NEW SERIES INVOLVING BINOMIAL COEFFICIENTS (III)

ZHI-WEI SUN

ABSTRACT. We evaluate some series with summands involving a single binomial coefficient $\binom{6k}{3k}$. For example, we prove that

$$\sum_{k=0}^{\infty} \frac{(63k^2 + 78k + 22)8^k}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = \frac{3\pi}{2}.$$

Motivated by Galois theory, we introduce the so-called Duality Principle for irrational series of Ramanujan's type or Zeilberger's type, and apply it to find 26 new irrational series identities. For example, we conjecture that

$$\sum_{k=1}^{\infty} \frac{(32(91\sqrt{33} - 523))^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} \left((91\sqrt{33} + 891)k - 33\sqrt{33} - 225 \right)$$
$$= 320 \left(\frac{11}{3}\sqrt{33} L_{-11}(2) - 27 L_{-3}(2) \right),$$

where $L_d(2) = \sum_{k=1}^{\infty} \frac{(\frac{d}{k})}{k^2}$ for any integer $d \equiv 0, 1 \pmod{4}$ with $(\frac{d}{k})$ the Kronecker symbol.

1. INTRODUCTION

In 1914, S. Ramanujan [14] proposed 17 conjectural series for $1/\pi$ including the identity

$$\sum_{k=0}^{\infty} (4k+1) \frac{\binom{2k}{k}^3}{(-64)^k} = \frac{2}{\pi}$$

first found by G. Bauer in 1859. The classical Ramanujan-type series for $1/\pi$ has the form:

$$\sum_{k=0}^{\infty} (ak+b) \frac{c_k}{m^k} = \frac{\sqrt{d}}{\pi},$$

where $a, b, m \in \mathbb{Z}$ with $am \neq 0$, d is a positive rational number, and c_k with $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$ is among

$$\binom{2k}{k}^3, \quad \binom{2k}{k}^2 \binom{3k}{k}, \quad \binom{2k}{k}^2 \binom{4k}{2k}, \quad \binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}. \tag{1.1}$$

Such series are called *rational series* since a, b, m are rational numbers. There are 36 known rational Ramanujan-type series for $1/\pi$ (cf.[7, Chapter 14]), and it is widely believed that the list is complete. By Stirling's formula,

$$\lim_{k\to\infty} \sqrt[k]{c_k} = \begin{cases} 64 & \text{if } c_k = \binom{2k}{k}^3, \\ 108 & \text{if } c_k = \binom{2k}{k}^2\binom{3k}{k}, \\ 256 & \text{if } c_k = \binom{2k}{k}^2\binom{4k}{2k}, \\ 1728 & \text{if } c_k = \binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}. \end{cases}$$

In 1993, D. Zeilberger [24] used the WZ method to obtain the identity

$$\sum_{k=1}^{\infty} \frac{21k-8}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{2}. \tag{1.2}$$

Along this line, J. Guillera [8] deduced the following identities of Zeilberger's type:

$$\sum_{k=1}^{\infty} \frac{(4k-1)(-64)^k}{k^3\binom{2k}{k}^3} = -16G, \quad \sum_{k=1}^{\infty} \frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3} = -2G, \tag{1.3}$$

and

$$\sum_{k=1}^{\infty} \frac{(3k-1)16^k}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{2}, \tag{1.4}$$

where G denotes the Catalan constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$. In 2010, the author [16] conjectured the following similar identities:

$$\sum_{k=1}^{\infty} \frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} = \frac{\pi^2}{2}, \quad \sum_{k=1}^{\infty} \frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} = 8\pi^2, \quad \sum_{k=1}^{\infty} \frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}} = 12\pi^2, \tag{1.5}$$

$$\sum_{k=1}^{\infty} \frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}} = K \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}} = -\frac{45}{2}K, \tag{1.6}$$

where

$$K := \sum_{k=1}^{\infty} \frac{(\frac{k}{3})}{k^2} = \sum_{n=0}^{\infty} \left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right)$$

with $(\frac{k}{3})$ the Legendre symbol. All the five conjectural identities have been confirmed, see [13, 9, 10]. In 2024, F. Calegari, V. Dimitrov and Y. Tang [4] proved that K is irrational while the Catalan constant G is not yet known to be irrational. Note that [21, (5.1)] gives the fastest way to compute the value of K.

In 1974 R. W. Gosper discovered the identity

$$\sum_{k=0}^{\infty} \frac{25k-3}{2^k\binom{3k}{k}} = \frac{\pi}{2}$$

which was later proved in [2] via integration. The author [19] evaluated some series with summands involving a single binomial coefficient $\binom{3k}{k}$ or $\binom{4k}{2k}$; for example, the author [19] proved that

$$\sum_{k=0}^{\infty} \frac{(49k+1)8^k}{3^k\binom{3k}{k}} = 81 + 16\sqrt{3}\,\pi \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{10k-1}{\binom{4k}{2k}} = \frac{4\sqrt{3}}{27}\pi.$$

In [21] the author focused on series of the type

$$\sum_{k=1}^{\infty} \frac{ak^2 + bk + c}{k(3k-1)(3k-2)m^k\binom{4k}{k}},$$

where a, b, c and $m \neq 0$ are rational numbers; for example, the author [21] established the identities

$$\sum_{k=1}^{\infty} \frac{(5k^2 - 4k + 1)8^k}{k(3k-1)(3k-2)\binom{4k}{k}} = \frac{3}{2}\pi$$

and

$$\sum_{k=1}^{\infty} \frac{415k^2 - 343k + 62}{k(3k-1)(3k-2)(-8)^k\binom{4k}{k}} = -3\log 2.$$

In 2014 W. Chu and W. Zhang [6] used hypergeometric transformations to deduce many series identities involving binomial coefficients. In particular, Examples 58, 115 and 117 of [6] yield some series identities with summands involving a single binomial coefficient $\binom{6k}{3k}$. For example, [6, Example 115] gives the identity

$$\sum_{k=0}^{\infty} \frac{21k^2 + 27k + 8}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = \frac{8\pi}{9\sqrt{3}}. \tag{1.7}$$

In this paper we deduce the following new results.

Theorem 1.1. *We have the identity*

$$\sum_{k=0}^{\infty} \frac{(63k^2 + 78k + 22)8^k}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = \frac{3\pi}{2}. \tag{1.8}$$

Theorem 1.2. *We have*

$$\sum_{k=0}^{\infty} \frac{(6k-1)(42k-1)\binom{6k}{3k}}{4096^k} = \frac{32}{27}\sqrt{3}, \tag{1.9}$$

$$\sum_{k=0}^{\infty} \frac{(42k+5)\binom{6k}{3k}}{(6k-5)4096^k} = -\frac{4}{9}\sqrt{3}, \tag{1.10}$$

and

$$\sum_{k=0}^{\infty} \frac{(252k^2 - 120k + 5)\binom{6k}{3k}}{(2k-1)(6k-1)(6k-5)4096^k} = -\frac{\sqrt{3}}{2}. \tag{1.11}$$

In the next section, we will prove Theorem 1.1 via integration, and deduce Theorem 1.2 by using the identity

$$\sum_{k=0}^{\infty} \frac{k(252k^2 - 264k + 61)\binom{6k}{3k}}{(2k-1)(6k-1)(6k-5)4096^k} = \frac{1}{12\sqrt{3}} \tag{1.12}$$

given by [6, Example 117].

