm{per}(A_{2n}^{-1}) = \sum_{\pi \in \mathfrak{D}_{2n}} (-1)^{\mathrm{exph}(\pi)}. \tag{3.9}$$

We give an example to illustrate the above relation.

Example 3.8. The six entries in A_6^{-1} that correspond to the derangement $\pi = 315624$ have been colored blue. Note that $\mathrm{exph}(\pi) = 3$ since π has excedances introduced by ϕ at 1 (case I), 4 (case III), and 5 (case II).

$$A_6^{-1} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 & -1 & -1 \end{bmatrix} \implies \mathrm{exph}(\pi) = 3 \text{ for } \pi = 315624.$$

Relying on the connection (3.9), we proceed to show that

$$\mathrm{per}(A_{2n}^{-1}) = (-1)^n E_{2n}. \tag{3.10}$$

Indeed, we will construct a bijection to show the following equidistribution result, which immediately implies (3.10) when we set $t = -1$ and $y = 0$, and apply (3.2). For any $\pi \in \mathfrak{S}_n$, let $\mathrm{Fix}(\pi) = \{i \in [n] : \pi(i) = i\}$ be the set of fixed points of π and $\mathrm{fix}(\pi) = |\mathrm{Fix}(\pi)|$ be its cardinality.

Theorem 3.9. *For each $n \ge 1$, we have*

$$\sum_{\pi \in \mathfrak{S}_{2n}} t^{\mathrm{exc}(\pi)} y^{\mathrm{fix}(\pi)} = \sum_{\pi \in \mathfrak{S}_{2n}} t^{\mathrm{exph}(\pi)} y^{\mathrm{fix}(\pi)}. \tag{3.11}$$

For each $x \in [2n]$, we define $y := \phi(x)$ by

$$y = \begin{cases} n+k & \text{if } x = 2k-1 \text{ for some } 1 \le k \le n, \\ k & \text{if } x = 2k \text{ for some } 1 \le k \le n. \end{cases} \tag{3.12}$$

Then ϕ is a permutation of $[2n]$, and it naturally induces a bijection on the symmetric group $\mathfrak{S}_{2n}$:

$$\begin{aligned} \Phi : \mathfrak{S}_{2n} &\to \mathfrak{S}_{2n} \\ \pi &\mapsto \sigma, \end{aligned}$$

where σ is obtained from π by replacing each i with $\phi(i)$ in the two-line notation of π. Take $\pi = 315462$ for example, we see

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 4 & 6 & 2 \end{pmatrix} \mapsto \begin{pmatrix} 4 & 1 & 5 & 2 & 6 & 3 \\ 5 & 4 & 6 & 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 1 & 5 & 6 & 3 \end{pmatrix},$$

so we have $\sigma = \Phi(\pi) = 421563$. One checks that $\mathrm{exc}(\pi) = \mathrm{exph}(\sigma) = 3$ and $\mathrm{fix}(\pi) = \mathrm{fix}(\sigma) = 1$.

Proof of Theorem 3.9. We show that the previous example is not a coincidence. Namely, if Φ maps π to σ, then we have

$$\mathrm{exc}(\pi) = \mathrm{exph}(\sigma) \tag{3.13}$$

and

$$\mathrm{fix}(\pi) = \mathrm{fix}(\sigma), \tag{3.14}$$

which readily imply (3.11). Since ϕ is a bijection, $i = \pi(i)$ if and only if $\phi(i) = \phi(\pi(i))$, and actually $\phi(\pi(i)) = \sigma(\phi(i))$. Consequently, i is a fixed point of π if and only if $\phi(i)$ is a fixed point of σ. This proves (3.14).

The proof of (3.13) is a case-by-case verification using the definition of the statistic "exph". We show one case here and leave the details of remaining cases to the interested reader. Suppose that $i < \pi(i)$ is an excedance of π such that $i = 2k-1$ is odd while $\pi(i) = 2j$ is even. According to (3.12), we have $\phi(i) = n+k$ while $\phi(\pi(i)) = j \geq k = \phi(i)-n$. So the pair $(\phi(i), \phi(\pi(i))) = (\phi(i), \sigma(\phi(i)))$ is in situation II and contributes 1 to $\mathrm{exph}(\sigma)$. $\square$

To complete the proof of Conjecture 1.3, it suffices now to show that

$$\mathrm{per}(A_{2n}) = \mathrm{per}(A_{2n}^{-1}). \tag{3.15}$$

This is done by utilizing the action $\phi_{k,l}$ on matrices introduced in Definition 3.4.

Lemma 3.10. *Let $A = [a_{i,j}]_{1\leq i,j\leq 2n}$ be a $2n \times 2n$ matrix over $\mathbb{R}$. Let U_n be the $n \times n$ matrix with all entries 1, and let*

$$\tilde{U}_{2n} = \begin{bmatrix} U_n & -U_n \\ -U_n & U_n \end{bmatrix}.$$

Then

$$\Bigg(\prod_{(k,l)\in S} \phi_{k,l}\Bigg)(A) = A \circ \tilde{U}_{2n},$$

where $S = \{(i,j) \in [2n] \times [2n] : i+j \in [n+1, 2n] \cup [3n+2, 4n]\}$. Consequently, $\mathrm{per}(A) = \mathrm{per}(A \circ \tilde{U}_{2n})$.

Proof. The number of times that the entry $a_{i,j}$ changes its sign under $\prod_{(k,l)\in S} \phi_{k,l}$ equals

$$\begin{cases} 2n-1, & \text{if } (i,j) \in [1,n] \times [n+1,2n] \cup [n+1,2n] \times [1,n]; \\ 2n-2, & \text{if } (i,j) \in [n+1,2n] \times [n+1,2n]; \\ 2n, & \text{if } (i,j) \in [1,n] \times [1,n]. \end{cases}$$

This proves the first statement in Lemma 3.10. The second statement follows from the first one and the equality (3.6). $\square$

Since

$$A_{2n}^{-1} = \begin{bmatrix} P_n - J_n & P_n \\ P_n & P_n + J_n \end{bmatrix} = A_{2n} \circ \tilde{U}_{2n},$$

the desired identity (3.15) follows from Lemma 3.10. This completes the proof of Conjecture 1.3.

4. Proof of Conjecture 1.6 and a variant of excedance

The proof of Conjecture 1.6 consists of three steps. Firstly, we adjust the original matrix R_n to get $\tilde{R}_n$ so that the modified permanent $\mathrm{per}(\tilde{R}_n)$, for n even (resp. odd), can be recognized as the sign balance of certain variant of the excedance statistic over permutations (resp. almost derangements). Next, with the aid of the celebrated *Foata–Strehl action* [18] on permutations and a simple bijection, we are able to identify these weighted sums as the so-called *central Poupard numbers*. Finally, we confirm the conjectured connections with the binomial transforms of Euler numbers, namely Eq. (1.7), utilizing the generating functions of Poupard numbers due to Foata and Han [16]. As a byproduct, we prove in subsection 4.3 a conjecture due to Peter Bala on a certain continued fraction expansion.

