Accepted by Bull. Aust. Math. Soc.

CHARACTERISTIC POLYNOMIALS OF THE MATRICES WITH (j,k)-ENTRY $q^{j\pm k}+t$

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Abstract. In this paper, we determine the characteristic polynomials of the matrices $[q^{j-k}+t]_{1\le j,k\le n}$ and $[q^{j+k}+t]_{1\le j,k\le n}$ for any complex number $q\ne 0,1$. As an application, for complex numbers a,b,c with $b\ne 0$ and $a^2\ne 4b$, and the sequence $(w_m)_{m\in\mathbb{Z}}$ with $w_{m+1}=aw_m-bw_{m-1}$ for all $m\in\mathbb{Z}$, we determine the exact value of $\det[w_{j-k}+c\delta_{jk}]_{1\le j,k\le n}$.

1. Introduction

For any integer $n\ge 3$, we have the determinant identity

$$\det[j-k]_{1\le j,k\le n}=0$$

since $(1-k)+(3-k)=2(2-k)$ for all $k=1,\ldots,n$. However, it is nontrivial to determine the characteristic polynomial $\det[xI_n-(j-k)]_{1\le j,k\le n}$ of the matrix $[j-k]_{1\le j,k\le n}$, where I_n is the identity matrix of order n.

For $j,k\in\mathbb{N}=\{0,1,2,\ldots\}$, the Kronecker symbol δ_{jk} takes 1 or 0 according as $j=k$ or not. In 2003, B. Cloitre [1] generated the sequence $\det[j-k+\delta_{jk}]_{1\le j,k\le n}$ $(n=1,2,3,\ldots)$ with initial fifteen terms as follows:

1, 2, 7, 21, 51, 106, 197, 337, 541, 826, 1211, 1717, 2367, 3186, 4201.

In 2013 C. Baker added a comment to [1] in which he claimed that

$$\det[j-k+\delta_{jk}]_{1\le j,k\le n}=1+\frac{n^2(n^2-1)}{12} \tag{1.1}$$

without any proof or linked reference. It seems that Baker found this by guessing the recurrence of the sequence via using the Maple package `gfun`.

Recall that the q-analogue of an integer m is given by

$$[m]_q=\frac{q^m-1}{q-1}.$$

Note that $\lim_{q\to 1}[m]_q=m$.

In our first theorem we determine the characteristic polynomial of the matrix $[q^{j-k}+t]_{1\le j,k\le n}$ for any complex number $q\ne 0,1$.

Key words and phrases. Characteristic polynomial, determinant, eigenvalue, Lucas sequence.
2020 *Mathematics Subject Classification.* Primary 11B39, 11C20, 15A18; Secondary 15A15, 15B05.
Supported by the National Natural Science Foundation of China (grant no. 12371004).

Theorem 1.1. *Let $n \geq 2$ be an integer, and let $q \neq 0, 1$ be a complex number. Then the characteristic polynomial of the matrix $P = [q^{j-k} + t]_{1 \leq j,k \leq n}$ is*

$$\det(xI_n - P) = x^{n-2}(x^2 - n(t+1)x + t(n^2 - q^{1-n}[n]_q^2)). \tag{1.2}$$

Putting $t = -1$ and replacing x by $(q-1)x$ in Theorem 1.1, we immediately obtain the following corollary.

Corollary 1.1. *Let $n \geq 2$ be an integer, and let $q \neq 0, 1$ be a complex number. For the matrix $P_q = [[j-k]_q]_{1 \leq j,k \leq n}$ we have*

$$\det(xI_n - P_q) = x^n + \frac{q^{1-n}[n]_q^2 - n^2}{(q-1)^2} x^{n-2}. \tag{1.3}$$