Recall that the usual harmonic numbers are defined by

$$H_n := \sum_{0 < k \leqslant n} \frac{1}{k} \quad (n = 0, 1, 2, \ldots).$$

For any integer $m \geqslant 2$, the harmonic numbers of order m are given by

$$H_n^{(m)} := \sum_{0 < k \leqslant n} \frac{1}{k^m} \quad (n = 0, 1, 2, \ldots).$$

Motivated by [17, 12, 22, 23], the author [20, 19, 21] posed many conjectural series identities involving harmonic numbers. For example, inspired by the Ramanujan series

$$\sum_{k=0}^{\infty} (42k + 5) \frac{\binom{2k}{k}^3}{4096^k} = \frac{16}{\pi},$$

the author [20, (110)] conjectured the identity

$$\sum_{k=0}^{\infty} (42k + 5) \frac{\binom{2k}{k}^3}{4096^k} \left(H_{2k}^{(3)} - \frac{43}{352} H_k^{(3)} \right) = \frac{555}{77} \cdot \frac{\zeta(3)}{\pi} - \frac{32}{11} G$$

which remains open. In Section 3, we will pose our conjectures on series involving $\binom{6k}{3k}$ some of which also involve harmonic numbers.

In Section 4, we will introduce the so-called Duality Principle for irrational series of Ramanujan's type and Zeilberger's type. Motivated by this Duality Principle, in Sections 5 and 6 we will propose totally 26 irrational conjectural series of Ramanujan's type or Zeilberger's type.

All the series in this paper converge at geometric rates and hence evaluations of them can be easily checked numerically via `Mathematica`.

2. PROOFS OF THEOREMS 1.1 AND 1.2

Lemma 2.1. *We have*

$$\sum_{k=0}^{\infty} \frac{(63k^2 + 18k + 2)8^k}{(6k+1)\binom{6k}{3k}} = \frac{3}{2}\pi + 4. \tag{2.1}$$

Proof. For any $a > 0$ and $b > 0$, it is well known that

$$B(a, b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

coincides with $\Gamma(a)\Gamma(b)/\Gamma(a+b)$. Thus

$$\sum_{k=0}^{\infty} \frac{(63k^2 + 18k + 2)8^k}{(6k+1)\binom{6k}{3k}}$$

$$= \sum_{k=0}^{\infty} (63k^2 + 18k + 2)8^k \frac{\Gamma(3k+1)^2}{\Gamma(6k+2)}$$

$$= \sum_{k=0}^{\infty} (63k^2 + 18k + 2)8^k B(3k+1, 3k+1)$$

$$= \int_0^1 \sum_{k=0}^{\infty} (63k^2 + 18k + 2)(2x(1-x))^{3k} dx$$

For $|z| < 1$, by calculus we have

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \quad \sum_{k=1}^{\infty} kz^{k-1} = \frac{d}{dz}(1-z)^{-1} = \frac{1}{(1-z)^2},$$

and

$$\sum_{k=0}^{\infty} k(k-1)z^k = z^2 \frac{d}{dz}(1-z)^{-2} = \frac{2z^2}{(1-z)^3}.$$

Thus, for $0 \leqslant x \leqslant 1$ we have

$$\sum_{k=0}^{\infty} (63k^2 + 18k + 2)(2x(1-x))^{3k}$$

$$= \sum_{k=0}^{\infty} (63k(k-1) + 81k + 2)(2x(1-x))^{3k}$$

$$= \frac{126(2x(1-x))^6}{(1-(2x(1-x))^3)^3} + \frac{81(2x(1-x))^3}{(1-(2x(1-x))^3)^2} + \frac{2}{(1-(2x(1-x))^3)}$$

$$= \frac{47(2x(1-x))^6 + 77(2x(1-x))^3 + 2}{(1-(2x(1-x))^3)^3}$$

$$= \frac{8(47(1-t^2)^6 + 616(1-t^2)^3 + 128)}{(8-(1-t^2)^3)^3},$$

where $t = 2x - 1$. Combining this with the last paragraph, we obtain

$$\sum_{k=0}^{\infty} \frac{(63k^2 + 18k + 2)8^k}{(6k+1)\binom{6k}{3k}}$$

$$= 8 \int_{-1}^1 \frac{47(1-t^2)^6 + 616(1-t^2)^3 + 128}{2(8-(1-t^2)^3)^3} dt$$

$$= 8 \int_0^1 \frac{47(1-t^2)^6 + 616(1-t^2)^3 + 128}{(8-(1-t^2)^3)^3} dt.$$

For

$$f(t) := \frac{48t(t^4 - 3t^2 + 10)}{(t^6 - 3t^4 + 3t^2 + 7)^2} + \frac{t(3t^4 - 10t^2 - 25)}{t^6 - 3t^4 + 3t^2 + 7} + 3\arctan t,$$

it is easy to verify that

$$\frac{d}{dt}f(t) = 4 \times \frac{47(1 - t^2)^6 + 616(1 - t^2)^3 + 128}{(8 - (1 - t^2)^3)^3}.$$

Thus

$$\sum_{k=0}^{\infty} \frac{(63k^2 + 18k + 2)8^k}{(6k + 1)\binom{6k}{3k}} = 2(f(1) - f(0)) = 2f(1) = 4 + \frac{3}{2}\pi$$

as desired. This ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.2. *For any $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} \frac{x^k(9(x - 64)k^3 + 18(x - 16)k^2 + (11x + 208)k + 2x + 40)}{(6k + 1)(6k + 5)\binom{6k}{3k}}$$
$$= 8 + \frac{(n + 1)(3n + 1)(3n + 2)x^{n+1}}{(6n + 1)(6n + 5)\binom{6n}{3n}}. \tag{2.2}$$

and

$$\sum_{k=0}^{n} \frac{x^k(9(x - 64)k^3 + 18(x - 48)k^2 + (11x - 368)k + 2x - 40)}{(2k + 1)(6k + 1)(6k + 5)\binom{6k}{3k}}$$
$$= -8 + \frac{(n + 1)(3n + 1)(3n + 2)x^{n+1}}{(2n + 1)(6n + 1)(6n + 5)\binom{6n}{3n}}. \tag{2.3}$$

This can be easily proved by induction on n.

