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PROOF OF A CONJECTURAL SUPERCONGRUENCE MODULO p^5

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ABSTRACT. In this paper, we prove the supercongruence

$$\sum_{n=0}^{(p-1)/2}\frac{6n+1}{256^n}\binom{2n}{n}^3\equiv p(-1)^{(p-1)/2}+(-1)^{(p-1)/2}\frac{7}{24}p^4B_{p-3}\ (\mathrm{mod}\ p^5)$$

for any prime $p>3$, where $B_0,B_1,\ldots$ are the Bernoulli numbers. This confirms a conjecture posed by Z.-W. Sun in 2019.

1. INTRODUCTION

In 1997, L. van Hamme [25] proposed many conjectural p-adic supercongruences motivated by corresponding Ramanujan-type series for $1/\pi$. For example, he conjectured the supercongruence

$$\sum_{n=0}^{(p-1)/2}\frac{6n+1}{256^n}\binom{2n}{n}^3\equiv(-1)^{(p-1)/2}p\ (\mathrm{mod}\ p^4)\tag{1.1}$$

for any prime $p>3$, inspired by the Ramanujan series (cf. [19])

$$\sum_{n=0}^{\infty}\frac{6n+1}{256^n}\binom{2n}{n}^3=\frac{4}{\pi}.$$

The congruence (1.1) was confirmed by L. Long [12] in 2011.

In 2011, Z.-W. Sun [21] formulated many conjectural supercongruences involving Bernoulli numbers or Euler numbers. Recall that the Bernoulli numbers $B_0,B_1,\ldots$ and the Euler numbers $E_0,E_1,\ldots$ are defined by

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}\ (|x|<2\pi)\ \text{and}\ \frac{2}{e^x+e^{-x}}=\sum_{n=0}^{\infty}E_n\frac{x^n}{n!}\ \left(|x|<\frac{\pi}{2}\right)$$

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respectively. For example, for any prime $p>3$, he conjectured that

$$\sum_{n=0}^{p-1}\frac{6n+1}{256^n}\binom{2n}{n}^3\equiv(-1)^{(p-1)/2}p-p^3E_{p-3}\pmod{p^4}. \tag{1.2}$$

This was later confirmed by G.-S. Mao and C.-W. Wen [15, Th. 1.2].

In 2019, Z.-W. Sun [23, Conj. 22] conjectured that: for any prime $p>3$ and positive odd integer m, we have

$$\frac{16^{m-1}}{(pm)^4\binom{m-1}{\frac{m-1}{2}}^3}\left(\sum_{n=0}^{(pm-1)/2}\frac{6n+1}{256^n}\binom{2n}{n}^3-(-1)^{(p-1)/2}p\sum_{r=0}^{(m-1)/2}\frac{6r+1}{256^r}\binom{2r}{r}^3\right)$$
$$\equiv(-1)^{(p-1)/2}\frac{7}{24}B_{p-3}\pmod{p}.$$

In this paper, we confirm this in the case $m=1$. Namely, we establish the following result.

Theorem 1.1. *Let $p>3$ be a prime. Then*

$$\sum_{n=0}^{(p-1)/2}\frac{6n+1}{256^n}\binom{2n}{n}^3\equiv(-1)^{(p-1)/2}\left(p+\frac{7}{24}p^4B_{p-3}\right)\pmod{p^5}. \tag{1.3}$$

For any prime $p>3$, Z.-W. Sun [21] also conjectured the congruence

$$\sum_{n=0}^{p-1}\frac{3n+1}{16^n}\binom{2n}{n}^3\equiv p+\frac{7}{6}p^4\pmod{p^5},$$

which was confirmed by C. Wang and D.-W. Hu [26] in 2020. For more studies of such congruences, one may consult [8, 10, 11, 18].

In the next section, we provide some known lemmas. We will use the WZ method to prove Theorem 1.1 in Section 3.

2. Some known lemmas

In 1862, J. Wolstenholme [27] proved the classical congruence

$$\binom{2p-1}{p-1}\equiv 1\pmod{p^3}$$

for any prime $p>3$. This was refined by J.W.L. Glaisher [3] in 1900.

Lemma 2.1 (Glaisher [3])**.** *For any prime $p>3$, we have*

$$\binom{2p-1}{p-1}\equiv 1-\frac{2}{3}p^3B_{p-3}\pmod{p^4}. \tag{2.1}$$

Remark 2.2. For modern references about (2.1), the reader may consult [5] and [16].

In 1895, F. Morley [17] got the following fundamental congruence:

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3}$$

for any prime $p > 3$. This was refined by L. Carlitz [1] in 1953.