4.1. **An exc variant and cyclic valley-hopping.** The first few terms of R_n $(n \geq 1)$ read as follows:

$$R_1 = \begin{bmatrix} -1 \end{bmatrix},\ R_2 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},\ R_3 = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 0 \end{bmatrix},\ R_4 = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix},$$

$$R_5 = \begin{bmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 & 1 \end{bmatrix},\ R_6 = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix}.$$

After multiplying the i-th row by -1 for each $\lceil \frac{n}{2} \rceil \leq i \leq n$, and rearranging the rows we get the matrix $\tilde{R}_n$ as illustrated by the following examples:

$$\tilde{R}_1 = \begin{bmatrix} {\color{red}1} \end{bmatrix},\ \tilde{R}_3 = \begin{bmatrix} 1 & 1 & 0 \\ {\color{red}1} & {\color{red}1} & {\color{red}1} \\ 0 & -1 & -1 \end{bmatrix},\ \tilde{R}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & -1 \\ {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ 1 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \end{bmatrix},$$

$$\tilde{R}_2 = \begin{bmatrix} 1 & 1 \\ {\color{red}1} & {\color{red}1} \end{bmatrix},\ \tilde{R}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ 1 & -1 & -1 & -1 \end{bmatrix},\ \tilde{R}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

Note that

$$\mathrm{per}(R_n) = (-1)^{\lceil \frac{n+1}{2} \rceil} \mathrm{per}(\tilde{R}_n). \tag{4.1}$$

The comparison between $\tilde{R}_n$ and P_n suggests the following variation of excedance statistic.

Definition 4.1. For $\pi \in \mathfrak{S}_n$, define

$$\widetilde{\mathrm{exc}}(\pi) := |\{i \in [n] : \pi(i) > i \text{ and } \pi(i) \neq \lceil (n+1)/2 \rceil\}|.$$

Note that $\mathrm{exc}(\pi) - \widetilde{\mathrm{exc}}(\pi)$ equals 1 or 0 depending on whether $\lceil \frac{n+1}{2} \rceil$ is an excedance top or not.

The two identities below follow directly from the pattern of $\tilde{R}_n$ and the definition of $\widetilde{\text{exc}}$.

$$\text{per}(\tilde{R}_{2m}) = \sum_{\pi \in \mathfrak{S}_{2m}} (-1)^{\widetilde{\text{exc}}(\pi)}, \tag{4.2}$$

$$\text{per}(\tilde{R}_{2m+1}) = \sum_{\pi \in \tilde{\mathfrak{D}}_{2m+1}} (-1)^{\widetilde{\text{exc}}(\pi)}, \tag{4.3}$$

where $\tilde{\mathfrak{D}}_{2m+1} = \{\pi \in \mathfrak{S}_{2m+1} : \text{Fix}(\pi) \subseteq \{m+1\}\}$. In order to understand the cancellation happened in computing the permanents $\text{per}(\tilde{R}_n)$, we adopt Foata's first fundamental transformation and the Foata–Strehl action on permutations that we recall below.

In this section, we write each permutation in its *standard cycle form* according to the following convention:

(i) each cycle has its largest letter in the leftmost position;

(ii) the cycles are listed from left to right in increasing order of their largest letters.

For instance, the cycle form of $\pi = 562437198$ is $\pi = (4)(715326)(98)$. Foata's "transformation fondamentale" $o : \mathfrak{S}_n \to \mathfrak{S}_n$ is defined as follows: For each $\pi \in \mathfrak{S}_n$, the one-line notation of $o(\pi)$ is obtained from the standard cycle form of π^{-1} by erasing all the parentheses. For example, $o(\pi) = 471532698$ for $\pi = 735412698$, since $\pi^{-1} = 562437198$ has the cycle form $(4)(715326)(98)$.

For a permutation $\pi \in \mathfrak{S}_n$, introduce the *set of excedance tops* of π and the *set of descent tops* of π by

$$\text{Exct}(\pi) := \{\pi(i) : 1 \le i < n, i < \pi(i)\} \quad \text{and} \quad \text{Dest}(\pi) = \{\pi(i) : 1 \le i < n, \pi(i) > \pi(i+1)\},$$

respectively. The following result is known.

Lemma 4.2. *For the bijection* $o : \mathfrak{S}_n \to \mathfrak{S}_n$, *we have* $\text{Exct}(\pi) = \text{Dest}(o(\pi))$ *for each* $\pi \in \mathfrak{S}_n$.

Fix a permutation $\pi \in \mathfrak{S}_n$ and a letter $x \in [n]$. The *x-factorization* of π is the unique decomposition $\pi = w_1 w_2 x w_3 w_4$, where w_2 (resp. w_3) is the maximal consecutive subword (possibly empty) immediately to the left (resp. right) of x whose letters are all smaller than x. The Foata–Strehl action [18] $\varphi_x : \mathfrak{S}_n \to \mathfrak{S}_n$ can be defined by

$$\varphi_x(\pi) = w_1 w_3 x w_2 w_4. \tag{4.4}$$

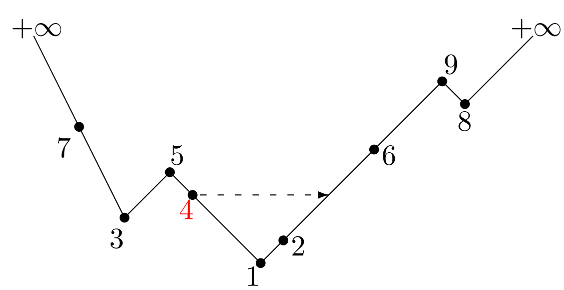

FIGURE 1. The Foata–Strehl action φ_x on 735412698 with $x = 4$.

For example, if $\pi = 735412698$ and $x = 4$, then the x-factorization yields $w_1 = 735$, $w_2 = \emptyset$, $w_3 = 12$, and $w_4 = 698$. Thus, $\varphi_x(\pi) = 735124698$; see Fig. 1 for a visualization of the action φ_x. Clearly φ_x is an involution on $\mathfrak{S}_n$ for every $x \in [n]$. Following Sun and Wang [31], we define the action $\psi_x : \mathfrak{S}_n \to \mathfrak{S}_n$ by $\psi_x(\pi) := o^{-1}(\varphi_x(o(\pi)))$. The action ψ_x is an involution on $\mathfrak{S}_n$ (as φ_x is an involution), and when x is properly chosen, it endows us with a sign-reversing bijection that proves Lemma 4.3 in the following.

Given a permutation $\pi = \pi(1)\pi(2)\cdots\pi(n)$ in $\mathfrak{S}_n$, we use the conventions $\pi(0) = \pi(n+1) = +\infty$ (resp. $\pi(0) = 0, \pi(n+1) = +\infty$) when n is even (resp. odd) and say that $\pi(i) \in [n]$ is:

- a *double ascent value* of π if $\pi(i-1) < \pi(i) < \pi(i+1)$;
- a *double descent value* of π if $\pi(i-1) > \pi(i) > \pi(i+1)$.

Denote by $\mathrm{Dasc}(\pi)$ (resp. $\mathrm{Ddes}(\pi)$) the set of double ascent (resp. descent) values of π. When $n = 2m$ is even, let $\mathfrak{S}^*_{2m}$ be the set of permutations π in $\mathfrak{S}_{2m}$ such that $\mathrm{Dasc}(o(\pi)) \cup \mathrm{Ddes}(o(\pi)) = \{m+1\}$. When $n = 2m+1$ is odd, let $\mathfrak{S}^*_{2m+1}$ be the set of permutations π in $\mathfrak{S}_{2m+1}$ such that $\mathrm{Dasc}(o(\pi)) \cup \mathrm{Ddes}(o(\pi)) = \{m+1\}$. Note that if $x \in \mathrm{Fix}(\pi)$, then the convention $o(\pi)(0) = 0$ guarantees that x is a double ascent value of $o(\pi)$. In particular, $\mathfrak{S}^*_{2m+1}$ is a subset of $\tilde{\mathfrak{D}}_{2m+1}$.