Remark 1.1. Fix an integer $n \geq 2$. Observe that

$$\begin{aligned}
\lim_{q \to 1} \frac{q^{1-n}[n]_q^2 - n^2}{(q-1)^2} &= \lim_{t \to 0} \frac{(t+1)^{1-n}(((t+1)^n - 1)/t)^2 - n^2}{t^2} \\
&= \lim_{t \to 0} \frac{(t+1)^{1-n}((\sum_{k=1}^n \binom{n}{k} t^{k-1})^2 - n^2) + ((t+1)^{1-n} - 1)n^2}{t^2} \\
&= \lim_{t \to 0} \left(\frac{(n + \binom{n}{2}t + \binom{n}{3}t^2 + \cdots)^2 - n^2}{(t+1)^{n-1}t^2} + n^2 \frac{1 - (t+1)^{n-1}}{(t+1)^{n-1}t^2} \right) \\
&= \binom{n}{2}^2 + 2n\binom{n}{3} + \lim_{t \to 0} \left(2n\binom{n}{2} \frac{t^{-1}}{(t+1)^{n-1}} - n^2 \frac{\sum_{k=1}^n \binom{n-1}{k} t^{k-2}}{(t+1)^{n-1}} \right) \\
&= \binom{n}{2}^2 + 2n\binom{n}{3} - n^2\binom{n-1}{2} = \frac{n^2(n^2-1)}{12}.
\end{aligned}$$

So, by Corollary 1.1 we have

$$\det[x\delta_{jk} - (j-k)]_{1 \leq j,k \leq n} = x^n + \frac{n^2(n^2-1)}{12} x^{n-2}, \tag{1.4}$$

which indicates that when $n > 2$ the n eigenvalues of $A_n = [j-k]_{1 \leq j,k \leq n}$ are

$$\lambda_1 = \frac{n\sqrt{n^2-1}}{2\sqrt{3}} i, \quad \lambda_2 = -\frac{n\sqrt{n^2-1}}{2\sqrt{3}} i, \quad \lambda_3 = \cdots = \lambda_n = 0.$$

Note that (1.1) follows from (1.4) with $x = -1$. Concerning the permanent of A_n, motivated by [3, Conj. 11.23] we conjecture that

$$\mathrm{per}(A_{p-1}) \equiv 3 \pmod{p} \quad \text{and} \quad \mathrm{per}(A_p) \equiv 1 + 4p \pmod{p^2}.$$

for any odd prime p. Inspired by (1.1), Z.-W. Sun [4] conjectured that for any positive integers m and n we have

$$\det[(j-k)^m + \delta_{jk}]_{1 \leq j,k \leq n} = 1 + n^2(n^2-1)f(n)$$

for certain polynomial $f(x) \in \mathbb{Q}[x]$ with $\deg f = (m+1)^2 - 4$.

Applying Corollary 1.1 with $q=-1$, we find that

$$\det(xI_n-P_{-1})=x^n+\frac{(-1)^{n-1}[n]_{-1}^2-n^2}{4}x^{n-2}$$

for any integer $n\ge 2$. In particular,

$$\det\left[\frac{1-(-1)^{j-k}}{2}+\delta_{j,k}\right]_{1\le j,k\le n}=\frac{9-(-1)^n-2n^2}{8}. \tag{1.5}$$

Applying Theorem 1.1 with $(t,x)=(-1,-2),(1,-1)$, we obtain the following result.

Corollary 1.2. *For any positive integer n, we have*

$$\det[2^{j-k}-1+2\delta_{jk}]_{1\le j,k\le n}=\frac{4^n-2^{n-1}n^2+1}{2}. \tag{1.6}$$

and

$$\det[2^{j-k}+1+\delta_{j,k}]_{1\le j,k\le n}=(n+1)^2-2^{1-n}(2^n-1)^2. \tag{1.7}$$

In contrast with Theorem 1.1, we also establish the following result.

Theorem 1.2. *Let $n\ge 2$ be an integer, and let $q\ne 0,1$ be a complex number. For the matrix $Q=[q^{j+k}+t]_{0\le j,k\le n-1}$, we have*

$$\det(xI_n-Q)=x^n-(nt+[n]_{q^2})x^{n-1}+(n[n]_{q^2}-[n]_q^2)tx^{n-2}. \tag{1.8}$$

The identity (1.8) with $q=2$ and $x=t=-1$ yields the following corollary.

Corollary 1.3. *For any positive integer n, we have*

$$\det[2^{j+k}-1+\delta_{jk}]_{0\le j,k\le n-1}=(2^n-1)^2-(n-1)\frac{4^n+2}{3}. \tag{1.9}$$

For complex numbers a and $b\ne 0$, the Lucas sequence $u_m=u_m(a,b)$ $(m\in\mathbb{Z})$ and its companion sequence $v_m=v_m(a,b)$ $(m\in\mathbb{Z})$ are defined as follows:

$$u_0=0,\ u_1=1,\ \text{and } u_{k+1}=au_k-bu_{k-1} \text{ for all } k\in\mathbb{Z};$$
$$v_0=2,\ v_1=a,\ \text{and } v_{k+1}=av_k-bv_{k-1} \text{ for all } k\in\mathbb{Z}.$$