Lemma 2.3 (L. Carlitz [1]). *For each odd prime p, we have*

$$(-1)^{(p-1)/2} \binom{p-1}{(p-1)/2} \equiv 4^{p-1} + \frac{p^3}{12} B_{p-3} \pmod{p^4}.$$

We also need the following result of E. Lehmer established in 1938.

Lemma 2.4 (E. Lehmer [9]). *For any prime $p > 3$, we have*

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k} \equiv -2 \sum_{k=1}^{(p-1)/2} \frac{1}{2k-1} \equiv -2q_p(2) + p\, q_p(2)^2 \pmod{p^2}, \qquad (2.2)$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

Let $a_1, a_2, \ldots, a_m$ be integers. For any integer $n \geq m$, we define the alternating multiple harmonic sum

$$H(a_1, a_2, \ldots, a_m; n) := \sum_{1 \leq k_1 < k_2 < \ldots < k_m \leq n} \prod_{i=1}^{m} \frac{\operatorname{sign}(a_i)^{k_i}}{k_i^{|a_i|}},$$

and call m and $\sum_{i=1}^{m} |a_i|$ its *depth* and *weight* respectively.

We need the following known results as lemmas.

Lemma 2.5 ([7]). *Let $a, r \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. For any prime $p > ar + 2$, we have*

$$H(\{a\}^r; p-1) \equiv \begin{cases} (-1)^r \frac{a(ar+1)}{2(ar+2)} p^2 B_{p-ar-2} \pmod{p^3} & \text{if } ar \text{ is odd,} \\ (-1)^{r-1} \frac{a}{ar+1} p B_{p-ar-1} \pmod{p^2} & \text{if } ar \text{ is even.} \end{cases}$$

Lemma 2.6 ([20]). *For any $a \in \mathbb{Z}^+$ and prime $p > a + 2$, we have*

$$H\left(a; \frac{p-1}{2}\right) \equiv \begin{cases} -2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2 q_p(2)^3 - \frac{7}{12}p^2 B_{p-3} \pmod{p^3} & \text{if } a = 1, \\ -\frac{2^a - 2}{a} B_{p-a} \pmod{p} & \text{if } a > 1 \text{ is odd,} \\ \frac{a(2^{a+1}-1)}{2(a+1)} p B_{p-a-1} \pmod{p^2} & \text{if } a \text{ is even.} \end{cases}$$

Lemma 2.7 ([6]). *Let $a, b \in \mathbb{Z}^+$ with $a+b$ odd. For any prime $p > a+b$, we have*

$$H\left(a, b; \frac{p-1}{2}\right) \equiv \frac{B_{p-a-b}}{2(a+b)} \left((-1)^b \binom{a+b}{a} + 2^{a+b} - 2\right) \pmod{p}.$$

Lemma 2.8 (R. Tauraso and J. Q. Zhao [24])**.** *For any prime $p>3$, we have*

$$H(1,-1;p-1)\equiv q_p(2)^2-p\,q_p(2)^3-\frac{13}{24}pB_{p-3}\pmod{p^2}. \tag{2.3}$$

We also need the following lemma involving the harmonic numbers

$$H_n:=H(1;n)=\sum_{0<k\le n}\frac{1}{k}\quad (n=0,1,2,\ldots)$$

and the second order harmonic numbers

$$H(2;n)=\sum_{0<k\le n}\frac{1}{k^2}\quad (n=0,1,2,\ldots).$$

Lemma 2.9. *Let $p>3$ be a prime. Then we have*

$$\sum_{k=1}^{(p-1)/2}\frac{H_k}{k^2}\equiv-\frac{B_{p-3}}{2}\pmod{p}, \tag{2.4}$$

$$\sum_{k=1}^{(p-1)/2}\frac{H_{2k}}{k^2}\equiv\frac{3}{2}B_{p-3}\pmod{p}, \tag{2.5}$$

$$\sum_{k=1}^{(p-1)/2}\frac{H_{2k}^2}{k}\equiv-\frac{2}{3}q_p(2)^3+\frac{2}{3}B_{p-3}\pmod{p}, \tag{2.6}$$

$$\sum_{k=0}^{(p-3)/2}\frac{H_k^2}{2k+1}\equiv\frac{B_{p-3}}{4}+4q_p(2)^3\pmod{p}, \tag{2.7}$$

$$\sum_{k=0}^{(p-1)/2}\frac{H_kH_{2k}}{k}\equiv\frac{5}{6}B_{p-3}-\frac{4}{3}q_p(2)^3\pmod{p}. \tag{2.8}$$

Also,

$$\sum_{k=1}^{(p-1)/2}\frac{H(2:k)}{k}\equiv-\frac{3}{2}B_{p-3}\pmod{p}, \tag{2.9}$$

$$\sum_{k=1}^{(p-1)/2}\frac{H(2;2k)}{k}\equiv-\frac{5}{4}B_{p-3}\pmod{p}. \tag{2.10}$$

Remark 2.10. The congruences (2.4) and (2.5) can be found in [13, Lemma 3.2] and [14, Lemma 2.4], respectively. For (2.6)–(2.8), the reader may consult [14, (3.12) and Theorem 1.3]. (2.9) and (2.10) can be found in [14, (2.2)-(2.3)].