Lemma 4.3. *We have*

$$\sum_{\pi\in\mathfrak{S}_{2m}} (-1)^{\widetilde{\mathrm{exc}}(\pi)} = \sum_{\pi\in\mathfrak{S}^*_{2m}} (-1)^{\widetilde{\mathrm{exc}}(\pi)} = (-1)^{m-1}|\mathfrak{S}^*_{2m}|, \tag{4.5}$$

$$\sum_{\pi\in\tilde{\mathfrak{D}}_{2m+1}} (-1)^{\widetilde{\mathrm{exc}}(\pi)} = \sum_{\pi\in\mathfrak{S}^*_{2m+1}} (-1)^{\widetilde{\mathrm{exc}}(\pi)} = (-1)^{m}|\mathfrak{S}^*_{2m+1}|. \tag{4.6}$$

Proof. For convenience, we use $A\triangle B$ to denote the symmetric difference of two sets A and B. In particular, when $B = \{b\}$ is a singleton, we have

$$A\triangle\{b\} = \begin{cases} A\setminus\{b\} & \text{if } b\in A,\\ A\cup\{b\} & \text{if } b\notin A.\end{cases}$$

Let us first deal with the even case. Take any $\pi \in \mathfrak{S}_{2m}$ and set $\sigma = o(\pi)$. Note that the convention $\sigma(0) = \sigma(2m+1) = +\infty$ forces the union $\mathrm{Dasc}(\sigma)\cup\mathrm{Ddes}(\sigma)$ to be non-empty. Hence there are two possibilities for this union:

(i) There exists some letter $x \neq m+1$ and $x \in \mathrm{Dasc}(\sigma)\cup\mathrm{Ddes}(\sigma)$.
(ii) $\mathrm{Dasc}(\sigma)\cup\mathrm{Ddes}(\sigma) = \{m+1\}$, i.e., $\pi \in \mathfrak{S}^*_{2m}$.

If we are in case (i), let x be the smallest such letter, then we see that

$$\mathrm{Dasc}(\varphi_x(\sigma)) = \mathrm{Dasc}(\sigma)\triangle\{x\}, \text{ and } \mathrm{Ddes}(\varphi_x(\sigma)) = \mathrm{Ddes}(\sigma)\triangle\{x\}.$$

Consequently, $\mathrm{Dest}(\varphi_x(\sigma)) = \mathrm{Dest}(\sigma)\triangle\{x\}$, and $\mathrm{Exct}(\psi_x(\pi)) = \mathrm{Exct}(\pi)\triangle\{x\}$ by Lemma 4.2. Since $x \neq m+1$, we deduce that $\widetilde{\mathrm{exc}}(\pi)$ and $\widetilde{\mathrm{exc}}(\psi_x(\pi))$ have opposite parity, so they cancel each other in the first summation of (4.5). All remaining permutations are in case (ii), thus we have proved the first equality in (4.5).

It is not difficult to see the following alternative description of $\mathfrak{S}^*_{2m}$.

A characterization of $\mathfrak{S}^*_{2m}$: Every permutation σ in $o(\mathfrak{S}^*_{2m})$ can be uniquely constructed from an up-down permutation $\hat\sigma$ (i.e., $\hat\sigma(1) < \hat\sigma(2) > \hat\sigma(3) < \cdots < \hat\sigma(2m-2) > \hat\sigma(2m-1)$) consisting of letters from $[2m]\setminus\{m+1\}$, by inserting $m+1$ in one of the following ways:

(e1) before $\hat\sigma(1)$ if $m+1 > \hat\sigma(1)$;
(e2) after $\hat\sigma(2m-1)$ if $m+1 > \hat\sigma(2m-1)$;
(e3) inbetween $\hat\sigma(i)$ and $\hat\sigma(i+1)$ such that $\hat\sigma(i) > m+1 > \hat\sigma(i+1)$;
(e4) inbetween $\hat\sigma(i)$ and $\hat\sigma(i+1)$ such that $\hat\sigma(i) < m+1 < \hat\sigma(i+1)$.

Recall that $\widetilde{\mathrm{exc}}$ rejects $m+1$ as an excedance top, so inserting $m+1$ does not affect the value of $\widetilde{\mathrm{exc}}$ in view of Lemma 4.2. Moreover, the excedance tops counted by $\widetilde{\mathrm{exc}}(o^{-1}(\hat\sigma))$ are precisely the peaks of $\hat\sigma$ (i.e., those $\hat\sigma(i)$ with $\hat\sigma(i-1) < \hat\sigma(i) > \hat\sigma(i+1)$, or to be precise, $\hat\sigma(2), \hat\sigma(4), \ldots, \hat\sigma(2m-2)$, since $\hat{\sigma}$ is an up-down permutation). So $\widetilde{\mathrm{exc}}(\pi) = m-1$ for every $\pi \in \mathfrak{S}^*_{2m}$, which takes care of the second equality in (4.5).

Next for the odd case, again set $\sigma = o(\pi)$ for a given $\pi \in \tilde{\mathfrak{D}}_{2m+1}$. New conventions $\sigma(0) = 0$ and $\sigma(2m+2) = +\infty$ force the union $\mathrm{Dasc}(\sigma) \cup \mathrm{Ddes}(\sigma)$ to be non-empty. We proceed as in the even case, to find the smallest $x \in \mathrm{Dasc}(\sigma) \cup \mathrm{Ddes}(\sigma) \setminus \{m+1\}$ if it exists, and apply ψ_x to explain the cancellations and establish the first equality of (4.6). To show the second equality of (4.6), we resort to the following equivalent description of $\mathfrak{S}^*_{2m+1}$.

> **A characterization of $\mathfrak{S}^*_{2m+1}$:** Every permutation σ in $o(\mathfrak{S}^*_{2m+1})$ can be uniquely constructed from a down-up permutation $\hat{\sigma}$ (i.e., $\hat{\sigma}(1) > \hat{\sigma}(2) < \hat{\sigma}(3) > \cdots < \hat{\sigma}(2m-1) > \hat{\sigma}(2m)$) consisting of letters from $[2m+1] \setminus \{m+1\}$, by inserting $m+1$ in one of the following ways:
> (o1) before $\hat{\sigma}(1)$ if $m+1 < \hat{\sigma}(1)$;
> (o2) after $\hat{\sigma}(2m)$ if $m+1 > \hat{\sigma}(2m)$;
> (o3) inbetween $\hat{\sigma}(i)$ and $\hat{\sigma}(i+1)$ such that $\hat{\sigma}(i) > m+1 > \hat{\sigma}(i+1)$;
> (o4) inbetween $\hat{\sigma}(i)$ and $\hat{\sigma}(i+1)$ such that $\hat{\sigma}(i) < m+1 < \hat{\sigma}(i+1)$.

It follows from the above description and Lemma 4.2 that $\widetilde{\mathrm{exc}}(\pi) = m$ for every $\pi \in \mathfrak{S}^*_{2m+1}$. This completes the proof of the lemma. □

4.2. **Poupard numbers and the proof of Conjecture 1.6.** In view of (4.1)-(4.3) and (4.5)-(4.6), we get

$$\mathrm{per}(R_{2m}) = |\mathfrak{S}^*_{2m}| \quad \text{and} \quad \mathrm{per}(R_{2m+1}) = -|\mathfrak{S}^*_{2m+1}|. \tag{4.7}$$

Therefore, it remains to enumerate $\mathfrak{S}^*_{2m}$ and $\mathfrak{S}^*_{2m+1}$. It turns out that they are in simple bijections with certain subsets of alternating permutations which were previously investigated by Foata and Han [16], in their course of deriving new combinatorial interpretations for the finite difference equation system introduced by Christiane Poupard.

