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PROBLEMS AND RESULTS ON DETERMINANTS INVOLVING LEGENDRE SYMBOLS

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Abstract. In this paper we investigate determinants whose entries are linear combinations of Legendre symbols. We deduce some new results in this direction: for example, we prove that for any prime $p\equiv 3\pmod 4$ we have

$$\det\left[x+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)-\left(\frac{k}{p}\right)\right]_{0\leqslant j,k\leqslant (p-1)/2}=4,$$

where $(\frac{\cdot}{p})$ is the Legendre symbol. We also pose many conjectures for further research. For example, for any prime $p>3$ we conjecture that

$$\det\left[\left(\frac{j+k}{p}\right)+\left(\frac{j-k}{p}\right)+\left(\frac{jk}{p}\right)\right]_{1\leqslant j,k\leqslant (p-1)/2}$$
$$=\begin{cases}(\frac{2}{p})p^{(p-5)/4}&\text{if }p\equiv 1\pmod 4,\\(-1)^{(h(-p)-1)/2}(1-(2-(\frac{2}{p}))h(-p))p^{(p-3)/4}&\text{if }p\equiv 3\pmod 4,\end{cases}$$

where $h(-p)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

1. Introduction

Let p be an odd prime, and let $(\frac{\cdot}{p})$ denote the Legendre symbol. For any integer $a\not\equiv 0\pmod p$, by the quadratic Gauss sum formula we have

$$\sum_{k=0}^{p-1}e^{2\pi iak^2/p}=\left(\frac{a}{p}\right)\sqrt{(-1)^{(p-1)/2}p}.$$

Let ε_p and $h(p)$ be the fundamental unit and the class number of the real quadratic field $\mathbb{Q}(\sqrt{p})$. When $p\equiv 1\pmod 4$, by Dirichlet's class number formula we have

$$\prod_{m=1}^{p-1}(1-e^{2\pi im/p})^{(\frac{m}{p})}=\varepsilon_p^{-2h(p)},$$

which implies that

$$\prod_{k=1}^{(p-1)/2}\left(1-e^{2\pi iak^2/p}\right)=\sqrt{p}\,\varepsilon_p^{-(\frac{a}{p})h(p)}$$

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for each integer $a \not\equiv 0 \pmod p$ (see, e.g., [9, Theorem 1.3(i)]). For convenience, we write

$$\varepsilon_p^{h(p)} = a_p + b_p\sqrt{p} \ \text{ with } 2a_p, 2b_p \in \mathbb{Z}. \tag{1.1}$$

For a matrix $A = [a_{jk}]_{1\leqslant j,k\leqslant n}$ over a field, let $\det A$ or $|a_{jk}|_{1\leqslant j,k\leqslant n}$ denote its determinant. In this paper we focus on determinants involving Legendre symbols.

Let $p = 2n + 1$ be an odd prime. In 2004, R. Chapman [2] used quadratic Gauss sums and Dirichlet's class number formula to determine the determinants of the matrices

$$C_p(x) = \left[x + \left(\frac{j+k-1}{p}\right)\right]_{1\leqslant j,k\leqslant n} \text{ and } C_p^*(x) = \left[x + \left(\frac{j+k-1}{p}\right)\right]_{1\leqslant j,k\leqslant n+1}.$$

By [2, Corollary 3], provided $p > 3$ we have

$$\det C_p(x) = \begin{cases} (-1)^{n/2}2^n(b_p - a_p x) & \text{if } p \equiv 1 \pmod 4, \\ -2^n x & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1.2}$$

and

$$\det C_p^*(x) = \begin{cases} (-1)^{n/2}2^n(pb_p x - a_p) & \text{if } p \equiv 1 \pmod 4, \\ 2^n & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.3}$$

Since $(n + 1 - j) + (n + 1 - k) - 1 \equiv -j - k \pmod p$, we also have

$$\det C_p(x) = \left|x + \left(\frac{-j-k}{p}\right)\right|_{1\leqslant j,k\leqslant n} = (-1)^n\left|(-1)^n x + \left(\frac{j+k}{p}\right)\right|_{1\leqslant j,k\leqslant n} \tag{1.4}$$

and

$$\det C_p^*(x) = \left|x + \left(\frac{-j-k}{p}\right)\right|_{0\leqslant j,k\leqslant n} = \left|(-1)^n x + \left(\frac{j+k}{p}\right)\right|_{0\leqslant j,k\leqslant n}. \tag{1.5}$$

Let p be an odd prime, and write

$$\varepsilon_p^{(2-(\frac{2}{p}))h(p)} = a_p' + b_p'\sqrt{p} \ \text{ with } 2a_p', 2b_p' \in \mathbb{Z}. \tag{1.6}$$

In 2003, Chapman conjectured that

$$\left|\left(\frac{j-k}{p}\right)\right|_{0\leqslant j,k\leqslant (p-1)/2} = \begin{cases} -a_p' & \text{if } p \equiv 1 \pmod 4, \\ 1 & \text{if } p \equiv 3 \pmod 4; \end{cases}$$

this challenging conjecture was finally confirmed by M. Vsemirnov [11, 12] in 2012-2013 via matrix decomposition and quadratic Gauss sums. Recently, L.-Y. Wang, H.-L. Wu and H.-X. Ni [13] extended this as follows:

$$\left|x + \left(\frac{j-k}{p}\right)\right|_{0\leqslant j,k\leqslant (p-1)/2} = \begin{cases} (\frac{2}{p})pb_p' x - a_p' & \text{if } p \equiv 1 \pmod 4, \\ 1 & \text{if } p \equiv 3 \pmod 4, \end{cases} \tag{1.7}$$

which was ever conjectured by the author.

Let $p = 2n + 1$ be an odd prime, and let $d \in \mathbb{Z}$. The author [8] initiated the study of the determinants

$$S(d,p) = \left| \left(\frac{j^2 + dk^2}{p}\right) \right|_{1 \leqslant j,k \leqslant n} \text{ and } T(d,p) = \left| \left(\frac{j^2 + dk^2}{p}\right) \right|_{0 \leqslant j,k \leqslant n}.$$

He proved that

$$S(d,p) = \begin{cases} \frac{2}{p-1} T(d,p) & \text{if } (\frac{d}{p}) = 1, \\ 0 & \text{if } (\frac{d}{p}) = -1. \end{cases},$$

and

$$\left(\frac{T(d,p)}{p}\right) = \begin{cases} (\frac{2}{p}) & \text{if } (\frac{d}{p}) = 1, \\ 1 & \text{if } (\frac{d}{p}) = -1. \end{cases}$$

We first state a basic result.