3. Proof of Theorem 1.1

To prove Theorem 1.1, we need a WZ pair found by J. Guillera [4, p. 42]. For $n, k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, we define

$$F(n,k) = \frac{(6n-2k+1)}{2^{8n-2k}} \frac{\binom{2n}{n}\binom{2n+2k}{n+k}\binom{2n-2k}{n-k}\binom{n+k}{n}}{\binom{2k}{k}} \tag{3.1}$$

and

$$G(n,k) = \frac{n^2\binom{2n}{n}\binom{2n+2k}{n+k}\binom{2n-2k}{n-k}\binom{n+k}{n}}{2^{8n-2k-4}(2n+2k-1)\binom{2k}{k}}. \tag{3.2}$$

Clearly $F(n,k) = G(n,k) = 0$ if $n < k$. It is easy to check that

$$F(n,k-1) - F(n,k) = G(n+1,k) - G(n,k) \tag{3.3}$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}^+$. The WZ pair $\langle F, G\rangle$ first appeared in [4, p. 42], and it was also used in [2] and [28, p. 9].

Summing (3.3) over $n \in \{0, \ldots, (p-1)/2\}$, we get

$$\sum_{n=0}^{(p-1)/2} F(n,k-1) - \sum_{n=0}^{(p-1)/2} F(n,k) = G\left(\frac{p+1}{2}, k\right) - G(0,k) = G\left(\frac{p+1}{2}, k\right).$$

Furthermore, summing both side of the above identity over $k \in \{1, \ldots, (p-1)/2\}$, we obtain

$$\sum_{n=0}^{(p-1)/2} F(n,0) = F\left(\frac{p-1}{2}, \frac{p-1}{2}\right) + \sum_{k=1}^{(p-1)/2} G\left(\frac{p+1}{2}, k\right). \tag{3.4}$$

Lemma 3.1. *Let $p > 3$ be a prime. Then*

$$F\left(\frac{p-1}{2}, \frac{p-1}{2}\right)$$
$$\equiv (-1)^{\frac{p-1}{2}} p\left(1 - pq_p(2) + p^2 q_p(2)^2 - p^3 q_p(2)^3 - \frac{7}{12}p^3 B_{p-3}\right) \pmod{p^5}.$$

Proof. By the definition of $F(n,k)$, we have

$$F\left(\frac{p-1}{2}, \frac{p-1}{2}\right) = \frac{2p-1}{2^{3p-3}}\binom{2p-2}{p-1}\binom{p-1}{(p-1)/2} = \frac{p\binom{2p-1}{p-1}\binom{p-1}{(p-1)/2}}{2^{3p-3}}.$$

This, together with Lemma 2.1, Lemma 2.3 and the equality $2^{p-1} = 1 + pq_p(2)$, yields that

$$F\left(\frac{p-1}{2}, \frac{p-1}{2}\right) \equiv \frac{p(1 - \frac{2}{3}p^3 B_{p-3})(-1)^{(p-1)/2}(4^{p-1} + \frac{1}{12}p^3 B_{p-3})}{(1 + p\, q_p(2))^3}$$
$$\equiv (-1)^{\frac{p-1}{2}} p\left(1 - pq_p(2) + p^2 q_p(2)^2 - p^3 q_p(2)^3 - \frac{7}{12}p^3 B_{p-3}\right) \pmod{p^5}.$$

This concludes the proof. □

Lemma 3.2. *For any prime $p>3$, we have*

$$\sum_{k=1}^{(p-1)/2}\frac{p/2-k}{(p+1-2k)(p+2k)}$$
$$\equiv \frac{1}{2}q_p(2)-\frac{p}{4}q_p(2)^2-2pq_p(2)+\frac{1}{6}p^2q_p(2)^3$$
$$+4p^2q_p(2)+p^2q_p(2)^2+\frac{7}{48}p^2B_{p-3} \pmod{p^3}.$$

Proof. In view of Lemma 2.4,

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k(2k-1)}=2\sum_{k=1}^{(p-1)/2}\frac{1}{2k-1}-H_{(p-1)/2}$$
$$\equiv 4q_p(2)-2pq_p(2)^2 \pmod{p^2} \tag{3.5}$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k(2k-1)^2}=H_{(p-1)/2}-2\sum_{k=1}^{(p-1)/2}\frac{1}{2k-1}+2\sum_{k=1}^{(p-1)/2}\frac{1}{(2k-1)^2}$$
$$\equiv -4q_p(2)-\frac{1}{2}H(2;(p-1)/2)\equiv -4q_p(2) \pmod{p}. \tag{3.6}$$