By the Binet formula,

$$(\alpha-\beta)u_m=\alpha^m-\beta^m \quad\text{and}\quad v_m=\alpha^m+\beta^m \quad\text{for all } m\in\mathbb{Z},$$

where

$$\alpha=\frac{a+\sqrt{a^2-4b}}{2}\quad\text{and}\quad\beta=\frac{a-\sqrt{a^2-4b}}{2} \tag{1.10}$$

are the two roots of the quadratic equation $x^2-ax+b=0$. Clearly $b^nu_{-n}=-u_n$ and $b^nv_{-n}=v_n$ for all $n\in\mathbb{N}$. For any positive integer n, it is known that

$$u_n=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1-k}{k}a^{n-1-2k}(-b)^k \quad\text{and}\quad v_n=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n}{n-k}\binom{n-k}{k}a^{n-2k}(-b)^k,$$

(cf. [5, p. 10]) which can be easily proved by induction. Note also that $u_m(2,1)=m$ for all $m\in\mathbb{Z}$.

For $P(z)=\sum_{k=0}^{n-1}a_kz^k\in\mathbb{C}[z]$, it is known (cf. [2, Lemma 9]) that

$$\det[P(x_j+y_k)]_{1\le j,k\le n}=a_{n-1}^n\prod_{r=0}^{n-1}\binom{n-1}{r}\times\prod_{1\le j<k\le n}(x_j-x_k)(y_k-y_j).$$

Thus, for any integer $n\ge 3$, and complex numbers a and $b\ne 0$, we have

$$(\alpha-\beta)^n\det[u_{j-k}(a,b)]_{1\le j,k\le n}=\det[v_{j-k}(a,b)]_{1\le j,k\le n}=0 \tag{1.11}$$

(where α and β are given by (1.10)), since

$$\det\left[\alpha^{j-k}\pm\beta^{j-k}\right]_{1\le j,k\le n}=\prod_{k=1}^n\alpha^{-k}\times\prod_{j=1}^n\beta^j\times\det\left[\left(\frac{\alpha}{\beta}\right)^j\pm\left(\frac{\alpha}{\beta}\right)^k\right]_{1\le j,k\le n}=0.$$

As an application of Theorem 1.1, we obtain the following new result.

Theorem 1.3. *Let a and $b\ne 0$ be complex numbers with $a^2\ne 4b$. Let $(w_m)_{m\in\mathbb{Z}}$ be a sequence of complex numbers with $w_{k+1}=aw_k-bw_{k-1}$ for all $k\in\mathbb{Z}$. For any complex number c and integer $n\ge 2$, we have*

$$\det[w_{j-k}+c\delta_{jk}]_{1\le j,k\le n}=c^n+c^{n-1}nw_0+c^{n-2}(w_1^2-aw_0w_1+bw_0^2)\frac{b^{1-n}u_n(a,b)^2-n^2}{a^2-4b}. \tag{1.12}$$

Remark 1.2. It is hard to guess the exact formula for $\det[w_{j-k}+c\delta_{jk}]_{1\le j,k\le n}$ in Theorem 1.3 via looking at various numerical examples.

Corollary 1.4. *Let a,b,c be complex numbers with $b\ne 0$ and $a^2\ne 4b$. For any integer $n\ge 2$, we have*

$$\det[u_{j-k}(a,b)+c\delta_{jk}]_{1\le j,k\le n}=c^n+c^{n-2}\frac{b^{1-n}u_n(a,b)^2-n^2}{a^2-4b}. \tag{1.13}$$

and

$$\det[v_{j-k}(a,b)+c\delta_{jk}]_{1\le j,k\le n}=c^{n-2}((n+c)^2-b^{1-n}u_n(a,b)^2). \tag{1.14}$$

For any $m\in\mathbb{Z}$, $u_m(-1,1)$ coincides with the Legendre symbol $(\frac{m}{3})$, and $v_m(1,-1)=\omega^m+\bar\omega^m$ where ω denotes the cubic root $(-1+\sqrt{-3})/2$ of unity. Applying Corollary 1.4 with $a=-1$ and $b=1$, we get the following result.