Theorem 1.1. (i) *Let p be an odd prime, and let $m, n \in \mathbb{Z}$ with $n \geqslant m + 3$. Then, for any complex numbers a, b, c, d, we have*

$$\left| a + b\left(\frac{j}{p}\right) + c\left(\frac{k}{p}\right) + d\left(\frac{jk}{p}\right) \right|_{m \leqslant j,k \leqslant n} = 0. \tag{1.8}$$

(ii) *Let $p > 5$ be a prime with $p \equiv 1 \pmod 4$. For any $\delta \in \{\pm 1\}$ and $m \in \{0, 1\}$, we have*

$$\left| x + \left(\frac{j^2 + k^2}{p}\right) + \delta\left(\frac{j^2 - k^2}{p}\right) \right|_{m \leqslant j,k \leqslant (p-1)/2} = 0. \tag{1.9}$$

Remark 1.1. In 1956, D. H. Lehmer [6] found all the eigenvalues of the determinant

$$\left| a + b\left(\frac{j}{p}\right) + c\left(\frac{k}{p}\right) + d\left(\frac{jk}{p}\right) \right|_{1 \leqslant j,k \leqslant p-1},$$

where p is an odd prime and a, b, c, d are complex numbers. As a supplement to Theorem 1.1(ii), we conjecture that

$$\left| x + \left(\frac{j^2 + k^2}{p}\right) + \left(\frac{j^2 - k^2}{p}\right) \right|_{1 \leqslant j,k \leqslant (p-1)/2} = \left(\frac{p-1}{2} x - 1\right) p^{(p-3)/4} \tag{1.10}$$

for any prime $p \equiv 3 \pmod 4$.

Now we state our central result.

Theorem 1.2. *Let p be an odd prime, and let $a_i, b_i \in \mathbb{Z}$ for all $i = 1, \ldots, m$. Let $c_1, \ldots, c_m$ be complex numbers, and set*

$$c = \sum_{s=1}^{m} c_s \left(\frac{a_s}{p}\right) \sum_{t=1}^{m} c_t \left(\frac{b_t}{p}\right).$$

(i) *For each* $n\in\{1,\ldots,p-1\}$, *we have*

$$\left|\sum_{i=1}^m c_i\left(\frac{a_ij+b_ik}{p}\right)+\left(\frac{jk}{p}\right)x\right|_{0\leqslant j,k\leqslant n}=\left|\sum_{i=1}^m c_i\left(\frac{a_ij+b_ik}{p}\right)\right|_{0\leqslant j,k\leqslant n} \tag{1.11}$$

and

$$\begin{aligned}&c\left|\sum_{i=1}^m c_i\left(\frac{a_ij+b_ik}{p}\right)+\left(\frac{jk}{p}\right)x\right|_{1\leqslant j,k\leqslant n}\\ =&\ c\left|\sum_{i=1}^m c_i\left(\frac{a_ij+b_ik}{p}\right)\right|_{1\leqslant j,k\leqslant n}-x\left|\sum_{i=1}^m c_i\left(\frac{a_ij+b_ik}{p}\right)\right|_{0\leqslant j,k\leqslant n}.\end{aligned} \tag{1.12}$$

(ii) *For any positive integer* n, *we have*

$$\begin{aligned}&c\left|x+\sum_{i=1}^m c_i\left(\frac{a_ij+b_ik}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z\right|_{0\leqslant j,k\leqslant n}\\ =&\left(y+\sum_{i=1}^m c_i\left(\frac{a_i}{p}\right)\right)\left(z+\sum_{i=1}^m c_i\left(\frac{b_i}{p}\right)\right)\left|\sum_{i=1}^m c_i\left(\frac{a_ij+b_ik}{p}\right)\right|_{0\leqslant j,k\leqslant n}\\ &+cx\left|\sum_{i=1}^m c_i\left(\frac{a_ij+b_ik}{p}\right)-\sum_{i=1}^m c_i\left(\frac{a_i}{p}\right)\left(\frac{j}{p}\right)-\sum_{i=1}^m c_i\left(\frac{b_i}{p}\right)\left(\frac{k}{p}\right)\right|_{1\leqslant j,k\leqslant n}.\end{aligned} \tag{1.13}$$

Applying Theorem 1.2 and using the known values of

$$\left|x+\left(\frac{j+k}{p}\right)\right|_{0\leqslant j,k\leqslant (p-1)/2}\quad\text{and}\quad\left|x+\left(\frac{j-k}{p}\right)\right|_{0\leqslant j,k\leqslant (p-1)/2}$$

(where p is an odd prime), we can deduce the following general result.

Theorem 1.3. *Let* p *be an odd prime.*

(i) *If* $p>3$, *then*

$$\begin{aligned}&\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z\right|_{0\leqslant j,k\leqslant (p-1)/2}\\ =&\begin{cases}(\frac{2}{p})2^{(p-1)/2}(pb_px-(y+1)(z+1)a_p) & \text{if } p\equiv 1\pmod 4,\\ (y+1)(z+1)2^{(p-1)/2} & \text{if } p\equiv 3\pmod 4.\end{cases}\end{aligned} \tag{1.14}$$

(ii) *We have*

$$\begin{aligned}&\left|x+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z\right|_{0\leqslant j,k\leqslant (p-1)/2}\\ =&\begin{cases}(\frac{2}{p})pb_p'x-(1+y)(1+z)a_p' & \text{if } p\equiv 1\pmod 4,\\ (1+y)(1-z) & \text{if } p\equiv 3\pmod 4.\end{cases}\end{aligned} \tag{1.15}$$

Clearly, this theorem has the following consequence.

Corollary 1.1. *Let p be a prime with $p\equiv 3 \pmod 4$. Then*

$$\left|x+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)-\left(\frac{k}{p}\right)\right|_{0\leqslant j,k\leqslant (p-1)/2}=4. \tag{1.16}$$

When $p>3$, we have

$$\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j}{p}\right)+\left(\frac{k}{p}\right)\right|_{0\leqslant j,k\leqslant (p-1)/2}=2^{(p+3)/2}. \tag{1.17}$$

We also have the following general result.