It is easy to see that

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)}{(p+1-2k)(p+2k)}$$
$$=\sum_{k=1}^{(p-1)/2}\frac{k-1/2}{(2k)(2p+1-2k)}=-\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{1}{k(1-\frac{2p}{2k-1})}$$
$$\equiv -\frac{1}{4}H_{(p-1)/2}-\frac{p}{2}\sum_{k=1}^{(p-1)/2}\frac{1}{k(2k-1)}-p^2\sum_{k=1}^{(p-1)/2}\frac{1}{k(2k-1)^2} \pmod{p^3}.$$

Then we immediately obtain the desired result by Lemma 2.6, (3.5) and (3.6). □

Lemma 3.3. *For any prime $p>3$, we have*

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H_k}{(p+1-2k)(p+2k)}$$
$$\equiv 2q_p(2)-q_p(2)^2-6pq_p(2)+2pq_p(2)^2+pq_p(2)^3+\frac{7}{12}pB_{p-3} \pmod{p^2}.$$

Proof. By Lemmas 2.5 and 2.6, and (2.3), we have

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{H_{2k}}{k}&=\sum_{k=1}^{p-1}\frac{(1+(-1)^k)H_k}{k}\\&=H(1,1;p-1)+H(1,-1;p-1)+\frac{1}{2}H\left(2;\frac{p-1}{2}\right)\\&\equiv q_p(2)^2-pq_p(2)^3+\frac{7}{24}pB_{p-3}\pmod{p^2}.\end{aligned}\tag{3.7}$$

Noting $2H(1,1;n)=H_n^2-H(2;n)$, we get

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{H_k}{k}&=H(1,1;(p-1)/2)+H(2;(p-1)/2)\\&=\frac{1}{2}H_{(p-1)/2}^2+\frac{1}{2}H(2;(p-1)/2)\\&\equiv 2q_p(2)^2-2pq_p(2)^3+\frac{7}{6}pB_{p-3}\pmod{p^2}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}H_{(p+1)/2-k}\equiv&\frac{2}{p+1-2k}+2pH(2;2k)-\frac{p}{2}H(2;k)\\&+H_{(p-1)/2}+2H_{2k}-H_k\pmod{p^2}.\end{aligned}$$

This, together with (3.5)-(3.8), (2.9) and (2.10), yields that

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{H_{(p+1)/2-k}}{k}\equiv&-8q_p(2)+4q_p(2)^2+4pq_p(2)^2+8pq_p(2)\\&-4pq_p(2)^3-\frac{7}{3}pB_{p-3}\pmod{p^2}\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{H_{(p+1)/2-k}}{k(2k-1)}&\equiv\sum_{k=1}^{(p-1)/2}\frac{H_k}{k(2k-1)}=2\sum_{k=1}^{(p-1)/2}\frac{H_k}{2k-1}-\sum_{k=1}^{(p-1)/2}\frac{H_k}{k}\\&\equiv-\sum_{k=1}^{(p-1)/2}\frac{H_{(p+1)/2-k}}{k}-\sum_{k=1}^{(p-1)/2}\frac{H_k}{k}\\&\equiv 8q_p(2)-6q_p(2)^2\pmod{p}.\end{aligned}\tag{3.9}$$

Since

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H_k}{(p+1-2k)(p+2k)}\\=&\sum_{k=1}^{(p-1)/2}\frac{(k-1/2)H_{(p+1)/2-k}}{(2k)(2p+1-2k)}=-\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{H_{(p+1)/2-k}}{k(1-\frac{2p}{2k-1})}\\\equiv&-\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{H_{(p+1)/2-k}}{k}-\frac{p}{2}\sum_{k=1}^{(p-1)/2}\frac{H_{(p+1)/2-k}}{k(2k-1)}\pmod{p^2},\end{aligned}$$

we immediately get the desired result by using (3.8) and (3.9). □

Lemma 3.4. *Let $p>3$ be a prime. Then*

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H_{2k}}{(p+1-2k)(p+2k)}\\&\equiv q_p(2)-\frac{1}{4}q_p(2)^2-3pq_p(2)+\frac{p}{2}q_p(2)^2+\frac{p}{4}q_p(2)^3+\frac{13}{32}pB_{p-3}\pmod{p^2}.\end{aligned}$$