`Proof of Theorem` 1.1. Putting $x = 8$ in (2.2) and (2.3), we get the identities

$$\sum_{k=0}^{n} \frac{8^k(63k^3 + 18k^2 - 37k - 7)}{(6k + 1)(6k + 5)\binom{6k}{3k}} = -1 - \frac{8^n(n + 1)(3n + 1)(3n + 2)}{(6n + 1)(6n + 5)\binom{6n}{3n}}$$

and

$$\sum_{k=0}^{n} \frac{8^k(63k^3 + 90k^2 + 35k + 3)}{(2k + 1)(6k + 1)(6k + 5)\binom{6k}{3k}} = 1 - \frac{8^n(n + 1)(3n + 1)(3n + 2)}{(2n + 1)(6n + 1)(6n + 5)\binom{6n}{3n}}.$$

Letting $n \to +\infty$ we obtain the identities

$$\sum_{k=0}^{\infty} \frac{8^k(63k^3 + 18k^2 - 37k - 7)}{(6k + 1)(6k + 5)\binom{6k}{3k}} = -1 \tag{2.4}$$

and

$$\sum_{k=0}^{\infty} \frac{8^k(63k^3 + 90k^2 + 35k + 3)}{(2k + 1)(6k + 1)(6k + 5)\binom{6k}{3k}} = 1. \tag{2.5}$$

Since
$$(6k+5)(63k^2+18k+2) - 6(63k^3+18k^2-37k-7) = 315k^2+324k+52,$$
in view of (2.1) and (2.4) we have
$$\sum_{k=0}^{\infty} \frac{8^k(315k^2+324k+52)}{(6k+1)(6k+5)\binom{6k}{3k}}$$
$$= \sum_{k=0}^{\infty} \frac{8^k(63k^2+18k+2)}{(6k+1)\binom{6k}{3k}} - 6\sum_{k=0}^{\infty} \frac{8^k(63k^3+18k^2-37k-7)}{(6k+1)(6k+5)\binom{6k}{3k}}$$
$$= \frac{3}{2}\pi + 4 - 6 \times (-1) = \frac{3}{2}\pi + 10.$$

As
$$(2k+1)(315k^2+324k+52) - 10(63k^3+90k^2+35k+3) = 63k^2+78k+22,$$
with the aid of (2.5) we deduce further that
$$\sum_{k=0}^{\infty} \frac{8^k(63k^2+78k+22)}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}}$$
$$= \sum_{k=0}^{\infty} \frac{8^k(315k^2+324k+52)}{(6k+1)(6k+5)\binom{6k}{3k}} - 10\sum_{k=0}^{\infty} \frac{8^k(63k^3+90k^2+35k+3)}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}}$$
$$= \frac{3}{2}\pi + 10 - 10 \times 1 = \frac{3}{2}\pi.$$

This proves the desired (1.8). $\qquad\qquad\square$

Lemma 2.3. *For any $n \in \mathbb{N}$, we have*
$$\sum_{k=0}^{n} \frac{(4536k^3-4500k^2+978k+5)\binom{6k}{3k}}{(2k-1)(6k-1)(6k-5)4096^k} = -\frac{\binom{6n}{3n}}{4096^n}. \qquad (2.6)$$

and
$$\sum_{k=0}^{n} \frac{(4536k^3-4644k^2+1074k+25)\binom{6k}{3k}}{(6k-5)4096^k} = -\frac{(2n+1)(6n+5)\binom{6n}{3n}}{4096^n}. \quad (2.7)$$

Proof. It is easy to show (2.6) and (2.7) by induction on n. $\qquad\square$

`Proof of Theorem` 1.2. Letting $n \to +\infty$ in (2.6), we get
$$\sum_{k=0}^{\infty} \frac{(4536k^3-4500k^2+978k+5)\binom{6k}{3k}}{(2k-1)(6k-1)(6k-5)4096^k} = 0. \qquad (2.8)$$

Since
$$4536k^3-4500k^2+978k+5 - 18k(252k^2-264k+61) = 252k^2-120k+5,$$
combining (2.8) and (1.12) we obtain
$$\sum_{k=0}^{\infty} \frac{(252k^2-120k+5)\binom{6k}{3k}}{(2k-1)(6k-1)(6k-5)} = 0 - 18 \times \frac{1}{12\sqrt{3}} = -\frac{\sqrt{3}}{2}.$$

This proves (1.11).

Observe that

$$4536k^3 - 4500k^2 + 978k + 5$$
$$= 9(2k-1)(6k-1)(42k+5) - 8(252k^2 - 120k + 5).$$

Thus, in view of (2.8) and (1.11), we have

$$\sum_{k=0}^{\infty} \frac{(42k+5)\binom{6k}{3k}}{(6k-5)4096^k} = \frac{8}{9} \sum_{k=0}^{\infty} \frac{(252k^2 - 120k + 5)\binom{6k}{3k}}{(2k-1)(6k-1)(6k-5)4096^k}$$
$$= \frac{8}{9} \times \left(-\frac{\sqrt{3}}{2} \right) = -\frac{4}{9}\sqrt{3}.$$

This proves (1.10).

Letting $n \to +\infty$ in (2.7), we get the identity

$$\sum_{k=0}^{\infty} \frac{(4536k^3 - 4644k^2 + 1074k + 25)\binom{6k}{3k}}{(6k-5)4096^k} = 0. \tag{2.9}$$

Since

$$4536k^3 - 4644k^2 + 1074k + 25 - 3(6k-5)(6k-1)(42k-1) = 8(42k+5),$$

combining (2.9) with (1.10), we obtain

$$\sum_{k=0}^{\infty} \frac{(6k-1)(42k-1)\binom{6k}{3k}}{4096^k} = -\frac{8}{3} \sum_{k=0}^{\infty} \frac{(42k+5)\binom{6k}{3k}}{6k-5} = \frac{32}{27}\sqrt{3}.$$

This proves (1.9).

In view of the above, we have completed the proof of Theorem 1.2.   □

3. Conjectural series with summands involving $\binom{6k}{3k}$

Conjecture 3.1. *We have*

$$\sum_{k=0}^{\infty} \frac{8^k((63k^2 + 78k + 22)H_{3k} + 12k + 8)}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = 6G - \frac{3}{2}\pi \log 2. \tag{3.1}$$

Remark 3.1. This is motivated by (1.8).

Conjecture 3.2. *We have*

$$\sum_{k=0}^{\infty} \frac{(3k+2)^2 16^k}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = \frac{\pi}{2\sqrt{3}}. \tag{3.2}$$

Also,

$$\sum_{k=0}^{\infty} \frac{(3k+2)^2 16^k H_k}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = \frac{5}{2}K - \frac{\pi \log 432}{6\sqrt{3}} \tag{3.3}$$

and

$$\sum_{k=0}^{\infty} \frac{16^k((3k+2)^2(3H_{3k} - 2H_{2k}) + \frac{2k(3k+2)}{2k+1})}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = \frac{5}{4}K + \frac{\pi}{6\sqrt{3}} \log \frac{27}{256}. \tag{3.4}$$

Remark 3.2. In the spirit of the proof of Theorem 1.1, we can show that (3.2) is equivalent to the identity

$$\sum_{k=0}^{\infty} \frac{16^k(54k^2 - 36k - 12)}{(6k+1)\binom{6k}{3k}} = -4 - \frac{\pi}{\sqrt{3}},$$

which we cannot prove via integration.

Conjecture 3.3. *We have*

$$\sum_{k=0}^{\infty} \frac{(21k^2 + 27k + 8)(H_{2k} + 2H_k) + \frac{(3k+1)(5k+4)}{6k+3}}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = 14K - \frac{16\log 3}{3\sqrt{3}}\pi \quad (3.5)$$

and

$$\sum_{k=0}^{\infty} \frac{(21k^2 + 27k + 8)H_{3k} + 2k + 13/9}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = \frac{8}{3}K - \frac{8\log 3}{9\sqrt{3}}\pi. \quad (3.6)$$

Remark 3.3. This is motivated by the identity (1.7).