Corollary 1.5. *For any integer $n\ge 2$ and complex number c, we have*

$$\det\left[\left(\frac{j-k}{3}\right)+c\delta_{j,k}\right]_{1\le j,k\le n}=c^n+c^{n-2}\left\lfloor\frac{n^2}{3}\right\rfloor. \tag{1.15}$$

Recall that those $F_m=u_m(1,-1)$ ($m\in\mathbb{Z}$) are the well-known Fibonacci numbers, and those $L_m=v_m(1,-1)$ ($m\in\mathbb{Z}$) are the Lucas numbers. Corollary 1.4 with $a=1$ and $b=-1$ yields the following result.

Corollary 1.6. *For any integer $n\ge 2$ and complex number c, we have*

$$\det[F_{j-k}+c\delta_{jk}]_{1\le j,k\le n}=c^n+\frac{c^{n-2}}{5}\left((-1)^{n-1}F_n^2-n^2\right)\tag{1.16}$$

and

$$\det[L_{j-k}+c\delta_{jk}]_{1\le j,k\le n}=c^{n-2}((n+c)^2+(-1)^nF_n^2).\tag{1.17}$$

Although we have Theorem 1.2 which is similar to Theorem 1.1, it seems impossible to use Theorem 1.2 to deduce a result similar to Theorem 1.3.

We are going to prove Theorems 1.1, 1.2 and 1.3 in Sections 2, 3 and 4, respectively.

2. Proof of Theorem 1.1

Lemma 2.1. *Let n be a positive integer, and let $q\ne 0$ and t be complex numbers with $n-[n]_q+t(q^{1-n}[n]_q-n)\ne 0$. Suppose that*

$$\gamma=\frac{n(t+1)\pm\sqrt{n^2(t-1)^2+4tq^{1-n}[n]_q^2}}{2}\quad\text{and}\quad y=\frac{\gamma-[n]_q-nt}{n-[n]_q+(q^{1-n}[n]_q-n)t}.\tag{2.1}$$

Then, for any positive integer j, we have

$$\sum_{k=1}^n(q^{j-k}+t)(1+y(q^{k-n}-1))=\gamma(1+y(q^{j-n}-1)).\tag{2.2}$$

Proof. As $\gamma^2-n(t+1)\gamma+(n^2-q^{1-n}[n]_q^2)t=0$, we have

$$[n]_q(n-[n]_q+(q^{1-n}[n]_q-n)t)=(\gamma-[n]_q-nt)(\gamma-n+[n]_q)$$

and hence

$$(\gamma-n+[n]_q)y=[n]_q.\tag{2.3}$$

Let $j\in\{1,2,3,\ldots\}$, and set

$$\Delta_j=\sum_{k=1}^n(q^{j-k}+t)(1+y(q^{k-n}-1))-\gamma(1+y(q^{j-n}-1)).$$

Then

$$\begin{aligned}&\Delta_j-t(1+y(q^{k-n}-1))+\gamma(1-y)\\&=q^{j-n}\left(\sum_{k=1}^n q^{n-k}(1+y(q^{k-n}-1))-\gamma y\right)\\&=q^{j-n}\left([n]_q(1-y)+ny-\gamma y\right)=0\end{aligned}$$

by (2.3). So $\Delta_1=\Delta_2=\cdots$.

Next we show that $\Delta_n=0$. Observe that

$$\sum_{k=1}^n(q^{n-k}+t)(1+(q^{k-n}-1)y)$$

$$\begin{aligned}
&= \sum_{k=1}^{n} \left(q^{n-k}(1-y) + t(1-y) + y + q^{k-n}ty\right) \\
&= [n]_q(1-y) + nt(1-y) + ny + q^{1-n}[n]_q ty \\
&= [n]_q + nt + y(n - [n]_q + (q^{1-n}[n]_q - n)t) \\
&= \gamma = \gamma(1 + y(q^{n-n} - 1))
\end{aligned}$$

by the definition of y. So $\Delta_n = 0$.

In view of the above, $\Delta_j = 0$ for all $j = 1, 2, 3, \ldots$. This concludes our proof. □

Proof of Theorem 1.1. It is easy to verify the desired result for $n = 2$. Below we assume that $n \geq 3$.