Theorem 1.4. *Let $p>3$ be a prime. Then*

$$\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z\right|_{1\leqslant j,k\leqslant (p-1)/2}$$
$$=\begin{cases}(\frac{2}{p})2^{(p-1)/2}((yz-x)a_p+(y+1)(z+1)b_p) & \text{if } p\equiv 1 \pmod 4,\\ 2^{(p-1)/2}(yz-x) & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{1.18}$$

Also,

$$\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z\right|_{1\leqslant j,k\leqslant (p+1)/2}$$
$$=\begin{cases}(\frac{2}{p})2^{(p-1)/2}(pb_p(x-yz)-a_p(y+1)(z+1)) & \text{if } p\equiv 1 \pmod 4,\\ 2^{(p-1)/2}(y+1)(z+1) & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{1.19}$$

We are going to prove Theorems 1.1-1.4 in the next section, and pose in Sections 3-5 many conjectures on determinants involving linear combinations of Legendre symbols.

2. Proofs of Theorems 1.1-1.4

`Proof of Theorem 1.1`. (i) We now prove part (i) of Theorem 1.1. As the four Legendre symbols

$$\left(\frac{m}{p}\right),\ \left(\frac{m+1}{p}\right),\ \left(\frac{m+2}{p}\right),\ \left(\frac{m+3}{p}\right)$$

cannot be pairwise distinct, there are $j,j'\in\{m,m+1,m+2,m+3\}$ with $j\neq j'$ such that $(\frac{j}{p})=(\frac{j'}{p})$. Thus

$$a+b\left(\frac{j}{p}\right)+c\left(\frac{k}{p}\right)+d\left(\frac{jk}{p}\right)=a+b\left(\frac{j'}{p}\right)+c\left(\frac{k}{p}\right)+d\left(\frac{j'k}{p}\right)$$

for all $k=m,\ldots,n$, and hence (1.8) holds.

(ii) We now turn to prove part (ii) of Theorem 1.1. Set $n=(p-1)/2$ and $q=n!$. By Wilson's theorem,

$$-1\equiv (p-1)!=\prod_{k=1}^{n}k(p-k)\equiv (-1)^n(n!)^2=q^2 \pmod p.$$

For each $k=1,\ldots,n$, there is a unique $r_k\in\{1,\ldots,n\}$ such that qk is congruent to r_k or $-r_k$ modulo p. Note that $r_k^2\equiv -k^2 \pmod p$ and $r_k\neq k$ since $q^2\equiv -1\not\equiv 1 \pmod p$. As $qr_k\equiv \pm q^2k\equiv \mp k \pmod p$, we also have $r_{r_k}=k$. For any $k\in\{1,\ldots,n\}$ and $j\in\{m,\ldots,n\}$, clearly

$$\begin{aligned}&x+\left(\frac{j^2+k^2}{p}\right)+\delta\left(\frac{j^2-k^2}{p}\right)-\delta\left(x+\left(\frac{j^2+r_k^2}{p}\right)+\delta\left(\frac{j^2-r_k^2}{p}\right)\right)\\=&x+\left(\frac{j^2+k^2}{p}\right)+\delta\left(\frac{j^2+r_k^2}{p}\right)-\delta\left(x+\left(\frac{j^2+r_k^2}{p}\right)+\delta\left(\frac{j^2+k^2}{p}\right)\right)\\=&(1-\delta)x.\end{aligned}$$

When $\delta=1$, this clearly implies the equality (1.9).

Now we consider the case $\delta=-1$. As $n=(p-1)/2\geqslant 4$, we may choose $k\in\{1,\ldots,n\}\setminus\{1,r_1\}$. Note that $1,r_1,k,r_k$ are distinct elements of $\{1,\ldots,n\}$ with

$$\begin{aligned}&x+\left(\frac{j^2+k^2}{p}\right)+\delta\left(\frac{j^2-k^2}{p}\right)-\delta\left(x+\left(\frac{j^2+r_k^2}{p}\right)+\delta\left(\frac{j^2-r_k^2}{p}\right)\right)\\=&2x=x+\left(\frac{j^2+1^2}{p}\right)+\delta\left(\frac{j^2-1^2}{p}\right)-\delta\left(x+\left(\frac{j^2+r_1^2}{p}\right)+\delta\left(\frac{j^2-r_1^2}{p}\right)\right)\end{aligned}$$

for all $j=m,\ldots,n$. So (1.9) holds.

In view of the above, we have completed the proof of Theorem 1.1. □

To prove Theorem 1.2, we need the following basic lemma which can be found in [10, Lemma 2.1].

Lemma 2.1. *Let $A=[a_{jk}]_{0\leqslant j,k\leqslant m}$ be a matrix over a field. Then*

$$\det[x+a_{jk}]_{0\leqslant j,k\leqslant m}-\det[a_{jk}]_{0\leqslant j,k\leqslant m}=x\det[b_{jk}]_{1\leqslant j,k\leqslant m},\tag{2.1}$$

where $b_{jk}=a_{jk}-a_{j0}-a_{0k}+a_{00}$.

`Proof of Theorem 1.2`. For convenience, we set

$$f(j,k)=\sum_{i=1}^m c_i\left(\frac{a_ij+b_ik}{p}\right)$$

for any $j,k=0,1,2,\ldots$.

(i) We first prove part (i) of Theorem 1.2. For $i=1,\ldots,m$ and $j,k=1,\ldots,n$, clearly

$$f(j,k)+\left(\frac{jk}{p}\right)x=\left(\frac{j}{p}\right)\left(\frac{k}{p}\right)\left(\left(\frac{jk}{p}\right)f(j,k)+x\right).$$

It follows that

$$x^2\left|f(j,k)+\left(\frac{jk}{p}\right)x\right|_{0\leqslant j,k\leqslant n}=c\prod_{j=1}^n\left(\frac{j}{p}\right)\times\prod_{k=1}^n\left(\frac{k}{p}\right)\times\det A_0=c\det A_0,\tag{2.2}$$

where A_0 is obtained from the matrix $A=[(\frac{jk}{p})f(j,k)+x]_{0\leqslant j,k\leqslant n}$ via replacing the first entry x in the first row by 0. If we expand $\det A_0$ and $\det A$ along their first rows, we immediately see that

$$\det A-\det A_0=x\left|\left(\frac{jk}{p}\right)f(j,k)+x\right|_{1\leqslant j,k\leqslant n}. \tag{2.3}$$