Conjecture 3.4. *We have*

$$\sum_{k=0}^{\infty} \frac{27^k((111k^2 + 133k + 36)(H_{2k} - 3H_k) - \frac{(3k+1)(3k+4)}{2k+1})}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = \frac{16\log 3}{\sqrt{3}}\pi - 48K$$

$$(3.7)$$

and

$$\sum_{k=0}^{\infty} \frac{27^k((111k^2 + 133k + 36)H_{3k} + 30k + 19)}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = 12K. \quad (3.8)$$

Remark 3.4. This is motivated by the identity

$$\sum_{k=0}^{\infty} \frac{27^k(111k^2 + 133k + 36)}{(2k+1)(6k+1)(6k+5)\binom{6k}{3k}} = \frac{16\pi}{3\sqrt{3}}$$

given by [6, Example 58].

Conjecture 3.5. *We have*

$$\sum_{k=0}^{\infty} \frac{((6k-1)(42k-1)(3H_{3k} - 10H_{2k} + 4H_k) - 1836k - 50)\binom{6k}{3k}}{4096^k}$$

$$(3.9)$$

$$= -\frac{32\sqrt{3}}{27}(32 + 14\log 2 - 9\log 3)$$

and

$$\sum_{k=0}^{\infty} \frac{((6k-1)(42k-1)(2H_{6k} - H_{3k}) + 24k + 8/3)\binom{6k}{3k}}{4096^k}$$

$$(3.10)$$

$$= \frac{32\sqrt{3}}{81}\left(8 + \log\frac{64}{27}\right).$$

Remark 3.5. This is motivated by the identity (1.9).

Conjecture 3.6. *We have*

$$\sum_{k=0}^{\infty} \frac{(57k^2 + 2k + 1)\binom{6k}{3k}}{216^k} = 9\sqrt{3} \tag{3.11}$$

and

$$\sum_{k=0}^{\infty} \frac{((57k^2 + 2k + 1)(2H_{6k} - H_{3k}) + 14k + 8/3)\binom{6k}{3k}}{216^k} = 3\sqrt{3}(8 + 3\log 3). \tag{3.12}$$

Remark 3.6. In the spirit of the proof of Theorem 1.2, (3.11) has the following equivalent form:

$$\sum_{k=0}^{\infty} \frac{(57k^2 - 40k + 5)\binom{6k}{3k}}{(2k-1)(6k-1)(6k-5)216^k} = -\frac{1}{\sqrt{3}}. \tag{3.13}$$

Conjecture 3.7. *We have*

$$\sum_{k=0}^{\infty} \frac{(6k-1)\binom{6k}{3k}}{256^k} = -\frac{2}{3\sqrt{3}} \tag{3.14}$$

and

$$\sum_{k=0}^{\infty} \frac{((6k-1)(3H_{3k} - H_k) - 4)\binom{6k}{3k}}{256^k} = \frac{4\sqrt{3}}{9}(2\log 2 - 3). \tag{3.15}$$

Also,

$$\sum_{k=0}^{\infty} \frac{k\binom{6k}{3k}(2H_{2k} - H_k - \frac{2}{6k-3})}{(6k-1)(6k-5)256^k} = \frac{\sqrt{3}}{324}\left(6 + \log\frac{4}{27}\right) \tag{3.16}$$

and

$$\sum_{k=0}^{\infty} \frac{k\binom{6k}{3k}(3H_{3k} - H_k - 4/(6k-3))}{(6k-1)(6k-5)256^k} = \frac{\sqrt{3}}{54}(3 - 2\log 2). \tag{3.17}$$

Remark 3.7. In the spirit of the proof of Theorem 1.2, (3.14) has the following equivalent form:

$$\sum_{k=0}^{\infty} \frac{k\binom{6k}{3k}}{(6k-1)(6k-5)256^k} = \frac{\sqrt{3}}{108}. \tag{3.18}$$

Conjecture 3.8. *We have*

$$\sum_{k=0}^{\infty} \frac{(84k^2 - 4k + 1)\binom{6k}{3k}}{512^k} = 4\sqrt{2} \tag{3.19}$$

and

$$\sum_{k=0}^{\infty} \frac{((84k^2 - 4k + 1)(2H_{6k} - H_{3k}) + 16k + 8/3)\binom{6k}{3k}}{512^k} = \frac{4\sqrt{2}}{3}(8 + 3\log 2). \tag{3.20}$$

Also,

$$\sum_{k=0}^{\infty} \frac{\binom{6k}{3k}\left((42k-5)(H_{6k}-\frac{H_{3k}}{2}) - \frac{1224k^3-1308k^2+366k-25}{(2k-1)(6k-1)(6k-5)}\right)}{(6k-1)(6k-5)512^k} = -\frac{2+\log 2}{2\sqrt{2}}.$$

(3.21)

Remark 3.8. In the spirit of the proof of Theorem 1.2, (3.19) has the following equivalent form:

$$\sum_{k=0}^{\infty} \frac{(42k-5)\binom{6k}{3k}}{(6k-1)(6k-5)512^k} = -\frac{\sqrt{2}}{2}.$$

(3.22)

Conjecture 3.9. *We have*

$$\sum_{k=0}^{\infty} \frac{(6k-1)^2\binom{6k}{3k}}{(-512)^k} = \frac{4}{27}\sqrt{6},$$

(3.23)

$$\sum_{k=0}^{\infty} \frac{(6k-1)^2(3H_{3k}-4H_{2k}+H_k)\binom{6k}{3k}}{(-512)^k} = \frac{4\sqrt{6}}{9}\log\frac{3}{4},$$

(3.24)

and

$$\sum_{k=0}^{\infty} \frac{((6k-1)^2(2H_{6k}-H_{3k})+8/3)\binom{6k}{3k}}{(-512)^k} = \frac{4\sqrt{6}}{81}\left(8+3\log\frac{2}{3}\right).$$

(3.25)

Also,

$$\sum_{k=0}^{\infty} \frac{\binom{6k}{3k}\left((36k^2-12k+5)H(k) + \frac{16P(k)}{(6k-1)(6k-3)(6k-5)}\right)}{(2k-1)(6k-1)(6k-5)(-512)^k}$$

$$= \sqrt{6}\left(2\log 2 - \frac{\log 3}{2} - 8\right),$$

(3.26)

where $H(k) = 4H_{6k}+H_{3k}-4H_{2k}+H_k$ and $P(k) = 936k^3-1476k^2+622k-75$.