If $n - [n]_q$ and $q^{1-n}[n]_q - n$ are both zero, then $q^{n-1} = 1$ and $n = [n]_q = 1$. As $n \geq 3$, there are infinitely many $t \in \mathbb{C}$ such that

$$\begin{cases} n - [n]_q + t(q^{1-n}[n]_q - n) \neq 0, \\ n^2(t-1)^2 + 4tq^{1-n}[n]_q^2 \neq 0. \end{cases}$$

Take any such a number t, and choose γ and y as in (2.1). Then γ given in (2.1) is an eigenvalue of the matrix $P = [q^{j-k} + t]_{1 \leq j,k \leq n}$, and the column vector $v = (v_1, \ldots, v_n)^T$ with $v_k = 1 + y(q^{k-n} - 1)$ is an eigenvector of P associated with the eigenvalue γ. Note that γ given by (2.1) has two different choices since $n^2(t-1)^2 + 4tq^{1-n}[n]_q^2 \neq 0$.

Let $s \in \{3, \ldots, n\}$. For $1 \leq k \leq n$, let us define

$$v_k^{(s)} = \begin{cases} q^{2-s}[s-2]_q & \text{if } k = 1, \\ -q^{2-s}[s-1]_q & \text{if } k = 2, \\ \delta_{sk} & \text{if } 3 \leq k \leq n. \end{cases}$$

It is easy to verify that

$$\sum_{k=1}^{n} v_k^{(s)} = 0 = \sum_{k=1}^{n} q^{j-k} v_k^{(s)} \quad \text{for all } j = 1, \ldots, n.$$

Thus 0 is an eigenvalue of the matrix $P = [q^{j-k} + t]_{1 \leq j,k \leq n}$, and the column vector $v^{(s)} = (v_1^{(s)}, \ldots, v_n^{(s)})^T$ is an eigenvector of P associated with the eigenvalue 0.

If $\sum_{s=3}^{n} c_s v^{(s)}$ is the zero column vector for some $c_3, \ldots, c_n \in \mathbb{C}$, then for each $k = 3, \ldots, n$ we have

$$c_k = \sum_{s=3}^{n} c_s \delta_{sk} = \sum_{s=3}^{n} c_s v_k^{(s)} = 0.$$

Thus the $n - 2$ column vectors $v^{(3)}, \ldots, v^{(n)}$ are linearly independent over $\mathbb{C}$.

By the above, the n eigenvalues of the matrix $P=[q^{j-k}+t]_{1\le j,k\le n}$ are the two values of γ given by (2.2), and $\lambda_3=\cdots=\lambda_n=0$. Thus the characteristic polynomial of P is

$$\begin{aligned}\det(xI_n-P)=&\left(x-\frac{n(t+1)}{2}-\frac{\sqrt{n^2(t-1)^2+4tq^{1-n}[n]_q^2}}{2}\right)\\ &\times\left(x-\frac{n(t+1)}{2}+\frac{\sqrt{n^2(t-1)^2+4tq^{1-n}[n]_q^2}}{2}\right)\prod_{s=3}^{n}(x-\lambda_s)\\ =&x^{n-2}\left(\left(x-\frac{n(t+1)}{2}\right)^2-\frac{n^2(t-1)^2+4tq^{1-n}[n]_q^2}{4}\right)\\ =&x^{n-2}(x^2-n(t+1)x+t(n^2-q^{1-n}[n]_q^2)).\end{aligned}$$

In light of the above, the identity (1.2) holds for infinitely many values of t. Note that both sides of (1.2) are polynomials in t for any fixed $x\in\mathbb{C}$. Thus, if we view both sides of (1.2) as polynomials in x and t, then the identity (1.2) still holds. This ends our proof. □

3. Proof of Theorem 1.2

The following lemma is quite similar to Lemma 2.1.

Lemma 3.1. *Let n be a positive integer, and let $q\ne 0$ and t be complex numbers with $[n]_{q^2}+(q^{1-n}t-q^{n-1})[n]_q-nt\ne 0$. Suppose that*

$$\gamma=\frac{nt+[n]_{q^2}\pm\sqrt{(nt-[n]_{q^2})^2+4t[n]_q^2}}{2}\quad\text{and}\quad z=\frac{\gamma-q^{n-1}[n]_q-nt}{[n]_{q^2}+(q^{1-n}t-q^{n-1})[n]_q-nt}.\tag{3.1}$$

Then, for every $j=0,1,2,\ldots$, we have

$$\sum_{k=0}^{n-1}(q^{j+k}+t)(1+z(q^{k-n+1}-1))=\gamma(1+z(q^{j-n+1}-1)).\tag{3.2}$$