Proof. It is easy to check that for each $0\le k\le p-1$, we have

$$H_{p-1-k}\equiv pH(2;k)+H_k\pmod{p^2},$$

hence

$$H_{p+1-2k}\equiv pH(2;2k-2)+H_{2k-2}\pmod{p^2}\tag{3.10}$$

for each $1\le k\le (p-1)/2$. So

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\frac{H_{p+1-2k}}{k}\\&\equiv p\sum_{k=1}^{(p-1)/2}\frac{H(2;2k-2)}{k}+\sum_{k=1}^{(p-1)/2}\frac{H_{2k-2}}{k}\\&=p\left(\sum_{k=1}^{(p-1)/2}\frac{H(2;2k)}{k}-\sum_{k=1}^{(p-1)/2}\frac{1}{k(2k-1)^2}-\frac{1}{4}H\left(3;\frac{p-1}{2}\right)\right)\\&\quad+\sum_{k=1}^{(p-1)/2}\frac{H_{2k}}{k}-\sum_{k=1}^{(p-1)/2}\frac{1}{k(2k-1)}-\frac{1}{2}H\left(2;\frac{p-1}{2}\right)\pmod{p^2}.\end{aligned}$$

Combining the result in the last paragraph with Lemma 2.6, (3.7), (3.5), (3.6) and (2.10), we get

$$\sum_{k=1}^{(p-1)/2}\frac{H_{p+1-2k}}{k}\equiv q_p(2)^2-4q_p(2)+4pq_p(2)+2pq_p(2)^2-pq_p(2)^3-\frac{13}{8}pB_{p-3}\pmod{p^2}.$$

Similarly, by using (3.10) and Lemma 2.4, we get

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{H_{p+1-2k}}{k(2k-1)}&\equiv\sum_{k=1}^{(p-1)/2}\frac{H_{2k-2}}{k(2k-1)}\\&=2\sum_{k=1}^{(p-1)/2}\frac{H_{2k}}{2k-1}-\sum_{k=1}^{(p-1)/2}\frac{H_{2k}}{k}\\&\quad-\sum_{k=1}^{(p-1)/2}\frac{1}{k(2k-1)^2}-\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{1}{k^2(2k-1)}\\&\equiv-\sum_{k=1}^{(p-1)/2}\frac{H_{p+1-2k}}{k}-\sum_{k=1}^{(p-1)/2}\frac{H_{2k}}{k}-\sum_{k=1}^{(p-1)/2}\frac{1}{k(2k-1)^2}\\&\quad-\frac{1}{2}\sum_{k=1}^{(p-1)/2}\left(\frac{4}{2k-1}-\frac{2}{k}-\frac{1}{k^2}\right)\\&\equiv4q_p(2)-2q_p(2)^2\pmod{p}.\end{aligned}\tag{3.11}$$

Since

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H_{2k}}{(p+1-2k)(p+2k)}\equiv-\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{H_{p+1-2k}}{k}-\frac{p}{2}\sum_{k=1}^{(p-1)/2}\frac{H_{p+1-2k}}{k(2k-1)}\pmod{p^2},$$

we obtain the desired result in view of the last paragraph. □

Lemma 3.5. *For any prime $p>3$, we have*

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H_k^2}{(p+1-2k)(p+2k)}\equiv4q_p(2)-6q_p(2)^2+2q_p(2)^3+\frac{1}{8}B_{p-3}\pmod{p}.$$

Proof. It is easy to verify that

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H_k^2}{(p+1-2k)(p+2k)}\equiv\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{H_k^2}{2k-1}=\frac{1}{2}\sum_{k=0}^{(p-3)/2}\frac{H_{k+1}^2}{2k+1}$$
$$=\frac{1}{2}\sum_{k=0}^{(p-3)/2}\frac{H_k^2}{2k+1}+\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{1}{(2k-1)k^2}+\sum_{k=1}^{(p-1)/2}\frac{H_{k-1}}{k(2k-1)} \pmod p.$$

Observe that

$$\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{1}{(2k-1)k^2}=2\sum_{k=1}^{(p-1)/2}\frac{1}{2k-1}-H_{(p-1)/2}-\frac{1}{2}H\left(2;\frac{p-1}{2}\right)$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{H_{k-1}}{k(2k-1)}$$
$$=2\sum_{k=1}^{(p-1)/2}\frac{H_{k-1}}{2k-1}-\sum_{k=1}^{(p-1)/2}\frac{H_{k-1}}{k}$$
$$=2\sum_{k=1}^{(p-1)/2}\frac{H_k}{2k-1}-2\sum_{k=1}^{(p-1)/2}\frac{1}{k(2k-1)}-\sum_{k=1}^{(p-1)/2}\frac{H_{k-1}}{k}$$
$$\equiv-\sum_{k=1}^{(p-1)/2}\frac{H_{(p+1)/2-k}}{k}-2\sum_{k=1}^{(p-1)/2}\frac{1}{k(2k-1)}-\sum_{k=1}^{(p-1)/2}\frac{H_{k-1}}{k} \pmod p.$$