Remark 3.9. In the spirit of the proof of Theorem 1.2, (3.23) has the following equivalent form:

$$\sum_{k=0}^{\infty} \frac{(36k^2-12k+5)\binom{6k}{3k}}{(2k-1)(6k-1)(6k-5)(-512)^k} = -\frac{\sqrt{6}}{2}.$$

(3.27)

4. Duality principle for irrational series

Ramanujan's list (cf. [14]) of series for $1/\pi$ contains only one irrational series:

$$\sum_{k=0}^{\infty} \left(k + \frac{31}{270+48\sqrt{5}}\right) \frac{\binom{2k}{k}^3}{(2^4(\sqrt{5}+1)^8)^k} = \frac{16}{3(5+7\sqrt{5})\pi},$$

i.e.,

$$\sum_{k=0}^{\infty} \left(6k(7\sqrt{5}+5)+5\sqrt{5}-1\right) \frac{\binom{2k}{k}^3}{(12+4\sqrt{5})^{4k}} = \frac{32}{\pi}.$$

(4.1)

In 1988 J. M. Borwein and P. B. Borwein [3] gave 7 pairs of new irrational series of Ramanujan's type; for example,

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{4k}{2k}(40k(5457\sqrt{2}+693121)-869892\sqrt{2}+1877581)}{19^{2k}(12(481+340\sqrt{2}))^{4k}} = 466578\frac{\sqrt{19}}{\pi}$$

(4.2)

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{4k}{2k}(40k(693121-5457\sqrt{2})+869892\sqrt{2}+1877581)}{19^{2k}(12(481-340\sqrt{2}))^{4k}} = 2332890\frac{\sqrt{19}}{\pi}.$$

Note that the series in (4.2) converges quickly with the geometric rate about 3.996×10^{-17}.

In 2012 A. M. Aldawoud [1] did a systematic search via Maple, and guessed 76 new irrational series of the Ramanujan type. However, some of her tables (such as the fourth row of Table 3.4) contain false data which might be caused by typos; for example, the tenth row of [1, Table 3.1] indicates that for

$$x = \frac{8145698488}{79267303539275} - \frac{570145303\sqrt{7}}{1467913 0284125} \text{ and } \lambda = \frac{90352}{1274007} - \frac{59000\sqrt{7}}{3822021}$$

we should have

$$\pi\sqrt{7(1-1728x)}\sum_{k=0}^{\infty}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}x^k(k+\lambda) = \frac{1}{4},$$

but our calculation shows that

$$\pi\sqrt{7(1-1728x)}\sum_{k=0}^{\infty}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}x^k(k+\lambda) \approx 0.24999999999474.$$

In 2023 J. M. Campbell [5] proved some irrational Ramanujan-type series of level 3 (with summands involving $\binom{2k}{k}^2\binom{3k}{k}$) conjectured by Aldawoud [1].

Let d be any positive integer which is not a square. By Galois theory, the Galois group of the field extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ has exactly two elements, and the nontrivial Galois automorphism σ_d in $\text{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ sends $a+b\sqrt{d}$ (with $a,b \in \mathbb{Q}$) to its conjugate $a-b\sqrt{d}$. Inspired by this, for an irrational series $\sum_{k=0}^{\infty} a_k$ with $a_k \in \mathbb{Q}(\sqrt{d})$, we define its *dual* by

$$\lim_{n \to +\infty} \sigma_d\left(\sum_{k=0}^{n} a_k\right) = \sum_{k=0}^{\infty} \sigma_d(a_k)$$

if the series $\sum_{k=0}^{\infty} \sigma_d(a_k)$ converges.

The dual series of the Ramanujan series in (4.1) is

$$\sum_{k=0}^{\infty}\left(6k(-7\sqrt{5}+5)-5\sqrt{5}-1\right)\frac{\binom{2k}{k}^3}{(12-4\sqrt{5})^{4k}},$$

and the identity (4.1) has the following dual:

$$\sum_{k=0}^{\infty} \left(6k(7\sqrt{5} - 5) + 5\sqrt{5} + 1 \right) \frac{\binom{2k}{k}^3}{(12 - 4\sqrt{5})^{4k}} = \frac{96}{\pi}, \tag{4.3}$$

which was noted in [11].

In the spirit of the author's recent papers [20, 18, 19, 21], we pose the following conjecture motivated by (4.1) and (4.3).

Conjecture 4.1. *We have*

$$\sum_{k=0}^{\infty} \left(6k(5 + 7\sqrt{5}) + 5\sqrt{5} - 1)H(k) + \frac{60(7 - 3\sqrt{5})}{2k + 1} \right) \frac{\binom{2k}{k}^3}{(12 + 4\sqrt{5})^{4k}} = \frac{16}{3}\pi \tag{4.4}$$

and

$$\sum_{k=0}^{\infty} \left(6k(5 - 7\sqrt{5}) - 5\sqrt{5} - 1)H(k) + \frac{60(7 + 3\sqrt{5})}{2k + 1} \right) \frac{\binom{2k}{k}^3}{(12 - 4\sqrt{5})^{4k}} = 1136\pi. \tag{4.5}$$

where

$$H(k) := 35H_k^{(2)} - 136H_{2k}^{(2)}.$$

For any integer $d \equiv 0, 1 \pmod 4$, we adopt the notation

$$L_d(2) := L\left(2, \left(\frac{d}{\cdot} \right) \right) = \sum_{k=1}^{\infty} \frac{\left(\frac{d}{k} \right)}{k^2}$$

with $\left(\frac{d}{k} \right)$ the Kronecker symbol. Thus $G = L_{-4}(2)$ and $K = L_{-3}(2)$.

For convenience, let us call the series of the form

$$\sum_{k=1}^{\infty} \frac{(ak - b)m^k}{k^3 c_k} = r_1 L_{n'}(2) + r_2 \sqrt{d} L_{(dn)'}(2)$$

with am nonzero and c_k among (1.1) a *Zeilberger-type series*, where $d \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, $n \in \mathbb{Z}$, $a, b, m \in \mathbb{Q}(\sqrt{d})$, $r_1, r_2 \in \mathbb{Q}$, and c' with $c \in \mathbb{Z}$ is defined by

$$c' = \begin{cases} c & \text{if } c \equiv 0, 1 \pmod 4, \\ 4c & \text{otherwise.} \end{cases}$$

When $d = 1$, we call such series *rational series of the Zeilberger type*. It seems that (1.2)-(1.6) are the only rational series of Zeilberger's type. Table 5 of [10] provides ten irrational series of Zeilberger's type one of which is the identity

$$\sum_{k=1}^{\infty} \frac{3(16\sqrt{5} + 35)k - 4(5\sqrt{5} + 11)}{k^3 \binom{2k}{k}^3} \left(\frac{1 - \sqrt{5}}{2} \right)^{8k} = \frac{\pi^2}{30}. \tag{4.6}$$

We find the dual of this identity:

$$\sum_{k=1}^{\infty} \frac{3(16\sqrt{5}-35)k - 4(5\sqrt{5}-11)}{k^3\binom{2k}{k}^3} \left(\frac{1+\sqrt{5}}{2}\right)^{8k} = \frac{71}{30}\pi^2, \qquad (4.7)$$

which will be proved in a forthcoming paper joint with Yajun Zhou.

We make the following conjecture similar to Conjecture 4.1.