Let $\mathfrak{A}_n$ denote the set of alternating (down-up) permutations. For any $\pi \in \mathfrak{A}_n$, we suppose $\pi(i) = n$ for a certain i $(1 \le i \le n)$ and use the convention $\pi(0) = \pi(n+1) = 0$. Following Foata and Han [16], we introduce

$$\mathrm{grn}(\pi) := \max\{\pi(i-1), \pi(i+1)\},$$

and call it the ***gr**eater **n**eighbour* of n in π. Also, let $\mathfrak{A}_{n,k} := \{\pi \in \mathfrak{A}_n : \mathrm{grn}(\pi) = k\}$ for each $0 \le k \le n-1$.

Lemma 4.4. *For each $n \ge 1$, there exists a bijection $f : \mathfrak{S}^*_n \to \mathfrak{A}_{n+1,\lfloor \frac{n+1}{2} \rfloor}$. In particular, we have $|\mathfrak{S}^*_{2m}| = |\mathfrak{A}_{2m+1,m}|$ and $|\mathfrak{S}^*_{2m+1}| = |\mathfrak{A}_{2m+2,m+1}|$.*

Proof. For the even case with $n = 2m$, we refer to the boxed characterization of $\mathfrak{S}^*_{2m}$ given in the proof of Lemma 4.3, and construct the image $f(\pi)$ for each $\pi \in \mathfrak{S}^*_{2m}$ (set $\sigma = o(\pi)$) accordingly. We first apply the *complement map*

$$\tau = \tau(1) \cdots \tau(n) \mapsto \tau^{\mathrm{c}} := (n+1-\tau(1)) \cdots (n+1-\tau(n)) \tag{4.8}$$

to σ, and then insert $2m+1$ as follows.

(e1) If $\sigma^{\mathrm{c}}(1) = m$, then we insert $2m+1$ before m to get a new permutation $f(\pi)$.
(e2) If $\sigma^{\mathrm{c}}(2m) = m$, then we insert $2m+1$ after m to get a new permutation $f(\pi)$.
(e3) When $\sigma^{\mathrm{c}}(i) < \sigma^{\mathrm{c}}(i+1) = m < \sigma^{\mathrm{c}}(i+2)$, we insert $2m+1$ between m and $\sigma^{\mathrm{c}}(i)$ to get a new permutation $f(\pi)$.
(e4) When $\sigma^{\mathrm{c}}(i) > \sigma^{\mathrm{c}}(i+1) = m > \sigma^{\mathrm{c}}(i+2)$, we insert $2m+1$ between m and $\sigma^{\mathrm{c}}(i+2)$ to get a new permutation $f(\pi)$.

It is readily verified that in all the four cases, $f(\pi)$ is indeed a down-up permutation of length $2m+1$, whose greater neighbor of $2m+1$ is m. So f is well-defined and we get its inverse simply by removing $2m+1$ and then applying the complement map and o^{-1}.

The map f for the odd case with $n = 2m+1$ can be constructed similarly without applying complement map before we insert $2m+2$. The details are omitted. □

The last step towards proving Conjecture 1.6 is to utilize the bivariate generating function for all Poupard numbers $|\mathfrak{A}_{n,k}|$. Set $g_n(k) := |\mathfrak{A}_{2n-1,k-1}|$ $(n \geq 1,\ 1 \leq k \leq 2n-1)$, and $h_n(k) := |\mathfrak{A}_{2n,k}|$ $(n \geq 1,\ 1 \leq k \leq 2n-1)$.

Theorem 4.5 (Theorem 1.2 in [16])**.** *We have*

$$1 + \sum_{n\geq 1}\sum_{1\leq k\leq 2n+1} g_{n+1}(k)\frac{x^{2n+1-k}}{(2n+1-k)!}\cdot\frac{y^{k-1}}{(k-1)!} = \sec(x+y)\cos(x-y) \tag{4.9}$$

and

$$1 + \sum_{n\geq 1}\sum_{1\leq k\leq 2n+1} h_{n+1}(k)\frac{x^{2n+1-k}}{(2n+1-k)!}\cdot\frac{y^{k-1}}{(k-1)!} = \sec^2(x+y)\cos(x-y). \tag{4.10}$$

Lemma 4.6. *For each $n = 0, 1, 2, \ldots,$ we have*

$$g_{n+1}(n+1) = \sum_{k=0}^{n}\binom{n}{k}E_{2k} \tag{4.11}$$

and

$$h_{n+1}(n+1) = \sum_{k=0}^{n}\binom{n}{k}E_{2k+1}. \tag{4.12}$$

Proof. Observe that

$$\begin{aligned}\sec(x+y)\cos(x-y) &= \left(\sum_{k\geq 0}E_{2k}\frac{(x+y)^{2k}}{(2k)!}\right)\left(\sum_{k\geq 0}(-1)^k\frac{(x-y)^{2k}}{(2k)!}\right)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n-k}\frac{E_{2k}(x+y)^{2k}}{(2k)!}\cdot\frac{(x-y)^{2n-2k}}{(2n-2k)!}.\end{aligned}$$

Combining this with (4.9), we get

$$g_{n+1}(n+1) = \sum_{k=0}^{n}(-1)^k E_{2k}\sum_{i=0}^{2k}\binom{n}{i}\binom{n}{2k-i}(-1)^i,$$

which gives (4.11) since

$$(-1)^k\binom{n}{k} = [x^{2k}](1-x^2)^n = [x^{2k}](1-x)^n(1+x)^n = \sum_{i=0}^{2k}\binom{n}{i}\binom{n}{2k-i}(-1)^i,$$

where $[x^m]f(x)$ denotes the coefficient of x^m in the power series expansion of $f(x)$. The second formula (4.12) follows from the same manipulation by noticing $\tan' x = \sec^2 x$. □

With all the needed pieces on hand, Conjecture 1.6 now follows from (4.7), and Lemmas 4.4 and 4.6.

4.3. **Bala's continued fraction conjecture.** Recall that the *descent number* of a word $w = w_1 w_2 \cdots w_n$ over $\mathbb{N}$ is

$$\mathrm{des}(w) := |\{i \in [n-1] : w_i > w_{i+1}\}|.$$

The classical *Eulerian polynomial* $A_n(t)$ may be defined by Euler's formula

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (k+1)^n t^k. \tag{4.13}$$

It is well known that $A_n(t)$ has the following two interpretations (see [30, Chap. 1]):

$$\sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)} = A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{exc}(\pi)}.$$

Consider a variation of the Eulerian polynomials: $\tilde{A}_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\widetilde{\mathrm{exc}}(\pi)}$ for $n \geq 1$. For convenience, we list the first few terms of $\tilde{A}_n(t)$ as follows:

$$\begin{aligned}
\tilde{A}_1(t) &= 1,\\
\tilde{A}_2(t) &= 2,\\
\tilde{A}_3(t) &= 2 + 4t,\\
\tilde{A}_4(t) &= 4 + 16t + 4t^2,\\
\tilde{A}_5(t) &= 4 + 48t + 60t^2 + 8t^3,\\
\tilde{A}_6(t) &= 8 + 160t + 384t^2 + 160t^3 + 8t^4,\\
\tilde{A}_7(t) &= 8 + 368t + 1952t^2 + 2176t^3 + 520t^4 + 16t^5,\\
\tilde{A}_8(t) &= 16 + 1152t + 9648t^2 + 18688t^3 + 9648t^4 + 1152t^5 + 16t^6.
\end{aligned}$$

We have the following result for $\tilde{A}_{2n}(t)$.