This, together with (2.2), Lemma 2.6, (3.5), (3.8) and (2.7), yields the desired result. □

Lemma 3.6. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H_kH_{2k}}{(p+1-2k)(p+2k)}\equiv 2q_p(2)-\frac{5}{2}q_p(2)^2+\frac{1}{2}q_p(2)^3+\frac{5}{16}B_{p-3} \pmod p.$$

Proof. By (2.5), (2.6), (3.6) and (3.11), we have

$$\sum_{k=1}^{(p-1)/2}\frac{H_{2k}H_{2k-2}}{k}=\sum_{k=1}^{(p-1)/2}\frac{H_{2k}^2}{k}-\sum_{k=1}^{(p-1)/2}\frac{H_{2k}}{k(2k-1)}-\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{H_{2k}}{k^2}$$
$$\equiv-4q_p(2)+2q_p(2)^2-\frac{2}{3}q_p(2)^3-\frac{1}{12}B_{p-3} \pmod p. \tag{3.12}$$

In view of (2.4), (2.8) and (3.9), we have

$$\sum_{k=1}^{(p-1)/2}\frac{H_kH_{2k-2}}{k}=\sum_{k=1}^{(p-1)/2}\frac{H_{2k}H_k}{k}-\sum_{k=1}^{(p-1)/2}\frac{H_k}{k(2k-1)}-\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{H_k}{k^2}$$
$$\equiv -8q_p(2)+6q_p(2)^2-\frac{4}{3}q_p(2)^3+\frac{13}{12}B_{p-3}\pmod p. \tag{3.13}$$

For each $1\le k\le (p-1)/2$, we clearly have

$$H_{(p+1)/2-k}\equiv H_{(p-1)/2}+2H_{2k-2}-H_{k-1}\pmod p.$$

This, together with (3.10), yields that

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H_kH_{2k}}{(p+1-2k)(p+2k)}$$
$$\equiv\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{H_kH_{2k}}{2k-1}\equiv-\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{H_{(p+1)/2-k}H_{p+1-2k}}{k}$$
$$\equiv\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{H_{2k-2}}{k(2k-1)}-\frac{1}{4}H_{(p-1)/2}\sum_{k=1}^{(p-1)/2}\frac{H_{2k-2}}{k}$$
$$-\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{H_{2k}H_{2k-2}}{k}+\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{H_kH_{2k-2}}{k}$$

modulo p holds. Combining this with (3.12), (3.13), (3.11), (3.7), (3.5) and Lemma 2.6, we immediately get the desired result. □

Lemma 3.7. *For any prime $p>3$, we have*

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H_{2k}^2}{(p+1-2k)(p+2k)}\equiv q_p(2)-q_p(2)^2+\frac{1}{6}q_p(2)^3+\frac{1}{3}B_{p-3}\pmod p.$$

Proof. Replacing k in (3.6) by $(p+1)/2-j$, we have

$$\sum_{j=1}^{(p-1)/2}\frac{1}{(2j-1)j^2}\equiv 8q_p(2)\pmod p.$$

In view of (2.5) and Lemma 2.6, we can deduce that

$$\sum_{k=1}^{(p-1)/2}\frac{H_{2k-2}}{k^2}\equiv-8q_p(2)+\frac{5}{2}B_{p-3}\pmod p.$$

This, together with (3.10), (3.11) and (3.12), yields that

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H_{2k}^2}{(p+1-2k)(p+2k)}\\ \equiv&\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{H_{2k}^2}{2k-1}\equiv-\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{H_{p+1-2k}^2}{k}\equiv-\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{H_{2k-2}^2}{k}\\ \equiv&-\frac{1}{4}\bigg(\sum_{k=1}^{(p-1)/2}\frac{H_{2k}H_{2k-2}}{k}-\sum_{k=1}^{(p-1)/2}\frac{H_{2k-2}}{k(2k-1)}-\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{H_{2k-2}}{k^2}\bigg)\\ \equiv&\, q_p(2)-q_p(2)^2+\frac{1}{6}q_p(2)^3+\frac{1}{3}B_{p-3}\pmod p.\end{aligned}$$

This ends the proof. □

Lemma 3.8. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H(2;2k)}{(p+1-2k)(p+2k)}\equiv q_p(2)-\frac{3}{16}B_{p-3}\pmod p$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H(2;k)}{(p+1-2k)(p+2k)}\equiv 4q_p(2)-\frac{7}{8}B_{p-3}\pmod p.$$