Conjecture 4.2. *We have*

$$\sum_{k=1}^{\infty} \frac{((1-\sqrt{5})/2)^{8k}}{k^3\binom{2k}{k}^3} \left(\left(3(16\sqrt{5}+35)k - 4(5\sqrt{5}+11)\right)\mathcal{H}(k) + \frac{15}{k}\right) = \frac{\pi^4}{72}$$

$$(4.8)$$

and

$$\sum_{k=1}^{\infty} \frac{((1+\sqrt{5})/2)^{8k}}{k^3\binom{2k}{k}^3} \left(\left(3(35-16\sqrt{5})k + 4(5\sqrt{5}-11)\right)\mathcal{H}(k) + \frac{15}{k}\right) = \frac{433}{72}\pi^4.$$

$$(4.9)$$

where

$$\mathcal{H}(k) := 4H_{2k-1}^{(2)} - 35H_{k-1}^{(2)}.$$

Let d be a positive integer which is not a square. Let c_k ($k \in \mathbb{N}$) be one of the four products in (1.1), and set $c = \lim_{k\to\infty} \sqrt[k]{c_k}$. For a Ramanujan-type series

$$\sum_{k=0}^{\infty} \frac{(ak+b)c_k}{m^k} = \frac{r\sqrt{n}}{\pi}$$

with n a positive algebraic integer in $\mathbb{Q}(\sqrt{d})$, $a, b, m \in \mathbb{Q}(\sqrt{d})$, $r \in \mathbb{Q}$ and $amr \neq 0$, if $|\sigma_d(m)| < c$ then we define the *dual* of the Ramanujan series $\sum_{k=0}^{\infty} \frac{(ak+b)c_k}{m^k}$ as the Zeilberger-type series

$$\sum_{k=1}^{\infty} \sigma_d\left(\frac{(ak-b)m^k}{k^3c_k}\right).$$

Conjecture 4.3 (Duality Principle). *Let d be a positive integer which is not a square. And let c_k ($k \in \mathbb{N}$) be one of the four products in (1.1). Let $a, b, m \in \mathbb{Q}(\sqrt{d})$ with $a \neq 0$ and $m \notin \mathbb{Q}$. Let c denote the positive integer $\lim_{k\to+\infty} \sqrt[k]{c_k}$.*

(i) Suppose that we have a Ramanujan-type series

$$\sum_{k=0}^{\infty} \frac{(ak+b)c_k}{m^k} = \frac{r\sqrt{n}}{\pi}$$

where $r \in \mathbb{Q} \setminus \{0\}$, and n is a positive algebraic integer in $\mathbb{Q}(\sqrt{d})$. If $\sum_{k=0}^{\infty} \sigma_d\left(\frac{(ak+b)c_k}{m^k}\right)$ converges (i.e., $|\sigma_d(m)| > c$) and $\sigma_d(n) > 0$, then

$$\sum_{k=0}^{\infty} \sigma_d\left(\frac{(ak+b)c_k}{m^k}\right) = \frac{r^*\sqrt{\sigma_d(n)}}{\pi}$$

for some rational number r^*. If $\sum_{k=0}^{\infty} \sigma_d(\frac{(ak+b)c_k}{m^k})$ diverges (i.e. $|\sigma_d(m)| < c$), and $m < 0$ and $n \in \mathbb{Z}^+$, then we have a Zeilberger-type series

$$\sum_{k=1}^{\infty} \sigma_d\left(\frac{(ak-b)m^k}{k^3 c_k}\right) = r_1 L_{(-n)'}(2) + r_2\sqrt{d}L_{(-dn)'}(2)$$

for some $r_1, r_2 \in \mathbb{Q}$.

(ii) Suppose that we have a Zeilberger-type series

$$\sum_{k=1}^{\infty} \frac{(ak-b)m^k}{k^3 c_k} = r_1 L_{n'}(2) + r_2\sqrt{d}L_{(dn)'}(2),$$

where $n \in \mathbb{Z}$ and $r_1, r_2 \in \mathbb{Q}$. If the series $\sum_{k=1}^{\infty} \sigma_d(\frac{(ak-b)m^k}{k^3 c_k})$ converges (i.e., $|\sigma_d(m)| < c$), then

$$\sum_{k=1}^{\infty} \sigma_d\left(\frac{(ak-b)m^k}{k^3 c_k}\right) = r_1^* L_{n'}(2) + r_2^*\sqrt{d}L_{(dn)'}(2)$$

for some $r_1^*, r_2^* \in \mathbb{Q}$.

5. New irrational series for $1/\pi$

In this section, we apply the Duality Principle to find 16 new irrational series for $1/\pi$.

Conjecture 5.1. *We have*

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(9k(51-11\sqrt{6}) + 2(54-19\sqrt{6}))}{(27(37102+15147\sqrt{6}))^k} = \frac{375}{8\pi}, \tag{5.1}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(2k(182-27\sqrt{6}) + 3(16-\sqrt{6}))}{8^k(13-7\sqrt{6})^{2k}} = \frac{250\sqrt{2}}{\pi}, \tag{5.2}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(30k(7\sqrt{10}-2) + 25\sqrt{10}+16)}{108^k(14-5\sqrt{10})^{2k}} = \frac{162\sqrt{6}}{\pi}, \tag{5.3}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(15k(377\sqrt{3}-378) + 4(122\sqrt{3}-27))}{(114-63\sqrt{3})^{3k}} = \frac{11979}{\pi}, \tag{5.4}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(30k(59228-7\sqrt{34}) + 125792+7157\sqrt{34})}{108^k(1876-325\sqrt{34})^{2k}} = 215622\frac{\sqrt{6}}{\pi}, \tag{5.5}$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(3k(20801\sqrt{265}-24115) + 7049\sqrt{265}-82795)}{(-15)^k(60(341\sqrt{265}+5551))^{2k}} = 25920\frac{\sqrt{15}}{\pi}. \tag{5.6}$$

Remark 5.1. (5.2)–(5.5) are duals of the conjectural identities

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(2k(182+27\sqrt{6})+3(16+\sqrt{6}))}{8^k(13+7\sqrt{6})^{2k}} = \frac{125\sqrt{2}}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(30k(7\sqrt{10}+2)+25\sqrt{10}-16)}{108^k(14+5\sqrt{10})^{2k}} = \frac{81\sqrt{6}}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(15k(377\sqrt{3}+378)+4(122\sqrt{3}+27))}{(114+63\sqrt{3})^{3k}} = \frac{11979}{4\pi},$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(30k(59228+7\sqrt{34})+125792-7157\sqrt{34})}{108^k(1876+325\sqrt{34})^{2k}} = 107811\frac{\sqrt{6}}{\pi}$$

given by rows two, five, nine and eleven of [1, Table 3.6], respectively. Also, (5.6) is the dual of the conjectural identity

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(3k(20801\sqrt{265}+24115)+7049\sqrt{265}+82795)}{(-15)^k(60(341\sqrt{265}-5551))^{2k}} = 360^2\frac{\sqrt{15}}{\pi}$$

given by the eighth row of [1, Table 3.8].