Theorem 4.7. *Let $\mathfrak{S}_n^{(2)}$ be the set of all permutations of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$. Then*

$$\tilde{A}_{2n}(t) = 2^n \sum_{\pi \in \mathfrak{S}_n^{(2)}} t^{\mathrm{des}(\pi)}. \tag{4.14}$$

The polynomial $\sum_{\pi \in \mathfrak{S}_n^{(2)}} t^{\mathrm{des}(\pi)}$ that we denote by $A_n^{(2)}(t)$ is called the nth 2-*Eulerian polynomial.* Analog to Euler's formula (4.13), MacMahon [26, Volume 2, p. 211] proved that

$$\frac{A_n^{(2)}(t)}{(1-t)^{2n+1}} = \sum_{k \geq 0} \binom{k+2}{2}^n t^k. \tag{4.15}$$

Many interesting properties of the 2-Eulerian polynomials have been extensively studied ever since [2, 8, 23]. Remarkably, Ardila [2] proved that 2-Eulerian polynomials are the h-polynomials of the dual bipermutahedron.

To prove Theorem 4.7, we need two lemmas.

Lemma 4.8. *Let $\tilde{A}(1, 0) := 1$ and $\tilde{A}(n, k) := |\{\pi \in \mathfrak{S}_n : \widetilde{\mathrm{exc}}(\pi) = k\}|$ for $n \geq 2$ and $0 \leq k \leq n-2$. Then $\tilde{A}(n, k)$ satisfies the following recurrence relation:*

$$\tilde{A}(n, k) = \begin{cases} (n-k-1)\tilde{A}(n-1, k-1) + (k+2)\tilde{A}(n-1, k), & \text{if } n \text{ is even;}\\ (n-k)\tilde{A}(n-1, k-1) + (k+1)\tilde{A}(n-1, k), & \text{if } n \text{ is odd.} \end{cases}$$

Consequently,

$$\tilde{A}(2n,k) = (k+2)(k+1)\tilde{A}(2n-2,k) + (2n-k-1)(2k+2)\tilde{A}(2n-2,k-1) + (2n-k)(2n-k-1)\tilde{A}(2n-2,k-2).$$

Proof. Consider a permutation $\pi \in \mathfrak{S}_{n-1}$ in cycle form (i.e., write π as a product of disjoint cycles). For any $a \in [n-1]$ with $\pi(a) \neq \lceil \frac{n}{2} \rceil$, we call the slot inbetween a and $\pi(a)$ a *decreasing (resp. increasing) slot* if $a \geq \pi(a)$ (resp. $a < \pi(a)$). Clearly, if $\widetilde{\mathrm{exc}}(\pi) = i$, then π has i increasing slots and $n-2-i$ decreasing slots.

If $n \geq 3$ is odd, then a permutation counted by $\tilde{A}(n,k)$ can be constructed

- either from π with $\widetilde{\mathrm{exc}}(\pi) = k-1$ by inserting n into one of its $n-k-1$ decreasing slots or just before $\lceil n/2 \rceil$,
- or from π with $\widetilde{\mathrm{exc}}(\pi) = k$ by inserting n into one of its k increasing slots or setting n as a 1-cycle.

When n is even, a permutation counted by $\tilde{A}(n,k)$ can be constructed

- either from π with $\widetilde{\mathrm{exc}}(\pi) = k-1$ by inserting 0 into one of its $n-k-1$ decreasing slots and then increasing all letters by 1,
- or from π with $\widetilde{\mathrm{exc}}(\pi) = k$ in one of the following three ways and then increasing all letters by 1:
  (1) inserting 0 into one of its k increasing slots;
  (2) inserting 0 just before $\lceil n/2 \rceil$;
  (3) setting 0 as a 1-cycle.

This proves the desired recurrence relation for $\tilde{A}(n,k)$. □

Lemma 4.9. *Write $A_n^{(2)}(t) = \sum_{k=0}^{2n-2} A^{(2)}(n,k)t^k$. Then*

$$A^{(2)}(n,k) = \binom{k+2}{2} A^{(2)}(n-1,k) + (2n-k-1)(k+1)A^{(2)}(n-1,k-1) + \binom{2n-k}{2} A^{(2)}(n-1,k-2).$$

Proof. Applying MacMahon's formula (4.15), we have

$$\frac{A_n^{(2)}(t)}{(1-t)^{2n+1}} = \sum_{k\geq 0} \binom{k+2}{2}^n t^k = \sum_{k\geq 0} \binom{k+2}{2}^{n-1} (k(k-1)/2 + 2k + 1)t^k$$
$$= \frac{t^2}{2}(A_{n-1}^{(2)}(t)(1-t)^{-2n+1})'' + 2t(A_{n-1}^{(2)}(t)(1-t)^{-2n+1})' + A_{n-1}^{(2)}(t)(1-t)^{-2n+1}.$$

Multiplying both sides by $(1-t)^{2n+1}$ gives

$$A_n^{(2)}(t) = (A_{n-1}^{(2)}(t))''t^2(1-t)^2/2 + (A_{n-1}^{(2)}(t))'t(1-t)((2n-3)t+2) + A_{n-1}^{(2)}(t)(2n^2t^2 - 5nt^2 + 3t^2 + 4nt - 4t + 1).$$

Extracting the coefficients of t^k from both sides of the above equation yields the desired recurrence relation for $A^{(2)}(n,k)$. □

Proof of Theorem 4.7. Comparing the recurrence relation for $\tilde{A}(2n,k)$ in Lemma 4.8 with that for $A^{(2)}(n,k)$ in Lemma 4.9, we get the desired (4.14). □

The following fundamental generating function is known as *Touchard's continued fraction* [36] (see also [27]):

$$\sum_{k\geq 0} q^{\binom{k+1}{2}} z^k = \cfrac{1}{1-z+\cfrac{(1-q)z}{1-z+\cfrac{(1-q^2)z}{1-z+\cfrac{(1-q^3)z}{\cdots}}}}.$$

Theorem 4.10. *The exponential generating function for $A_n^{(2)}(t)$ has the following continued fraction expansion:*

$$\sum_{n\geq 0} \frac{tA_n^{(2)}(t)}{n!} z^n = t-1+\cfrac{1-t}{1-t+\cfrac{(1-e^{(1-t)^2 z})t}{1-t+\cfrac{(1-e^{2(1-t)^2 z})t}{1-t+\cfrac{(1-e^{3(1-t)^2 z})t}{\cdots}}}}. \tag{4.16}$$

Proof. By MacMahon's formula (4.15), we have

$$tA_n^{(2)}(t) = (1-t)^{2n+1} \sum_{k\geq 0} \binom{k+2}{2}^n t^{k+1}.$$

Multiplying both sides by $\frac{z^n}{n!}$ and summing over all $n \geq 0$ gives

$$\begin{aligned} \sum_{n\geq 0} \frac{tA_n^{(2)}(t)}{n!} z^n &= \sum_{n\geq 0} \frac{(1-t)^{2n+1} z^n}{n!} \sum_{k\geq 0} \binom{k+2}{2}^n t^{k+1} \\ &= (1-t)\sum_{k\geq 0} t^{k+1} \sum_{n\geq 0} \frac{(\binom{k+2}{2}(1-t)^2 z)^n}{n!} \\ &= (1-t)\sum_{k\geq 0} t^{k+1} (e^{(1-t)^2 z})^{\binom{k+2}{2}}. \end{aligned}$$

Applying Touchard's continued fraction yields (4.16). □

The number $2^{-n}\sum_{k=0}^{n}\binom{n}{k}E_{2k}$ is called *generalized Euler number of type* 2^n as in [29, A005799]. Combining (1.7), (4.1), (4.2) and (4.14), we get

$$(-1)^{n+1} A_n^{(2)}(-1) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} E_{2k}, \quad \text{for } n \geq 1. \tag{4.17}$$

It then follows from Theorem 4.10 the following exponential generating function for $2^{-n}\sum_{k=0}^{n}\binom{n}{k}E_{2k}$, which was conjectured by Peter Bala (2019) in [29, A005799].