Proof. Since $\gamma^2-(nt+[n]_{q^2})\gamma+t(n[n]_{q^2}-[n]_q^2)=0$, we have

$$(\gamma-[n]_{q^2}+q^{n-1}[n]_q)z=q^{n-1}[n]_q.\tag{3.3}$$

Let $j\in\{0,1,2,\ldots\}$, and set

$$R_j=\sum_{k=0}^{n-1}(q^{j+k}+t)(1+z(q^{k-n+1}-1))-\gamma(1+z(q^{j-n+1}-1)).$$

It is easy to see that

$$R_j-\sum_{k=0}^{n-1}t(1+z(q^{k-n+1}-1))+\gamma(1-z)=q^{j-n+1}\left(q^{n-1}[n]_q(1-z)+z[n]_{q^2}-\gamma z\right)=0$$

with the aid of (3.3). So $R_0 = R_1 = \cdots$. As

$$\sum_{k=0}^{n-1}(q^{n-1+k}+t)(1+z(q^{k-n+1}-1)) = \gamma = \gamma(1+z(q^{(n-1)-n+1}-1))$$

we get $R_{n-1} = 0$. So the desired result follows. □

Proof of Theorem 1.2. It is easy to verify the desired result for $n = 2$. Below we assume that $n \geq 3$.

If $[n]_{q^2} - q^{n-1}[n]_q$ and $q^{1-n}[n]_q - n$ are both zero, then $[n]_q \neq 0$ and

$$(q^n+1)[n]_q = (q+1)[n]_{q^2} = (q+1)q^{n-1}[n]_q = (q^n+q^{n-1})[n]_q,$$

hence $q^{n-1} = 1$ and $n = [n]_q = 1$. As $n \geq 3$, there are infinitely many $t \in \mathbb{C}$ such that

$$\begin{cases} [n]_{q^2} + (q^{1-n}t - q^{n-1})[n]_q - nt \neq 0, \\ (nt - [n]_{q^2})^2 + 4t[n]_q^2 \neq 0. \end{cases}$$

Take any such a number t, and choose γ and z as in (3.1). Then γ given in (3.1) is an eigenvalue of the matrix $Q = [q^{j+k}+t]_{0\leq j,k\leq n-1}$, and the column vector $v = (v_0, \ldots, v_{n-1})^T$ with $v_k = 1 + z(q^{k-n+1}-1)$ is an eigenvector of Q associated with the eigenvalue γ. Note that γ given by (3.1) has two different choices since $(nt - [n]_{q^2})^2 + 4t[n]_q^2 \neq 0$.

Let $s \in \{3, \ldots, n\}$. For $k \in \{0, \ldots, n-1\}$, let us define

$$v_k^{(s)} = \begin{cases} q[s-2]_q & \text{if } k = 0, \\ -[s-1]_q & \text{if } k = 1, \\ \delta_{s,k+1} & \text{if } 2 \leq k \leq n-1. \end{cases}$$

It is easy to verify that

$$\sum_{k=0}^{n-1} v_k^{(s)} = 0 = \sum_{k=0}^{n-1} q^{j+k} v_k^{(s)} \quad \text{for all } j = 1, \ldots, n.$$

Thus 0 is an eigenvalue of the matrix $Q = [q^{j+k}+t]_{0\leq j,k\leq n-1}$, and the column vector $v^{(s)} = (v_0^{(s)}, \ldots, v_{n-1}^{(s)})^T$ is an eigenvector of Q associated with the eigenvalue 0.

If $\sum_{s=3}^n c_s v^{(s)}$ is the zero column vector for some $c_3, \ldots, c_n \in \mathbb{C}$, then for each $k = 2, \ldots, n-1$ we have

$$c_{k+1} = \sum_{s=3}^n c_s \delta_{s,k+1} = \sum_{s=3}^n c_s v_k^{(s)} = 0.$$

Thus the $n-2$ column vectors $v^{(3)}, \ldots, v^{(n)}$ are linearly independent over $\mathbb{C}$.