Conjecture 5.2. *We have*

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{4k}{2k}(5k(32\sqrt{3}-17)+18\sqrt{3}-2)}{(36(746-425\sqrt{3}))^k} = \frac{121\sqrt{3}}{\pi}, \tag{5.7}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{4k}{2k}((352\sqrt{2}+57)k+2(23\sqrt{2}+15))}{(162(884-627\sqrt{2}))^k} = \frac{441}{2\pi}, \tag{5.8}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{4k}{2k}2^k(35k(2176-13\sqrt{7})+8(98\sqrt{7}+841))}{(9(325-119\sqrt{7}))^{2k}} = \frac{171^2}{\pi}, \tag{5.9}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{4k}{2k}(40k(3\sqrt{2}+19)-12\sqrt{2}+71)}{(48(57+40\sqrt{2}))^{2k}} = \frac{98\sqrt{3}}{\pi}, \tag{5.10}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k\binom{2k}{k}^2\binom{4k}{2k}(20k(323-124\sqrt{3})+1329-708\sqrt{3})}{(32(340\sqrt{3}+589))^{2k}} = \frac{968}{3\pi}, \tag{5.11}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{4k}{2k}(20k(13\sqrt{33}+11)+33\sqrt{33}+119)}{(-3072)^k(75-13\sqrt{33})^{2k}} = 384\frac{\sqrt{3}}{\pi}, \tag{5.12}$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{4k}{2k}(260k(\sqrt{57}+323)+513\sqrt{57}+7331)}{(18432(71825\sqrt{57}-542267))^k} = 18816\frac{\sqrt{3}}{\pi}. \tag{5.13}$$

Remark 5.2. (5.7), (5.8), (5.9) and (5.10) are the duals of the conjectural identities

$$\sum_{k=0}^{\infty} \left(5k(17 + 32\sqrt{3}) + 2(9\sqrt{3} + 1)\right) \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{746 - 425\sqrt{3}}{527076}\right)^k = \frac{1}{2\sqrt{3}\,\pi},$$

$$\sum_{k=0}^{\infty} \left((352\sqrt{2} - 57)k + 2(23\sqrt{2} - 15)\right) \binom{2k}{k}^2 \binom{4k}{2k} \left(\frac{627\sqrt{2} - 884}{777924}\right)^k = \frac{441}{4\pi},$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 \binom{4k}{2k} 2^k (35k(2176 - 13\sqrt{7}) + 8(98\sqrt{7} + 841))}{(9(325 + 119\sqrt{7}))^{2k}} = \frac{171^2}{2\pi},$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 \binom{4k}{2k} (40k(19 - 3\sqrt{2}) + 12\sqrt{2} + 71)}{(48(57 - 40\sqrt{2}))^{2k}} = \frac{294\sqrt{3}}{\pi},$$

given by rows two, four, five and six of [1, Table 3.3], respectively. As the six row of [1, Table 3.3] actually gave a wrong value of λ, we find the last identity by using the PSLQ algorithm (cf. [15]). Moreover, (5.12) is the dual of the conjectural identities

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 \binom{4k}{2k} (20k(13\sqrt{33} - 11) + 33\sqrt{33} - 119)}{(-3072)^k (75 + 13\sqrt{33})^{2k}} = 128\frac{\sqrt{3}}{\pi},$$

given by the third row of [1, Table 3.5], and (5.13) is the dual of the conjectural identities

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 \binom{4k}{2k} (260k(323 - \sqrt{57}) - 513\sqrt{57} + 7331)}{(-18432(71825\sqrt{57} + 542267))^k} = 6272\frac{\sqrt{3}}{\pi}$$

given by fifth row of [1, Table 3.5]. Note also that the dual of (5.11) is the Zeilberger-type identity given by the last row of [10, Table 5].

Conjecture 5.3. *We have*

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}(92158\sqrt{5}k + 5(1393\sqrt{5} + 1600))}{96^{3k}(165393\sqrt{5} - 369830)^k}$$
$$= \frac{32}{5\pi}\sqrt{30(3777190 - 165393\sqrt{5})}, \tag{5.14}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}(20124k + 125\sqrt{13} + 1339)}{60^{3k}(15965 - 4428\sqrt{13})^k}$$
$$= \frac{10}{3\pi}\sqrt{195(14230 - 41\sqrt{13})}, \tag{5.15}$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}2^k(13629k + 220\sqrt{5} + 1052)}{(135(274207975 - 122629507\sqrt{5}))^k}$$

$$= \frac{3}{2\pi}\sqrt{3(5607725 + 661856\sqrt{5})}. \tag{5.16}$$

Remark 5.3. (5.14), (5.15) and (5.16) are the duals of the conjectural identities

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}(92158\sqrt{5}k + 5(1393\sqrt{5} - 1600))}{(-96)^{3k}(165393\sqrt{5} + 369830)^k}$$

$$= \frac{32}{15\pi}\sqrt{30(3777190 + 165393\sqrt{5})},$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}(20124k - 125\sqrt{13} + 1339)}{60^{3k}(15965 + 4428\sqrt{13})^k} = \frac{5}{3\pi}\sqrt{195(14230 + 41\sqrt{13})}$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}2^k(13629k - 220\sqrt{5} + 1052)}{(135(274207975 + 122629507\sqrt{5}))^k}$$

$$= \frac{1}{2\pi}\sqrt{3(5607725 - 661856\sqrt{5})}$$

given by the second row of [1, Tsable 3.2], rows five and six of [1, Table 3.1], respectively.

6. New irrational series of Zeilberger's type

In this section, we apply the Duality Principle to find 10 new irrational series for $1/\pi$. They provide fast converging series for computing the *L*-values

$$L_{-8}(2), \ L_{-11}(2), \ L_{-24}(2), \ L_{-39}(2), \ L_{-68}(2) \text{ and } L_{-111}(2).$$

Conjecture 6.1. *We have*

$$\sum_{k=1}^{\infty} \frac{(24(12 - 7\sqrt{3}))^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}\left((3\sqrt{3} + 7)k - \sqrt{3} - 2\right) = \frac{5}{6}\left(9\sqrt{3}K - 16G\right), \tag{6.1}$$

$$\sum_{k=1}^{\infty} \frac{(54(265 - 153\sqrt{3}))^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}\left(3k(17\sqrt{3} + 27) - 16\sqrt{3} - 27\right)$$

$$= 135\left(G - \frac{11}{16}\sqrt{3}K\right), \tag{6.2}$$

$$\sum_{k=1}^{\infty} \frac{(27(37102 - 15147\sqrt{6}))^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}\left(9k(11\sqrt{6} + 51) - 2(19\sqrt{6} + 54)\right)$$

$$= \frac{5625}{2}\left(\sqrt{6}L_{-24}(2) - 3G\right), \tag{6.3}$$

$$\sum_{k=1}^{\infty} \frac{(-1728)^k(18-5\sqrt{13})^{2k}}{k^3\binom{2k}{k}^2\binom{3k}{k}} \left(15k(9\sqrt{13}+26)-39\sqrt{13}-134)\right)$$
$$= 54\left(80K-13\sqrt{13}L_{-39}(2)\right), \tag{6.4}$$

$$\sum_{k=1}^{\infty} \frac{(32(91\sqrt{33}-523))^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} \left((91\sqrt{33}+891)k-33\sqrt{33}-225\right)$$
$$= 320\left(\frac{11}{3}\sqrt{33}L_{-11}(2)-27K\right), \tag{6.5}$$

$$\sum_{k=1}^{\infty} \frac{(-3)^k(24(2737-450\sqrt{37}))^{2k}(15k(2898\sqrt{37}+3145)-6438\sqrt{37}-30355)}{k^3\binom{2k}{k}^2\binom{3k}{k}}$$
$$= 71874\left(380K-37\sqrt{37}L_{-111}(2)\right), \tag{6.6}$$

and

$$\sum_{k=1}^{\infty} \frac{(-3/5)^k(12m)^{2k}}{k^3\binom{2k}{k}^2\binom{3k}{k}} \left(15k(323\sqrt{145}+13195)-2407\sqrt{145}-35375\right)$$
$$= 23328\left(29\sqrt{145}L_{-87}(2)-375L_{-15}(2)\right) \tag{6.7}$$

where $m = 2975 - 247\sqrt{145}$.