Corollary 4.11. *We have*

$$\sum_{n\geq 0} \frac{2^{-n}\sum_{k=0}^{n}\binom{n}{k}E_{2k}}{n!} z^n = \cfrac{2}{2-\cfrac{1-e^{-4z}}{2-\cfrac{1-e^{-8z}}{2-\cfrac{1-e^{-12z}}{\cdots}}}}.$$

A permutation $\pi \in \mathfrak{S}_n^{(2)}$ is *alternating* if

$$\pi(1) \leq \pi(2) > \pi(3) \leq \pi(4) > \pi(5) \leq \cdots.$$

Let Alt_n be the set of all alternating permutations in $\mathfrak{S}_n^{(2)}$. On one hand, Gessel [19, Eq. (6)] proved that

$$|\mathrm{Alt}_n| = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} E_{2k}.$$

On the other hand, Lin–Ma–Ma–Zhou [23, Corollary 2.15] interpreted the γ-coefficients $\gamma_{n,k}^{(2)}$ appearing in the expansion (see the survey of Athanasiadis [3] on the theme of γ-positivity)

$$A_n^{(2)}(t) = \sum_{k=0}^{n-1} \gamma_{n,k}^{(2)} t^k (1+t)^{2(n-1)-2k} \tag{4.18}$$

as some class of *weakly increasing trees* (see [23] for the definition). It follows from the above expansion and Eq. (4.17) that

$$\gamma_{n,n-1}^{(2)} = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} E_{2k}.$$

Thus, we have the following interesting equinumerosity.

Corollary 4.12. *The number of alternating permutations in $\mathfrak{S}_n^{(2)}$ equals the number of weakly increasing trees on $\{1, 1, 2, 2, \ldots, n-1, n-1, n\}$ with n leaves and without young leaves.*

It would be interesting to find a bijective proof of Corollary 4.12, which would provide an alternative approach to the even case of Conjecture 1.6.

4.4. **Combinatorics of the γ-positivity of $\tilde{A}_{2m}(t)$.** The rest of this section is devoted to a group action proof of the γ-positivity of $\tilde{A}_{2m}(t)$ that results in a new interpretation of $\gamma_{n,k}^{(2)}$ defined in (4.18).

For any $\pi \in \mathfrak{S}_{2m}$, note that $|\mathrm{Dest}(\pi)| = \mathrm{des}(\pi)$. Introduce a variant of descents of π:

$$\widetilde{\mathrm{des}}(\pi) := |\mathrm{Dest}(\pi) \setminus \{m+1\}|.$$

By Lemma 4.2, we have

$$\tilde{A}_{2m}(t) = \sum_{\pi \in \mathfrak{S}_{2m}} t^{\widetilde{\mathrm{des}}(\pi)}.$$

With the convention $\pi(0) = \pi(2m+1) = +\infty$, for $i \in [2m]$, the letter $\pi(i)$ is called a *valley* (resp. *peak*) of π if $\pi(i-1) > \pi(i) < \pi(i+1)$ (resp. $\pi(i-1) < \pi(i) > \pi(i+1)$). Denote by $\mathrm{Val}(\pi)$ and $\mathrm{Peak}(\pi)$ the set of valleys and the set of peaks of π, respectively.

Theorem 4.13. *The polynomial $\tilde{A}_{2m}(t)$ has the γ-positive expansion*

$$\tilde{A}_{2m}(t) = \sum_{\pi \in \mathfrak{S}_{2m}} t^{\widetilde{\mathrm{des}}(\pi)} = \sum_{k=0}^{m-1} |\tilde{D}_{2m,k}| t^k (1+t)^{2m-2-2k}, \tag{4.19}$$

where

$$\tilde{D}_{2m,k} := \{\pi \in \mathfrak{S}_{2m} : \mathrm{Ddes}(\pi) \setminus \{m+1\} = \emptyset, m+1 \notin \mathrm{Val}(\pi) \text{ and } \widetilde{\mathrm{des}}(\pi) = k\}.$$

Example 4.14. For instance, we have

$$\tilde{D}_{4,0} = \{1234, 3124, 1324, 2314\} \text{ and } \tilde{D}_{4,1} = \{1342, 1423, 1432, 2341, 2413, 2431, 3142, 3241\},$$

and so

$$\tilde{A}_4(t) = 4(1+t)^2 + 8t.$$

An immediate consequence of Theorems 4.7 and 4.13 is the following permutation interpretation of $\gamma_{n,k}^{(2)}$ (see [23, Corollary 2.15] for another interpretation of $\gamma_{n,k}^{(2)}$ in terms of trees).

Corollary 4.15. *Let $\gamma_{n,k}^{(2)}$ be defined in* (4.18)*. Then $\gamma_{n,k}^{(2)} = 2^{-n}|\tilde{D}_{2n,k}|$ for $n \geq 1$.*

In order to prove Theorem 4.13, we need the following interesting equidistribution.

Lemma 4.16. *There exists a bijection η preserving the number of descents between*

$$\mathcal{P}_m := \{\pi \in \mathfrak{S}_{2m} : \mathrm{Ddes}(\pi) = \emptyset, m+1 \text{ is a peak}\}$$

and

$$\mathcal{V}_m := \{\pi \in \mathfrak{S}_{2m} : \mathrm{Ddes}(\pi) = \emptyset, m+1 \text{ is a valley}\}.$$

The proof of the above lemma can be considered as a nice application of the classical *Françon–Viennot bijection* [15] that encodes permutations as Laguerre histories.

Recall that a *Motzkin path* of length n is a lattice path in the first quadrant starting from $(0,0)$, ending at $(n,0)$, and using three possible steps:

$$U = (1,1) \text{ (up step)}, \; L = (1,0) \text{ (level step)}, \text{ and } D = (1,-1) \text{ (down step)}.$$

A Motzkin path in which each level step is further distinguished into two different types L_0 (in blue) and L_1 (in red) is called a 2-*Motzkin path.* Thus, each 2-Motzkin path can be represented as a word over the alphabet $\{U, D, L_0, L_1\}$. A *Laguerre history* of length n is a pair (w, μ), where $w = w_1 \cdots w_n$ is a 2-Motzkin path and $\mu = (\mu_1, \cdots, \mu_n)$ is a sequence of weights satisfying $0 \leq \mu_i \leq h_i(w)$, and $h_i(w)$ denotes the *height* of the starting point of the i-th step of w. Denote by $\mathfrak{L}_n$ the set of all Laguerre histories of length n.

Using the convention $\pi(0) = \pi(n+1) = 0$, the Françon–Viennot bijection $\phi_{FV} : \mathfrak{S}_n \to \mathfrak{L}_{n-1}$ that we need is the following modified version (see [24]) defined as $\phi_{FV}(\pi) = (w, \mu) \in \mathfrak{L}_{n-1}$, where for each $k \in [n]$ with $i = \pi(k) \leq n-1$:

$$w_i = \begin{cases} U & \text{if } \pi(k-1) > \pi(k) = i < \pi(k+1), \\ D & \text{if } \pi(k-1) < \pi(k) = i > \pi(k+1), \\ L_0 & \text{if } \pi(k-1) < \pi(k) = i < \pi(k+1), \\ L_1 & \text{if } \pi(k-1) > \pi(k) = i > \pi(k+1), \end{cases}$$

and μ_i is the number of (2–13) patterns with i representing the 2, i.e.,

$$\mu_i = (2\text{–}13)_i(\pi) := |\{j : j-1 > k \text{ and } \pi(j-1) < \pi(k) = i < \pi(j)\}|.$$

For example, if $\pi = 21637548 \in \mathfrak{S}_8$, then $\phi_{FV}(\pi) = (w, \mu)$, where $w = UDUUL_1DD$ and $\mu = (0,1,0,0,1,2,1)$; see Fig. 2 below for an illustration, where the L_1 step is colored red. It was known that ϕ_{FV} is a bijection and the reader is referred to [24] for a recursive description of its inverse ϕ_{FV}^{-1}.