Proof. In view of (2.10), (3.6) and Lemma 2.6, we have

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H(2;2k)}{(p+1-2k)(p+2k)}\\ \equiv&\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{H(2;2k)}{2k-1}\equiv-\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{H(2;p+1-2k)}{k}\equiv\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{H(2;2k-2)}{k}\\ =&\frac{1}{4}\bigg(\sum_{k=1}^{(p-1)/2}\frac{H(2;2k)}{k}-\sum_{k=1}^{(p-1)/2}\frac{1}{k(2k-1)^2}-\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{1}{k^3}\bigg)\\ \equiv&\, q_p(2)-\frac{3}{16}B_{p-3}\pmod p.\end{aligned}\tag{3.14}$$

By Lemmas 2.5 and 2.6, we have

$$H(2;(p+1)/2-k)=\sum_{j=1}^{(p+1)/2-k}\frac{1}{j^2}\equiv 4\sum_{j=k}^{(p-1)/2}\frac{1}{(2j-1)^2}$$
$$=4\left(H(2;p-1)-\frac{1}{4}H\left(2;\frac{p-1}{2}\right)-H(2;2k-2)+\frac{1}{4}H(2;k-1)\right)$$
$$\equiv H(2;k-1)-4H(2;2k-2) \pmod p.$$

Hence,

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H(2;k)}{(p+1-2k)(p+2k)}$$
$$\equiv\frac{1}{2}\sum_{k=1}^{(p-1)/2}\frac{H(2;(p+1)/2-k)}{p-2k}$$
$$\equiv\sum_{k=1}^{(p-1)/2}\frac{H(2;2k-2)}{k}-\frac{1}{4}\sum_{k=1}^{(p-1)/2}\frac{H(2;k-1)}{k} \pmod p.$$

In view of (3.14), we have

$$\sum_{k=1}^{(p-1)/2}\frac{H(2;2k-2)}{k}\equiv 4q_p(2)-\frac{3}{4}B_{p-3} \pmod p.$$

With the aid of Lemma 2.7, we get

$$\sum_{k=1}^{(p-1)/2}\frac{H(2;k-1)}{k}=H(2,1;(p-1)/2)\equiv\frac{1}{2}B_{p-3} \pmod p.$$

So

$$\sum_{k=1}^{(p-1)/2}\frac{(p/2-k)H(2;k)}{(p+1-2k)(p+2k)}\equiv 4q_p(2)-\frac{7}{8}B_{p-3} \pmod p.$$

Therefore the proof of Lemma 3.8 is complete. □

Lemma 3.9. *For any primes $p>3$, we have*

$$\sum_{k=1}^{(p-1)/2}G\left(\frac{p+1}{2},k\right)$$
$$\equiv(-1)^{(p-1)/2}p^2\left(q_p(2)-pq_p(2)^2+p^2q_p(2)^3+\frac{7}{8}p^2B_{p-3}\right) \pmod{p^5}.$$

Proof. For any complex number a, let $(a)_0 = 1$ and $(a)_n = a(a+1)\ldots(a+n-1)$ for $n\in\mathbb{Z}^+$. By the definition of $G(n,k)$, we have

$$\begin{aligned}G(n,k)&=\frac{n^2\binom{2n}{n}\binom{2n+2k}{n+k}\binom{2n-2k}{n-k}\binom{n+k}{n}}{2^{8n-4-2k}(2n+2k-1)\binom{2k}{k}}=\frac{n^2\binom{2n}{n}\left(\frac12\right)_{n+k}\left(\frac12\right)_{n-k}\binom{n}{k}}{2^{4n-4-2k}n!^2(2n+2k-1)\binom{2k}{k}}\\&=\frac{n^2\binom{2n}{n}\left(\frac12\right)_n\left(\frac12\right)_{n-1}\left(\frac12+n\right)_k\binom{n}{k}}{2^{4n-4-2k}n!^2\left(\frac12+n-k\right)_{k-1}(2n+2k-1)\binom{2k}{k}}\\&=\frac{n\binom{2n}{n}^2\binom{2n-2}{n-1}\left(\frac12+n\right)_k\binom{n}{k}}{2^{8n-6-2k}\left(\frac12+n-k\right)_{k-1}(2n+2k-1)\binom{2k}{k}},\end{aligned}\tag{3.15}$$

where we have used the equalities

$$\frac{\left(\frac12\right)_{n+k}}{(n+k)!}=\frac{\binom{2n+2k}{n+k}}{4^{n+k}},\quad \frac{\left(\frac12\right)_{n-k}}{(n-k)!}=\frac{\binom{2n-2k}{n-k}}{4^{n-k}},$$

$$\left(\frac12\right)_{n+k}=\left(\frac12\right)_n\left(\frac12+n\right)_k,\quad \frac{\left(\frac12\right)_n}{(n)!}=\frac{\binom{2n}{n}}{4^n}$$

and

$$\left(\frac12\right)_{n-k}\left(\frac12+n-k\right)_{k-1}=\left(\frac12\right)_{n-1}\quad (1\le k\le n).$$