By Lemma 2.1,

$$\det A=\left|\left(\frac{jk}{p}\right)f(j,k)\right|_{0\leqslant j,k\leqslant n}+x\left|\left(\frac{jk}{p}\right)f(j,k)\right|_{1\leqslant j,k\leqslant n}=x\left|f(j,k)\right|_{1\leqslant j,k\leqslant n}.$$

Combining this with (2.2) and (2.3), we obtain

$$\begin{aligned}&x^2\left|f(j,k)+\left(\frac{jk}{p}\right)x\right|_{0\leqslant j,k\leqslant n}\\&=c\left(x\left|f(j,k)\right|_{1\leqslant j,k\leqslant n}-x\left|\left(\frac{jk}{p}\right)f(j,k)+x\right|_{1\leqslant j,k\leqslant n}\right)\end{aligned}$$

and hence

$$\begin{aligned}&x\left|f(j,k)+\left(\frac{jk}{p}\right)x\right|_{0\leqslant j,k\leqslant n}\\&=c\left(\left|f(j,k)\right|_{1\leqslant j,k\leqslant n}-\left|\left(\frac{jk}{p}\right)f(j,k)+x\right|_{1\leqslant j,k\leqslant n}\right).\end{aligned} \tag{2.4}$$

Applying Lemma 2.1, we find that

$$\left|\left(\frac{jk}{p}\right)f(j,k)+x\right|_{1\leqslant j,k\leqslant n}=\left|\left(\frac{jk}{p}\right)f(j,k)\right|_{1\leqslant j,k\leqslant n}+x\det D,$$

where $D=[d_{jk}]_{2\leqslant j,k\leqslant n}$ with

$$d_{jk}=\left(\frac{jk}{p}\right)f(j,k)-\left(\frac{j}{p}\right)f(j,1)-\left(\frac{k}{p}\right)f(1,k)+f(1,1).$$

Therefore

$$\left|\left(\frac{jk}{p}\right)f(j,k)+x\right|_{1\leqslant j,k\leqslant n}=\left|f(j,k)\right|_{1\leqslant j,k\leqslant n}+x\det D.$$

Combining this with (2.4), we immediately get

$$\left|f(j,k)+\left(\frac{jk}{p}\right)x\right|_{0\leqslant j,k\leqslant n}=-c\det D$$

and hence (1.11) follows. In light of (1.11) and (2.4),

$$c|f(j,k)|_{1\leqslant j,k\leqslant n}-x|f(j,k)|_{0\leqslant j,k\leqslant n}=c\left|f(j,k)+\left(\frac{jk}{p}\right)x\right|_{1\leqslant j,k\leqslant n},$$

which gives (1.12).

(ii) Now we turn to prove part (ii) of Theorem 1.2. Let

$$a_{jk}=f(j,k)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z$$

for $j,k=0,\ldots,n$. It is easy to see that

$$a_{jk}-a_{j0}-a_{0k}+a_{00}=f(j,k)-\sum_{i=1}^m c_i\left(\frac{a_i}{p}\right)\left(\frac{j}{p}\right)-\sum_{i=1}^m c_i\left(\frac{b_i}{p}\right)\left(\frac{k}{p}\right).$$

Thus, in view of Lemma 2.1,

$$\begin{aligned}&|x+a_{jk}|_{0\leqslant j,k\leqslant n}-|a_{jk}|_{0\leqslant j,k\leqslant n}\\ &=x\left|f(j,k)-\sum_{i=1}^m c_i\left(\frac{a_i}{p}\right)\left(\frac{j}{p}\right)-\sum_{i=1}^m c_i\left(\frac{b_i}{p}\right)\left(\frac{k}{p}\right)\right|_{1\leqslant j,k\leqslant n}.\end{aligned}$$

So we have reduced (1.13) to the equality

$$c\,|a_{jk}|_{0\leqslant j,k\leqslant n}=\left(y+\sum_{i=1}^m c_i\left(\frac{a_i}{p}\right)\right)\left(z+\sum_{i=1}^m c_i\left(\frac{b_i}{p}\right)\right)|f(j,k)|_{0\leqslant j,k\leqslant n}. \tag{2.5}$$

For $k=0,\ldots,n$, clearly

$$a_{0k}=f(0,k)+\left(\frac{k}{p}\right)z=\left(z+\sum_{i=1}^m c_i\left(\frac{b_i}{p}\right)\right)\left(\frac{k}{p}\right)$$

and

$$a_{jk}-a_{0k}=f(j,k)+\left(\frac{j}{p}\right)y-\sum_{i=1}^m c_i\left(\frac{b_i}{p}\right)\left(\frac{k}{p}\right)$$

for all $j=1,\ldots,n$. Thus

$$\sum_{i=1}^m c_i\left(\frac{b_i}{p}\right)\times|a_{jk}|_{0\leqslant j,k\leqslant n}=\left(z+\sum_{i=1}^m c_i\left(\frac{b_i}{p}\right)\right)\left|f(j,k)+\left(\frac{j}{p}\right)y\right|_{0\leqslant j,k\leqslant n}. \tag{2.6}$$

For $j=0,\ldots,n$, apparently

$$f(j,0)+\left(\frac{j}{p}\right)y=\left(y+\sum_{i=1}^m c_i\left(\frac{a_i}{p}\right)\right)\left(\frac{j}{p}\right)$$

and

$$f(j,k)+\left(\frac{j}{p}\right)y-\left(f(j,0)+\left(\frac{j}{p}\right)y\right)=f(j,k)-\sum_{i=1}^m c_i\left(\frac{a_i}{p}\right)\left(\frac{j}{p}\right)$$

for all $k=1,\ldots,n$. Therefore

$$\sum_{i=1}^m c_i\left(\frac{a_i}{p}\right)\times\left|f(j,k)+\left(\frac{j}{p}\right)y\right|_{0\leqslant j,k\leqslant n}=\left(y+\sum_{i=1}^m c_i\left(\frac{a_i}{p}\right)\right)|f(j,k)|_{0\leqslant j,k\leqslant n}.$$

Combining this with (2.6), we immediately obtain the desired (2.5).

In view of the above, we have completed the proof of Theorem 1.2. □

Recall that an $n \times n$ matrix $A = [a_{jk}]_{1\leqslant j,k\leqslant n}$ over a field is called *skew-symmetric* if $a_{jk} + a_{kj} = 0$ for all $j, k = 1, \ldots, n$.