Proof of Lemma 4.16. Recall the complement map $\pi \mapsto \pi^{\mathrm{c}}$ defined in (4.8). It is convenient to consider the two sets

$$\mathcal{P}_m^{\mathrm{c}} := \{\pi^{\mathrm{c}} : \pi \in \mathcal{P}_m\} \quad \text{and} \quad \mathcal{V}_m^{\mathrm{c}} := \{\pi^{\mathrm{c}} : \pi \in \mathcal{V}_m\}.$$

Note that the letter $m+1$ in a permutation $\pi \in \mathcal{P}_m$ becomes the letter m in π^{c}. We aim to construct a bijection $\eta' : \mathcal{P}_m^{\mathrm{c}} \to \mathcal{V}_m^{\mathrm{c}}$ that preserves the number of ascents of permutations and then set η to be the map $\pi \mapsto (\eta'(\pi^{\mathrm{c}}))^{\mathrm{c}}$. To do this, we will introduce an involution Θ on $\mathfrak{L}_{2m-1}$.

Let $(w,\mu) \in \mathfrak{L}_{2m-1}$ be a Laguerre history. For any up step $w_i = U$, there is a unique down step $w_{i'} = D$ to the right of w_i, closest to w_i, whose ending point has the same height as the starting point of w_i. We call $w_{i'}$ (resp. w_i) the associated down (resp. up) step of w_i (resp. $w_{i'}$). For example, for the Laguerre history in the left part of Fig. 2, the associated down step of w_4 is w_6 and the remaining two associated pairs are (w_1, w_2) and (w_3, w_7). Now define $\Theta(w,\mu) = (w^*, \mu^*)$, where

$$w^*_{2m-i} = \begin{cases} U & \text{if } w_i = D, \\ D & \text{if } w_i = U, \\ w_i & \text{if } w_i \text{ is a level step,} \end{cases}$$

and

$$\mu^*_{2m-i} = \begin{cases} \mu_{i'} & \text{if } w_i = U \text{ (resp. } w_i = D\text{) whose associated down (resp. up) step is } w_{i'}, \\ \mu_i & \text{if } w_i \text{ is a level step.} \end{cases}$$

See Fig. 2 for an instance of Θ, where $w = UDUUL_1DD$ and $\mu = (0,1,0,0,1,2,1)$.

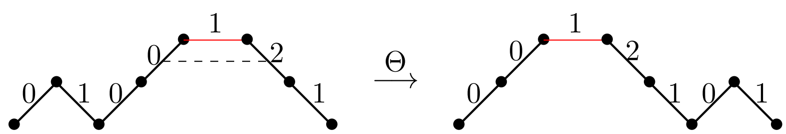

FIGURE 2. The construction of Θ: a red level step represents L_1.

It is clear from the construction that $\Theta : (w,\mu) \mapsto (w^*,\mu^*)$ is an involution on $\mathfrak{L}_{2m-1}$ for which

$$\text{the } m\text{-th step of } w \text{ is an up step } \Longleftrightarrow \text{ the } m\text{-th step of } w^* \text{ is a down step.}$$

It is routine to check that $\eta' := \phi_{FV}^{-1} \circ \Theta \circ \phi_{FV}$ is a bijection between $\mathcal{P}_m^c$ and $\mathcal{V}_m^c$ preserving the number of ascents. $\square$

Proof of Theorem 4.13. Recall the Foata–Strehl action φ_x defined in (4.4). Brändén [5] modified the Foata–Strehl action φ_x as

$$\varphi'_x(\pi) = \begin{cases} \varphi_x(\pi), & \text{if } x \text{ is a double ascent/descent value of } \pi; \\ \pi, & \text{otherwise.} \end{cases}$$

For our purpose, for any $x \in [2m]$ we consider the restricted version of φ'_x defined by

$$\widetilde{\varphi}_x(\pi) = \begin{cases} \varphi'_x(\pi) & \text{if } x \neq m+1, \\ \pi & \text{if } x = m+1, \end{cases}$$

where $\pi \in \mathfrak{S}_{2m}$. For our discussions below, the reader is advised to envision the restricted action $\widetilde{\varphi}_x$ using the so-called valley-hopping interpretation as in Fig. 1.

Since φ'_x's are involutions on $\mathfrak{S}_{2m}$ and they commute, so do $\widetilde{\varphi}_x$'s. Thus, for any subset $S \subseteq [2m]$, we can define the mapping $\widetilde{\varphi}_S : \mathfrak{S}_{2m} \to \mathfrak{S}_{2m}$ by

$$\widetilde{\varphi}_S := \prod_{x \in S} \widetilde{\varphi}_x.$$

This induces a $\mathbb{Z}_2^{2m}$-action on $\mathfrak{S}_{2m}$ via the mappings $\widetilde{\varphi}_S$ with $S \subseteq [2m]$. Let $\mathrm{Orb}(\pi)$ be the orbit of π under this action. If $x \in [2m]$ is a double descent/ascent value of π different from $m+1$, then x becomes a double ascent/descent value of $\widetilde{\varphi}_x(\pi)$ and so

$$\widetilde{\mathrm{des}}(\widetilde{\varphi}_x(\pi)) = \begin{cases} \widetilde{\mathrm{des}}(\pi) + 1 & \text{if } x \neq m+1 \text{ is a double ascent value of } \pi, \\ \widetilde{\mathrm{des}}(\pi) - 1 & \text{if } x \neq m+1 \text{ is a double descent value of } \pi. \end{cases}$$

Therefore, if we use $\hat{\pi}$ to denote the unique permutation in $\mathrm{Orb}(\pi)$ with $\mathrm{Ddes}(\hat{\pi}) \setminus \{m+1\} = \emptyset$, then

$$\sum_{\sigma \in \mathrm{Orb}(\pi)} t^{\widetilde{\mathrm{des}}(\sigma)} = t^{\widetilde{\mathrm{des}}(\hat{\pi})} (1+t)^{|\mathrm{Dasc}(\hat{\pi}) \setminus \{m+1\}|}.$$

Now we need to consider the following two cases.