It is easy to check that

$$\begin{aligned}\frac{\left(\frac p2+1\right)_k}{\left(\frac p2-k\right)_k}&\equiv\frac{k!\left(1+\frac p2H_k+\frac{p^2}{4}\sum_{1\le i<j\le k}\frac{1}{ij}\right)}{(-1)^kk!(\left(1-\frac p2H_k+\frac{p^2}{4}\sum_{1\le i<j\le k}\frac{1}{ij}\right)}\\&\equiv(-1)^k\left(1+pH_k+\frac{p^2}{2}H_k^2\right)\pmod{p^3}.\end{aligned}$$

In view of [22, (4.4)], $\binom{(p-1)/2}{k}(-4)^k/\binom{2k}{k}$ is congruent to

$$\begin{aligned}&1-p\sum_{j=1}^k\frac{1}{2j-1}+\frac{p^2}{2}\left(\left(\sum_{j=1}^k\frac{1}{2j-1}\right)^2-\sum_{j=1}^k\frac{1}{(2j-1)^2}\right)\\&=1-p\left(H_{2k}-\frac12H_k\right)+\frac{p^2}{2}\left(\left(H_{2k}-\frac12H_k\right)^2-H(2;2k)+\frac14H(2;k)\right)\end{aligned}$$

modulo p^3. By (3.15), we have

$$\begin{aligned}
&\sum_{k=1}^{\frac{p-1}{2}} G\left(\frac{p+1}{2},k\right)\\
&= \frac{(p+1)^2\binom{p+1}{(p+1)/2}^2\binom{p-1}{(p-1)/2}}{2^{4p-1}} \sum_{k=1}^{\frac{p-1}{2}} \frac{(p/2+1)_k\binom{(p-1)/2}{k}4^k(p/2-k)}{(p/2-k)_k\binom{2k}{k}(p+1-2k)(p+2k)}\\
&\equiv \frac{(p+1)^2\binom{p+1}{(p+1)/2}^2\binom{p-1}{(p-1)/2}}{2^{4p-1}} \sum_{k=1}^{(p-1)/2} \frac{(p/2-k)a_{p,k}}{(p+1-2k)(p+2k)} \pmod{p^5},
\end{aligned}$$

where $a_{p,k}$ denotes the expression

$$1+\frac{3p}{2}H_k-pH_{2k}+\frac{9p^2}{8}H_k^2-\frac{3p^2}{2}H_kH_{2k}+\frac{p^2}{2}H_{2k}^2-\frac{p^2}{2}\left(H(2;2k)-\frac{H(2;k)}{4}\right).$$

In light of Lemmas 3.2–3.8 and Lemma 2.3, we have

$$\begin{aligned}
&\sum_{k=1}^{(p-1)/2} G\left(\frac{p+1}{2},k\right)\\
&\equiv \frac{(p+1)^2\binom{p+1}{\frac{p+1}{2}}^2\binom{p-1}{\frac{p-1}{2}}}{2^{4p-1}}\left(\frac{1}{2}q_p(2)-\frac{3}{2}pq_p(2)^2+3p^2q_p(2)^3+\frac{7}{16}p^2B_{p-3}\right)\\
&\equiv \frac{2p^2\binom{p-1}{(p-1)/2}^3}{2^{4p-4}}\left(\frac{1}{2}q_p(2)-\frac{3}{2}pq_p(2)^2+3p^2q_p(2)^3+\frac{7}{16}p^2B_{p-3}\right)\\
&\equiv 2(-1)^{(p-1)/2}p^2 4^{p-1}\left(\frac{1}{2}q_p(2)-\frac{3}{2}pq_p(2)^2+3p^2q_p(2)^3+\frac{7}{16}p^2B_{p-3}\right)\\
&\qquad\qquad \pmod{p^5}.
\end{aligned}$$

Then we obtain the desired result by noting that

$$4^{p-1}=(1+p\,q_p(2))^2=1+2p\,q_p(2)+p^2q_p(2)^2.$$

This ends the proof. □

Proof of Theorem 1.1. Combining Lemmas 3.1 and 3.9 with (3.4), we immediately get

$$\sum_{n=0}^{(p-1)/2} F(n,0) \equiv p(-1)^{(p-1)/2}+(-1)^{(p-1)/2}\frac{7}{24}p^4B_{p-3} \pmod{p^5},$$

which is equivalent to our desired result.

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