Suppose that $A = [a_{jk}]_{1\leqslant j,k\leqslant n}$ is a skew-symmetric matrix over $\mathbb{Z}$. Note that

$$\det A = |a_{kj}|_{1\leqslant j,k\leqslant n} = |-a_{jk}|_{1\leqslant j,k\leqslant n} = (-1)^n \det A.$$

Thus $\det A = 0$ if n is odd. By a theorem of Cayley, $\det A$ is a square if n is even (cf. [5]).

Lemma 2.2. *Let p be an odd prime. Then*

$$\begin{aligned}&\left|\left(\frac{j+k}{p}\right)-\left(\frac{j}{p}\right)-\left(\frac{k}{p}\right)\right|_{1\leqslant j,k\leqslant (p-1)/2}\\ &=\begin{cases}(\frac{2}{p})2^{(p-1)/2}pb_p & \text{if } p\equiv 1 \pmod 4,\\ 0 & \text{if } p>3 \text{ and } p\equiv 3 \pmod 4.\end{cases}\end{aligned}\tag{2.7}$$

We also have

$$\left|\left(\frac{j-k}{p}\right)-\left(\frac{j}{p}\right)-\left(\frac{-k}{p}\right)\right|_{1\leqslant j,k\leqslant (p-1)/2}=\begin{cases}(\frac{2}{p})pb'_p & \text{if } p\equiv 1 \pmod 4,\\ 0 & \text{if } p\equiv 3 \pmod 4.\end{cases}\tag{2.8}$$

Proof. Let $n = (p-1)/2$ and $\delta \in \{\pm 1\}$. Define $a_{jk} = (\frac{j+\delta k}{p})$ for $j, k = 0, \ldots, n$. Then

$$a_{jk} - a_{j0} - a_{0k} + a_{00} = \left(\frac{j+\delta k}{p}\right)-\left(\frac{j}{p}\right)-\left(\frac{\delta k}{p}\right).$$

Thus, by Lemma 2.1 we have

$$\begin{aligned}&\det[1+a_{jk}]_{0\leqslant j,k\leqslant n}-\det[a_{jk}]_{0\leqslant j,k\leqslant n}\\ &=\left|\left(\frac{j+\delta k}{p}\right)-\left(\frac{j}{p}\right)-\left(\frac{\delta k}{p}\right)\right|_{1\leqslant j,k\leqslant n}.\end{aligned}\tag{2.9}$$

Combining (1.3) and (1.5), we obtain

$$\left|x+\left(\frac{j+k}{p}\right)\right|_{0\leqslant j,k\leqslant n}=\begin{cases}(\frac{2}{p})2^n(pb_px-a_p) & \text{if } p\equiv 1 \pmod 4,\\ 2^n & \text{if } p>3 \ \&\ p\equiv 3 \pmod 4.\end{cases}\tag{2.10}$$

So we know the exact value of $|x + a_{jk}|_{0\leqslant j,k\leqslant n}$ in the case $\delta = 1$. When $\delta = -1$, the equality (1.7) gives the exact value of $|x + a_{jk}|_{0\leqslant j,k\leqslant n}$. Since $|x + a_{jk}|_{0\leqslant j,k\leqslant n}$ is evaluated, we immediately obtain the exact value of

$$\left|\left(\frac{j+\delta k}{p}\right)-\left(\frac{j}{p}\right)-\left(\frac{\delta k}{p}\right)\right|_{1\leqslant j,k\leqslant n}$$

by using (2.9). Therefore (2.7) and (2.8) hold. In the case $p \equiv 3 \pmod 4$, we may prove (2.8) without using (1.7) since the matrix in (2.8) is skew-symmetric and of odd order. This ends our proof. □

`Proof of Theorem 1.3`. Set $n=(p-1)/2$. Combining Theorem 1.2(ii), (2.10), (1.7) and Lemma 2.2, we immediately obtain the desired results. □

`Proof of Theorem 1.4`. Let $n\in\{(p-1)/2,(p+1)/2\}$, and set

$$a_{jk}=x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y$$

for $j,k=1,\ldots,n$. Observe that

$$a_{jk}-a_{j1}+\frac{a_{j1}}{y+1}=\left(\frac{j+k-1}{p}\right)+\frac{x}{y+1}$$

for all $1<j\leqslant n$ and $1<k\leqslant n$. Thus

$$\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y\right|_{1\leqslant j,k\leqslant n}=(y+1)\left|\left(\frac{j+k-1}{p}\right)+\frac{x}{y+1}\right|_{1\leqslant j,k\leqslant n}.$$

Combining this with (1.2) and (1.3), we have

$$\begin{aligned}&\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y\right|_{1\leqslant j,k\leqslant (p-1)/2}\\&=\begin{cases}(\frac{2}{p})2^{(p-1)/2}(b_p(y+1)-a_px)&\text{if }p\equiv 1\pmod 4,\\-2^{(p-1)/2}x&\text{if }p\equiv 3\pmod 4,\end{cases}\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}&\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y\right|_{1\leqslant j,k\leqslant (p+1)/2}\\&=\begin{cases}(\frac{2}{p})2^{(p-1)/2}(pb_px-a_p(y+1))&\text{if }p\equiv 1\pmod 4,\\2^{(p-1)/2}(y+1)&\text{if }p\equiv 3\pmod 4.\end{cases}\end{aligned}\tag{2.12}$$

Let

$$b_{jk}=x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z$$

for $j,k=1,\ldots,n$. Note that

$$b_{jk}-b_{1k}+\frac{b_{1k}}{z+1}=\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\frac{x-yz}{z+1}$$

for all $j,k=1,\ldots,n$. Thus

$$|b_{jk}|_{1\leqslant j,k\leqslant n}=(z+1)\left|\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\frac{x-yz}{z+1}\right|_{1\leqslant j,k\leqslant n}.$$

Combining this with (2.11) and (2.12), we immediately obtain the desired identities (1.18) and (1.19). Note that both sides of the equalities (1.18) and (1.19) are polynomials in x,y,z. If we view y and z as complex numbers, to handle the case $y=-1$ or $z=-1$ we may take limits. This concludes our proof of Theorem 1.4. □

3. Conjectures on determinants involving $(\frac{j\pm k}{p})$, $(\frac{j}{p})$, $(\frac{k}{p})$ and $(\frac{jk}{p})$

Conjecture 3.1. *Let p be an odd prime.*

(i) *If $p>3$, then*

$$\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-1)/2}$$
$$=\begin{cases}(\frac{2}{p})2^{(p-1)/2}(pb_px+a_p(wx-(y+1)(z+1))) & \text{if } p\equiv 1 \pmod 4,\\ 2^{(p-1)/2}((y+1)(z+1)-wx) & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{3.1}$$

(ii) *We have*

$$\left|x+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-1)/2}$$
$$=\begin{cases}a_p'(wx-(y+1)(z+1))+(\frac{2}{p})pb_p'x & \text{if } p\equiv 1 \pmod 4,\\ wx+(1+y)(1-z) & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{3.2}$$

Remark 3.1. Conjecture 3.1 in the case $wx=0$ follows from Theorems 1.2 and 1.3.