- If $m+1$ is a double descent or a double ascent value of $\hat{\pi}$, then $|\mathrm{Dasc}(\hat{\pi}) \setminus \{m+1\}| = 2m - 2 - 2\widetilde{\mathrm{des}}(\hat{\pi})$ and so in this case
$$\sum_{\sigma \in \mathrm{Orb}(\pi)} t^{\widetilde{\mathrm{des}}(\sigma)} = t^{\widetilde{\mathrm{des}}(\hat{\pi})} (1+t)^{2m-2-2\widetilde{\mathrm{des}}(\hat{\pi})}.$$
- If $m+1$ is a peak of $\hat{\pi}$, then $|\mathrm{Dasc}(\hat{\pi}) \setminus \{m+1\}| = 2m - 3 - 2\widetilde{\mathrm{des}}(\hat{\pi})$. On the other hand, $m+1$ is a valley of $\eta(\hat{\pi})$ by Lemma 4.16, $\mathrm{Ddes}(\eta(\hat{\pi})) \setminus \{m+1\} = \emptyset$, $\widetilde{\mathrm{des}}(\eta(\hat{\pi})) = \widetilde{\mathrm{des}}(\hat{\pi}) + 1$ and
$$|\mathrm{Dasc}(\eta(\hat{\pi})) \setminus \{m+1\}| = 2m - 1 - 2\widetilde{\mathrm{des}}(\eta(\hat{\pi})).$$
Thus, we have
$$\begin{aligned} \sum_{\sigma \in \mathrm{Orb}(\pi) \biguplus \mathrm{Orb}(\eta(\hat{\pi}))} t^{\widetilde{\mathrm{des}}(\sigma)} &= \sum_{\sigma \in \mathrm{Orb}(\pi)} t^{\widetilde{\mathrm{des}}(\sigma)} + \sum_{\sigma \in \mathrm{Orb}(\eta(\hat{\pi}))} t^{\widetilde{\mathrm{des}}(\sigma)} \\ &= t^{\widetilde{\mathrm{des}}(\hat{\pi})} (1+t)^{2m-3-2\widetilde{\mathrm{des}}(\hat{\pi})} + t^{\widetilde{\mathrm{des}}(\hat{\pi})+1} (1+t)^{2m-3-2\widetilde{\mathrm{des}}(\hat{\pi})} \\ &= t^{\widetilde{\mathrm{des}}(\hat{\pi})} (1+t)^{2m-2-2\widetilde{\mathrm{des}}(\hat{\pi})}. \end{aligned}$$

Combining the above two cases, we obtain the desired γ-positive expansion for $\tilde{A}_{2m}(t)$. □

5. Concluding remarks

In this paper, we study permanents of the floor function of some fractions and the sign function of some trigonometric functions and establish their intriguing connections with several classical combinatorial sequences. It would be interesting to investigate the combinatorics of permanents of the floor function or the sign function of other elementary functions.

In the course of proving several permanent conjectures, we introduce the crucial action $\phi_{k,l}$ in Definition 3.4 on matrices that preserves the permanents. For any $S \subseteq [n] \times [n]$, define the transformation matrix T_S with respect to S by

$$\Big(\prod_{(k,l) \in S} \phi_{k,l} \Big)(A) = A \circ T_S,$$

where A is any $n \times n$ matrix over $\mathbb{R}$. Let us consider the set of transformation matrices

$$\mathcal{T}_n := \{T_S : S \subseteq [n] \times [n]\}.$$

For instance, $\mathcal{T}_2$ consists of four matrices

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Proposition 5.1. *For any positive integer n, the transformation group $\mathcal{T}_n$ is isomorphic to the group $\mathbb{Z}_2^{2n-2}$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. Consequently, $|\mathcal{T}_n| = 2^{2n-2}$.*

Sketch of the proof. Under the Hadamard product, $\mathcal{T}_n$ forms an abelian group whose non-identity elements always have order 2. By the well-known Structure Theorem for Finite Abelian Groups, $\mathcal{T}_n \cong \mathbb{Z}_2^{\ell(n)}$ for some $\ell(n) \in \mathbb{N}$. It remains to show that $\ell(n) = 2n-2$. This will be done once we can show that there are exactly $2n-2$ generators of $\mathcal{T}_n$.

For any $S \subseteq [n] \times [n]$, we define $\phi_S := \prod_{(k,l)\in S} \phi_{k,l}$, and call S a *kernel* if ϕ_S is the identity action. If S is a kernel, then for any $(a,b) \in S$ we have $\phi_{(a,b)} = \phi_{S\setminus\{(a,b)\}}$. Consider the subset $\mathsf{G}_n = \mathsf{L}_n \uplus \mathsf{R}_n$, the disjoint union of

$$\mathsf{L}_n := \{(i,i), (i+1,i) : 1 \le i \le \lfloor n/2 \rfloor\} \quad \text{and} \quad \mathsf{R}_n := \{(i-1,i), (i,i) : \lfloor n/2 \rfloor + 1 < i \le n\}.$$

Both L_n and R_n form (nearly) half of the border strip around the diagonal of the $n \times n$ grid, and $|\mathsf{G}_n| = 2n-2$. See Fig. 3 for examples of G_n with $n = 7, 8$.

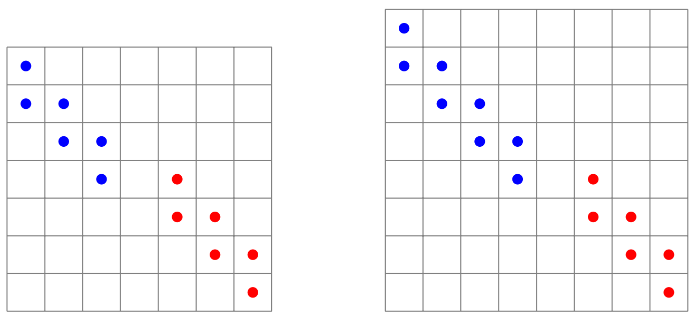

FIGURE 3. Examples of G_n for $n = 7, 8$, where L_n and R_n correspond respectively to the blue and the red dots.

Now we claim that the elements in $\{\phi_{(a,b)} : (a,b) \in \mathsf{G}_n\}$ form a set of generators for all the actions in $\{\phi_S : S \subseteq [n] \times [n]\}$, which would complete the proof. To prove this, one needs to check:

- The elements in $\{\phi_{(a,b)} : (a,b) \in \mathsf{G}_n\}$ generate all $\phi_{(i,j)}$ for $(i,j) \in [n]\times[n]$. To see this, note that the set $\{(i,j), (i+1,j), (i,j+1), (i+1,j+1)\}$ is a kernel for any $(i,j) \in [n-1]\times[n-1]$ and the set $\{(1,1), (2,2), \ldots, (n,n)\}$ is a kernel too.
- Any nonempty subset of G_n could not be a kernel. This can be proved by induction on n, since G_n can be embedded in G_{n+1}; compare G_7 and G_8 in Fig. 3.

The tedious details are left to the interested reader. □

Recall that a polynomial $f(z_1 \ldots, z_m) \in \mathbb{R}[z_1, \ldots, z_m]$ is said to be *stable*, if $f(z_1, \ldots, z_m) \neq 0$ whenever $z_1, \ldots, z_m \in \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. It is well known that the stability of the multivariate generating functions implies that their univariate counterparts, obtained by diagonalization, have only real zeros. By using the theory of stability, Brändén, Haglund, Visontai and Wagner [6] proved that $\operatorname{per}(zU_n + A)$ is a polynomial in z with only real zeros if A is a matrix $[a_{i,j}]_{1\leqslant i,j\leqslant n}$ with $a_{1,j} \geqslant a_{2,j} \geqslant \ldots \geqslant a_{n,j}$ for all $j = 1, \ldots, n$. Can the techniques in their paper be employed to find multivariable extension of the variation of the Eulerian polynomials $\sum_{\pi\in\mathfrak{S}_n} z^{\widetilde{\operatorname{exc}}(\pi)}$ that possesses certain nice stable property?

An interpretation for the γ-coefficients $\gamma_{n,k}^{(2)}$, defined in (4.18), of the 2-Eulerian polynomials in terms of weakly increasing trees on $\{1, 1, 2, 2, \ldots, n-1, n-1, n\}$ without young leaves has been found in [23]. Corollary 4.12 asserts that the diagonal coefficient $\gamma_{n,n-1}^{(2)}$ enumerates alternating multipermutations in $\mathfrak{S}_n^{(2)}$, which may shed some light on finding an interpretation for $\gamma_{n,k}^{(2)}$ in terms of certain class of permutations on $\{1, 1, 2, 2, \ldots, n, n\}$.

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