Remark 6.1. (6.5) provides a fast way to compute $L_{-11}(2)$. (6.3) is the dual of (5.1), and (6.7) is the dual of the conjectural identity

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}(15k(13195-323\sqrt{145})+35375-2407\sqrt{145})}{(-3/5)^k(12(2975+247\sqrt{145}))^{2k}} = \frac{5184\sqrt{15}}{\pi}$$

which corresponds to the corrected form of the fourth row of [1, Table 3.8] containing typos.

Conjecture 6.2. *We have*

$$\sum_{k=1}^{\infty} \frac{(8(457-325\sqrt{2}))^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}} \left(5k(13\sqrt{2}+32)-3(6\sqrt{2}+11)\right)$$
$$= 490\sqrt{2}L_{-8}(2)-980G \tag{6.8}$$

and

$$\sum_{k=1}^{\infty} \frac{(-4)^k(84(1121-338\sqrt{11}))^{4k}(5980ak-b)}{k^3\binom{2k}{k}^2\binom{4k}{2k}}$$
$$= 36120^2\left(36G-11\sqrt{11}L_{-11}(2)\right), \tag{6.9}$$

where

$$a = 11092\sqrt{11}+19437 \quad and \quad b = 11937508\sqrt{11}+37515813.$$

Also,

$$\sum_{k=1}^{\infty} \frac{(-4)^k (12(41 - 10\sqrt{17}))^{4k}}{k^3 \binom{2k}{k}^2 \binom{4k}{2k}} \left(140k(92\sqrt{17} + 2091) - 6868\sqrt{17} - 36591 \right)$$
$$= 415812(17\sqrt{17}L_{-68}(2) - 90G). \tag{6.10}$$

Remark 6.2. (6.8) is the dual of the conjectural identity

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(8(457 + 325\sqrt{2}))^k} \left(5k(32 - 13\sqrt{2}) + 3(11 - 6\sqrt{2}) \right) = \frac{49}{2\pi}$$

given by the first row of [1, Table 3.3], and (6.9) is the dual of the Ramanujan-type identity

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 \binom{4k}{2k}(5980a'k + b')}{(-4)^k ((84(1121 + 338\sqrt{11}))^{4k}} = 1806^2 \frac{2}{\pi}$$

with $a' = 11092\sqrt{11} - 19437$ and $b' = 11937508\sqrt{11} - 37515813$ listed in the last row of [1, Table 3.5] and first proved in [3]. Also, (6.10) is the dual of the conjectural identity

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 \binom{4k}{2k}(140k(2091 - 92\sqrt{17}) + 36591 - 6868\sqrt{17})}{(-4)^k (12(41 + 10\sqrt{17}))^{4k}} = \frac{25992}{\pi}$$

given by the 7th row of [1, Table 3.5].

REFERENCES

[1] A. M. Aldawoud, *Ramanujan-type series for* $1/\pi$ *with quadratic irrationals*, Master Thesis, Massey Univ., New Zealand, 2012.

[2] G. Almkvist, C. Krattenthaler and J. Petersson, *Some new series for* π, Experiment. Math. **12** (2003), 441–456.

[3] J. M. Borwein and P. B. Borwein, *More Ramanujan-type series for* $1/\pi$, in: Ramanujan Revisited (Urbana-champaign, III, 1987), pp. 359–374, Acdemic Press, Boston, 1988.

[4] F. Calegari, V. Dimitrov and Y. Tang, *The linear independence of* 1, $\zeta(2)$ *and* $L(2, \chi_{-3})$, preprint, arXiv:2408.15403, 2024.

[5] J. M. Campbell, *Proofs of conjectures on Ramanujan-type series of level* 3, Acta Math. Sci. **45** (2025), 1482–1496.

[6] W. Chu and W. Zhang, *Accelerating Dougall's* $_5F_4$*-sum and infinite series involving* π, Math. Comp. **83** (2014), 475–512.

[7] S. Cooper, Ramanujan's Theta Functions, Springer, Cham, 2017.

[8] J. Guillera, *Hypergeometric identities for 10 extended Ramanujan-type series*, Ramanujan J. **15** (2008), 219–234.

[9] J. Guillera, *WZ-proofs of "divergent" Ramanujan-type series*, in: Advances in Combinatorics (eds., I. Kotsireas and E. Zima), Springer, 2013, 187–195.

[10] J. Guillera and M. Rogers, *Ramanujan series upside-down*, J. Austral. Math. Soc. **97** (2014), 78–106.

[11] J. Guillera, *When Ramanujan meets Fibonacci and Lucas*, Amer. Math. Monthly 127 (2020), 159.

[12] V.J.W. Guo and X. Lian, *Some q-congruences on double basic hypergeometric sums*, J. Difference Equ. Appl. **27** (2021), 453–461.

[13] Kh. Hessami Pilehrood and T. Hessami Pilehrood, *Bivariate identities for values of the Hurwitz zeta function and supercongruences*, Electron. J. Combin. **18** (2012), #P35, 30pp.

[14] S. Ramanujan, *Modular equations and approximations to π*, Quart. J. Math. (Oxford) (2) **45** (1914), 350–372.

[15] A. Straub, *A gentle introduction to PSLQ*, 2010. Available from the website https://arminstraub.com/downloads/math/pslq.pdf.

[16] Z.-W. Sun, *Super congruences and Euler numbers*, Sci. China Math. **54** (2011), 2509–2535.

[17] Z.-W. Sun, *New series for some special values of L-functions*, Nanjing Univ. J. Math. Biquarterly **32** (2015), 189–218.

[18] Z.-W. Sun, *Infinite series involving binomial coefficients and harmonic numbers*, Sci. Sinica Math. **54** (2024), 765–774.

[19] Z.-W. Sun, *New series involving binomial coefficients (I)*, Nanjing Univ. J. Math. Biquarterly **41** (2024), 57–95.

[20] Z.-W. Sun, *Series with summands involving harmonic numbers*, in: M. B. Nathanson (ed.), Combinatorial and Additive Number Theory VI, Springer, Cham, 2025, pp. 413–460.

[21] Z.-W. Sun, *New series involving binomial coefficients (II)*, Acta Math. Sin. (Engl. Ser.) **42** (2026), to appear. See also arXiv:2307.03086.

[22] C. Wei, *On two double series for π and their q-analogues*, Ramanujan J. **60** (2023), 615–625.

[23] C. Wei and G. Ruan, *Some double series for π and their q-ananlogues*, J. Math. Anal. Appl. **537** (2024), Article ID 128309.

[24] D. Zeilberger, *Closed form (pun intended!)*, Contemporary Math. **143** (1993), 579–607.

SCHOOL OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, PEOPLE'S REPUBLIC OF CHINA

E-mail address: `zwsun@nju.edu.cn`

Homepage: `http://maths.nju.edu.cn/~zwsun`