Conjecture 3.2. *Let p be an odd prime, and set $v=wx-(y+1)(z+1)$.*

(i) *If $p>3$, then*

$$\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-3)/2}$$
$$=\begin{cases}(\frac{2}{p})2^{(p-3)/2}((pb_p-2a_p)x+(a_p-2b_p)v) & \text{if } p\equiv 1 \pmod 4,\\ 2^{(p-3)/2}(v-2x) & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{3.3}$$

(ii) *We have*

$$\left|x+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-3)/2}$$
$$=\begin{cases}-a_p'x-(\frac{2}{p})b_p'v & \text{if } p\equiv 1 \pmod 4,\\ x & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{3.4}$$

Remark 3.2. In light of Theorem 1.2, in the case $wx=0$ we can reduce Conjecture 3.2 to the case $y=z=0$. For any prime $p\equiv 3 \pmod 4$, clearly $|(\frac{j-k}{p})|_{1\leqslant j,k\leqslant (p-1)/2}=0$ since the matrix is skew-symmetric and of odd order; the author [8] conjectured that

$$\left|x+\left(\frac{j-k}{p}\right)\right|_{1\leqslant j,k\leqslant (p-1)/2}=x,\quad \text{i.e.,}\quad \left|x+\left(\frac{j-k}{p}\right)\right|_{0\leqslant j,k\leqslant (p-3)/2}=x.$$

In view of Lemma 2.1 or Theorem 1.2(ii), for any prime $p>3$, part (i) of Conjecture 3.2 implies that

$$\begin{aligned}&\left|\left(\frac{j+k}{p}\right)-\left(\frac{j}{p}\right)-\left(\frac{k}{p}\right)\right|_{1\leqslant j,k\leqslant (p-3)/2}\\ =&\begin{cases}(\frac{2}{p})2^{(p-3)/2}(pb_p-2a_p)&\text{if } p\equiv 1 \pmod 4,\\ -2^{(p-1)/2}&\text{if } p\equiv 3 \pmod 4,\end{cases}\end{aligned}\tag{3.5}$$

while part (ii) of Conjecture 3.2 implies that

$$\left|\left(\frac{j-k}{p}\right)-\left(\frac{j}{p}\right)-\left(\frac{-k}{p}\right)\right|_{1\leqslant j,k\leqslant (p-3)/2}=\begin{cases}-a_p'&\text{if } p\equiv 1 \pmod 4,\\ 1&\text{if } p\equiv 3 \pmod 4.\end{cases}\tag{3.6}$$

Conjecture 3.3. *Let $p\geqslant 5$ be a prime. Then*

$$\begin{aligned}&\left|x+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-5)/2}\\ =&\begin{cases}(\frac{2}{p})(2a_p'-pb_p')x+(a_p'-2b_p')((1+y)(1+z)-wx)&\text{if } p\equiv 1 \pmod 4,\\ wx+(1+y)(1-z)&\text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}\tag{3.7}$$

Remark 3.3. In light of Theorem 1.2, in the case $wx=0$ we can reduce this conjecture to the case $y=z=0$. In a unpublished preprint written in 2003, for each prime $p\geqslant 5$ with $p\equiv 3 \pmod 4$, R. Chapman conjectured that

$$\left|\left(\frac{j-k}{p}\right)\right|_{1\leqslant j,k\leqslant (p-3)/2}=1,\ \text{i.e.,}\ \left|\left(\frac{j-k}{p}\right)\right|_{0\leqslant j,k\leqslant (p-5)/2}=1.$$

In view of Lemma 2.1 of Theorem 1.2(ii), Conjecture 3.3 implies that

$$\left|\left(\frac{j-k}{p}\right)-\left(\frac{j}{p}\right)-\left(\frac{k}{p}\right)\right|_{1\leqslant j,k\leqslant (p-5)/2}=\left(\frac{2}{p}\right)(2a_p'-pb_p')\tag{3.8}$$

for any prime $p>5$ with $p\equiv 1 \pmod 4$, and that

$$\left|\left(\frac{j-k}{p}\right)-\left(\frac{j}{p}\right)+\left(\frac{k}{p}\right)\right|_{1\leqslant j,k\leqslant (p-5)/2}=0\tag{3.9}$$

for any prime $p>5$ with $p\equiv 3 \pmod 4$. The equality (3.9) is easy since the matrix is skew-symmetric and of odd order.

Conjecture 3.4. *For any prime $p\geqslant 7$ with $p\equiv 3 \pmod 4$, we have*

$$\left|x+\left(\frac{j-k}{p}\right)\right|_{0\leqslant j,k\leqslant (p-7)/2}=\left\lfloor\frac{p-2}{3}\right\rfloor^2 x.\tag{3.10}$$

Remark 3.4. Surprisingly, this concise conjecture has not been found before.