By the above, the n eigenvalues of the matrix $Q = [q^{j+k}+t]_{0\leq j,k\leq n-1}$ are the two values of γ given by (3.2), and $\lambda_3 = \cdots = \lambda_n = 0$. Thus the characteristic polynomial of Q is

$$\det(xI_n - Q) = \left(x - \frac{nt+[n]_{q^2}}{2} - \frac{\sqrt{(nt-[n]_{q^2})^2+4t[n]_q^2}}{2}\right)$$

$$\times\left(x-\frac{nt+[n]_{q^2}}{2}+\frac{\sqrt{(nt-[n]_{q^2})^2+4t[n]_q^2}}{2}\right)\prod_{s=3}^{n}(x-\lambda_s)$$
$$=x^{n-2}\left(\left(x-\frac{nt+[n]_{q^2}}{2}\right)^2-\frac{(nt-[n]_{q^2})^2+4t[n]_q^2}{4}\right)$$
$$=x^n-(nt+[n]_{q^2})x^{n-1}+(n[n]_{q^2}-[n]_q^2)tx^{n-2}.$$

In light of the above, the identity (1.8) holds for infinitely many values of t. Note that both sides of (1.8) are polynomials in t for any fixed $x\in\mathbb{C}$. Thus, if we view both sides of (1.8) as polynomials in x and t, then the identity (1.8) still holds. This concludes our proof. □

4. Proof of Theorem 1.3

Proof of Theorem 1.3. If $w_0=w_1=0$ or $n=2$, then the desired result can be easily verified. Below we assume that $n\geq 3$ and $\{w_0,w_1\}\neq\{0\}$.

(i) Let α and β be the two roots of the quadratic equation $z^2-az+b=0$. Note that $\alpha\beta=b\neq 0$. Also, $\alpha\neq\beta$ since $\Delta=a^2-4b$ is nonzero.

It is well known that there are constants $c_1,c_2\in\mathbb{C}$ such that

$$w_m=c_1\alpha^m+c_2\beta^m\quad\text{for all } m\in\mathbb{Z}.$$

As $c_1+c_2=w_0$ and $c_1\alpha+c_2\beta=w_1$, we find that

$$c_1=\frac{w_1-\beta w_0}{\alpha-\beta}\ \text{ and }\ c_2=\frac{\alpha w_0-w_1}{\alpha-\beta}.\tag{4.1}$$

Since w_0 or w_1 is nonzero, one of c_1 and c_2 is nonzero. Without any loss of generality, we assume $c_1\neq 0$.

Let W denote the matrix $[w_{j-k}+c\delta_{jk}]_{1\leq j,k\leq n}$. Then

$$\begin{aligned}\det(W)&=\det\left[c_1\alpha^{j-k}+c_2\beta^{j-k}+c\delta_{jk}\right]_{1\leq j,k\leq n}\\&=c_1^n\prod_{j=1}^{n}\beta^j\times\prod_{k=1}^{n}\beta^{-k}\times\det\left[\left(\frac{\alpha}{\beta}\right)^{j-k}+\frac{c_2+c\delta_{jk}\beta^{k-j}}{c_1}\right]_{1\leq j,k\leq n}\\&=c_1^n\det\left[q^{j-k}+t-x\delta_{jk}\right]_{1\leq j,k\leq n}=(-c_1)^n\det\left[x\delta_{jk}-q^{j-k}-t\right]_{1\leq j,k\leq n}\end{aligned}$$

where $q=\alpha/\beta\neq 0,1$, and $t=c_2/c_1$ and $x=-c/c_1$. Applying Theorem 1.1, we deduce that

$$\begin{aligned}\det(W)&=(-c_1)^nx^{n-2}(x^2-n(t+1)x+t(n^2-q^{1-n}[n]_q^2))\\&=c^{n-2}\left(c^2+nc(c_1+c_2)+c_1c_2\left(n^2-\frac{\alpha^{1-n}}{\beta^{1-n}}\left(\frac{(\alpha/\beta)^n-1}{\alpha/\beta-1}\right)^2\right)\right)\\&=c^n+nw_0c^{n-1}+c^{n-2}c_1c_2\left(n^2-(\alpha\beta)^{1-n}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2\right)\end{aligned}$$

$$= c^n + nw_0c^{n-1} + c^{n-2}c_1c_2\left(n^2 - b^{1-n}u_n(a,b)^2\right).$$

In view of (4.1),

$$c_1c_2 = \frac{(w_1-\beta w_0)(\alpha w_0 - w_1)}{(\alpha-\beta)^2} = \frac{-w_1^2+(\alpha+\beta)w_0w_1-\alpha\beta w_0^2}{\Delta} = -\frac{w_1^2-aw_0w_1+bw_0^2}{a^2-4b}.$$

Therefore the desired (1.12) follows. □

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