Conjecture 3.5. *Let $p>3$ be a prime.*

(i) *If $p\equiv 1 \pmod 4$, then*

$$\begin{aligned}&\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-1)/2}\\ =&\left(\frac{2}{p}\right)2^{(p-1)/2}(a_p(w-x)+b_p+(b_p-1)(y+z)-c_p(wx-yz)),\end{aligned}\tag{3.11}$$

where $c_p=(p+1)b_p-2$. When $p\equiv 3 \pmod 4$, we have

$$\begin{aligned}&\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-1)/2}\\ =&-2^{(p-1)/2}(w+x+(-1)^{(h(-p)-1)/2}(y+z+2yz-2wx)),\end{aligned}\tag{3.12}$$

where $h(-p)$ denotes the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

(ii) *If $p\equiv 1 \pmod 4$, then*

$$\begin{aligned}&\left|x+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-1)/2}\\ =&\,a_p'(w-x)+\left(\frac{2}{p}\right)(b_p'+(b_p'-1)(y+z)+c_p'(yz-wx)),\end{aligned}\tag{3.13}$$

where $c_p'=(p+1)b_p'-2$. When $p\equiv 3 \pmod 4$, we have

$$\begin{aligned}&\left|x+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-1)/2}\\ =&\,w+x-(-1)^{(h(-p)-1)/2}(y+z).\end{aligned}\tag{3.14}$$

Conjecture 3.6. *Let $p\geqslant 5$ be a prime.*

(i) *If $p\equiv 1 \pmod 4$, then*

$$\begin{aligned}&\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-3)/2}\\ =&\left(\frac{2}{p}\right)2^{(p-3)/2}(b_p-a_px+(a_p-2b_p)w+(b_p-1)(y+z)+d_p(yz-wx)),\end{aligned}\tag{3.15}$$

where $d_p=(p+1)b_p-2(a_p+1)$. When $p\equiv 3 \pmod 4$, we have

$$\begin{aligned}&\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-3)/2}\\ =&\,2^{(p-3)/2}(w+x+2(wx-yz)+(-1)^{(h(-p)-1)/2}(y+z+2yz-2wx)).\end{aligned}\tag{3.16}$$

(ii) *If* $p\equiv 1 \pmod 4$, *then*

$$\left|x+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-3)/2}$$
$$=e_p'+\left(\frac{2}{p}\right)((2a_p'-pb_p')x-b_p'w)+(e_p'+1)(y+z)+2(b_p'-1)(wx-yz), \tag{3.17}$$

where $e_p'=a_p'-2b_p'$. *When* $p\equiv 3 \pmod 4$, *we have*

$$\begin{aligned}&\left|x+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-3)/2}\\ &=1+\left(1-(-1)^{(h(-p)-1)/2}\left(\frac{2}{p}\right)\right)(2(wx-yz)+y-z).\end{aligned} \tag{3.18}$$

Conjecture 3.7. *Let* $p>3$ *be a prime.*

(i) *We have*

$$\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-1)/2}$$
$$=\begin{cases}(\frac{2}{p})p^{(p+3)/4}x & \text{if } p\equiv 1 \pmod 4,\\ (-1)^{(h(-p)-1)/2}p^{(p-3)/4}(px+(2-(\frac{2}{p}))h(-p)v) & \text{if } p\equiv 3 \pmod 4,\end{cases} \tag{3.19}$$

where $v=(y+2)z-wx$.

(ii) *If* $p\equiv 1 \pmod 4$, *then*

$$\left|\left(\frac{j+k}{p}\right)-\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z\right|_{0\leqslant j,k\leqslant (p-1)/2}=4p^{(p-5)/4}x_pyz \tag{3.20}$$

for some $x_p\in\mathbb{Z}$ *only depending on* p.

Remark 3.5. Our computation indicates that

$$x_5=1,\ x_{13}=-3,\ x_{17}=2,\ x_{29}=7,\ x_{37}=-7,\ x_{41}=6,\ x_{53}=3,\ x_{61}=15.$$

Conjecture 3.8. *Let* $p>3$ *be a prime.*

(i) *If* $p\equiv 1 \pmod 4$, *then*

$$\left|\left(\frac{j+k}{p}\right)-\left(\frac{j-k}{p}\right)\right|_{1\leqslant j,k\leqslant (p-1)/2}=(-p)^{(p-1)/4} \tag{3.21}$$

and

$$\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-1)/2}$$
$$=(-p)^{(p-5)/4}\left(\left(\frac{p-1}{2}\right)^2wx-\left(\frac{p-1}{2}y-1\right)\left(\frac{p-1}{2}z-1\right)\right). \tag{3.22}$$

(ii) *When* $p\equiv 3\pmod 4$, *we have*

$$(-1)^{\frac{h(-p)+1}{2}}\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-1)/2}$$
$$=p^{(p-3)/4}\left(\frac{p-1}{2}y-1+\left(2-\left(\frac{2}{p}\right)\right)h(-p)(w+x)-\left(\frac{2}{p}\right)\frac{16q_p}{p}wx\right) \tag{3.23}$$

for some integer q_p *only depending on* p.

Remark 3.6. Our computation indicates that

$$q_7=q_{11}=1,\ q_{19}=9,\ q_{23}=15,\ q_{31}=24,\ q_{43}=27,\ q_{47}=72,\ q_{59}=62,$$
$$q_{67}=51,\ q_{71}=259,\ q_{79}=82,\ q_{83}=18,\ q_{103}=349,\ q_{107}=-68,\ q_{127}=478.$$

Conjecture 3.9. *Let* $p>3$ *be a prime. If* $p\equiv 1\pmod 4$, *then*

$$\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-3)/2}$$
$$=\left(\frac{2}{p}\right)p^{(p-5)/4}(px-wx+(y+2)(z+2)). \tag{3.24}$$

When $p\equiv 3\pmod 4$, *there is an integer* $m_p\in\mathbb{Z}$ *only depending on* p *such that*

$$\left|x+\left(\frac{j+k}{p}\right)+\left(\frac{j-k}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-3)/2}$$
$$=(-1)^{(h(-p)+1)/2}p^{(p-7)/4}(p-2m_pw)x. \tag{3.25}$$

Remark 3.7. Our computation indicates that

$$m_7=2,\ m_{11}=1,\ m_{19}=-3,\ m_{23}=-1,\ m_{31}=3,\ m_{43}=1,\ m_{47}=0,\ m_{59}=8.$$

4. Conjectures on determinants involving $(\frac{j\pm k\pm 1}{p})$ or $(\frac{j^2\pm k^2}{p})$

Conjecture 4.1. *Let* $p>3$ *be a prime.*

(i) *If* $p\equiv 1\pmod 4$, *then*

$$\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-1)/2}$$
$$=\left(\frac{2}{p}\right)2^{(p-1)/2}((yz-(w+1)x)a_p+(w(1-x)+(y+1)(z+1))b_p). \tag{4.1}$$

When $p\equiv 3\pmod 4$, *we have*

$$\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-1)/2} \tag{4.2}$$
$$=2^{(p-1)/2}(yz-(w+1)x).$$

(ii) *If* $p\equiv 1\pmod 4$, *then*

$$\begin{aligned}&\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p+1)/2}\\ =&\left(\frac{2}{p}\right)2^{(p-1)/2}(pb_p((w+1)x-yz)+a_p(w(x-1)-(y+1)(z+1))).\end{aligned}\tag{4.3}$$

When $p\equiv 3\pmod 4$, *we have*

$$\begin{aligned}&\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p+1)/2}\\ =&\ 2^{(p-1)/2}(w(1-x)+(y+1)(z+1)).\end{aligned}\tag{4.4}$$

Remark 4.1. In the case $w=0$, this reduces to Theorem 1.4.

Conjecture 4.2. *Let* $p>3$ *be a prime. If* $p\equiv 1\pmod 4$, *then*

$$\begin{aligned}&\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-3)/2}\\ =&\left(\frac{2}{p}\right)2^{(p-5)/2}\left(\left(2-\left(\frac{2}{p}\right)\right)a_p-pb_p\right)x\\ &+\left(\frac{2}{p}\right)2^{(p-5)/2}\left(a_p+\left(\left(\frac{2}{p}\right)-2\right)b_p\right)(w+y+z+1)\\ &+\left(\frac{2}{p}\right)2^{(p-5)/2}\left((p-1)b_p+\left(\left(\frac{2}{p}\right)-1\right)(a_p+b_p)\right)(yz-wx).\end{aligned}\tag{4.5}$$

When $p\equiv 3\pmod 4$, *we have*

$$\begin{aligned}&\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-3)/2}\\ =&\ 2^{(p-5)/2}\left(\left(\left(\frac{2}{p}\right)-2\right)x-w-y-z-1+\left(1-\left(\frac{2}{p}\right)\right)(yz-wx)\right).\end{aligned}\tag{4.6}$$

Remark 4.2. Let $p=2n+1>3$ be a prime. As

$$\begin{aligned}\left|x+\left(\frac{j+k-1}{p}\right)\right|_{1\leqslant j,k\leqslant n-1}&=\left|x+\left(\frac{n+1-s+(n+1-t)-1}{p}\right)\right|_{2\leqslant s,t\leqslant n}\\ &=(-1)^{n(n-1)}\left|(-1)^n x+\left(\frac{j+k}{p}\right)\right|_{2\leqslant j,k\leqslant n},\end{aligned}$$

Conjecture 4.2 implies that

$$\begin{aligned}&\left|x+\left(\frac{j+k}{p}\right)\right|_{2\leqslant j,k\leqslant (p-1)/2}\\ =&\begin{cases}(\frac{2}{p})2^{(p-5)/2}(a_p-pb_px+(2-(\frac{2}{p}))(a_px-b_p)) & \text{if } p\equiv 1 \pmod 4,\\ 2^{(p-5)/2}((2-(\frac{2}{p}))x-1) & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned} \tag{4.7}$$

Conjecture 4.3. *Let $p>3$ be a prime. If $p\equiv 1 \pmod 4$, then*

$$\begin{aligned}&\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-1)/2}\\ =&\left(\frac{2}{p}\right)2^{(p-3)/2}(pb_p-2a_p)(w+y+z+1) \qquad (4.8)\\ &+\left(\frac{2}{p}\right)2^{(p-3)/2}\left((2b_p-a_p)px+((p-2)a_p-pb_p)(yz-wx)\right).\end{aligned}$$

If $p\equiv 3 \pmod 4$, then

$$\begin{aligned}&\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-1)/2} \qquad (4.9)\\ =&\ 2^{(p-3)/2}\left(2w(1-x)+2(y+1)(z+1)+p((w+1)x-yz)\right).\end{aligned}$$

Remark 4.3. Let $p=2n+1>3$ be a prime. As

$$\begin{aligned}\left|x+\left(\frac{j+k-1}{p}\right)\right|_{0\leqslant j,k\leqslant n}&=\left|x+\left(\frac{n+1-s+(n+1-t)-1}{p}\right)\right|_{1\leqslant s,t\leqslant n+1}\\ &=(-1)^{n(n+1)}\left|(-1)^nx+\left(\frac{j+k}{p}\right)\right|_{1\leqslant j,k\leqslant n+1},\end{aligned}$$

Conjecture 4.3 implies that

$$\begin{aligned}&\left|x+\left(\frac{j+k}{p}\right)\right|_{1\leqslant j,k\leqslant (p+1)/2}\\ =&\begin{cases}(\frac{2}{p})2^{(p-3)/2}(pb_p-2a_p+(2b_p-a_p)px) & \text{if } p\equiv 1 \pmod 4,\\ 2^{(p-3)/2}(2-px) & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned} \tag{4.10}$$

Conjecture 4.4. *For any prime $p>3$ with $p\equiv 3 \pmod 4$, we have*

$$\begin{aligned}&\left|x+\left(\frac{j+k-1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-3)/2}\\ =&\ 2^{(p-5)/2}\left(\left(\frac{p}{2}\left(2-\left(\frac{2}{p}\right)\right)-4\right)x+\left(\frac{p}{2}-2-\left(\frac{2}{p}\right)\right)(w+y+z+1)\right)\\ &+2^{(p-5)/2}\left(\frac{p}{2}\left(\left(\frac{2}{p}\right)-1\right)+2-\left(\frac{2}{p}\right)\right)(yz-wx).\end{aligned} \tag{4.11}$$

Remark 4.4. Let $p=2n+1>3$ be a prime. As

$$\begin{aligned}\left|x+\left(\frac{j+k-1}{p}\right)\right|_{0\leqslant j,k\leqslant n-1}=&\left|x+\left(\frac{n+1-s+(n+1-t)-1}{p}\right)\right|_{2\leqslant s,t\leqslant n+1}\\ =&(-1)^n\left|(-1)^nx+\left(\frac{j+k}{p}\right)\right|_{2\leqslant j,k\leqslant n+1},\end{aligned}$$

when $p\equiv 3\pmod 4$ Conjecture 4.4 implies that

$$\begin{aligned}&\left|x+\left(\frac{j+k}{p}\right)\right|_{2\leqslant j,k\leqslant (p+1)/2}\\ =&\ 2^{(p-5)/2}\left(2+\left(\frac{2}{p}\right)-\frac{p}{2}+\left(\frac{p}{2}\left(2-\left(\frac{2}{p}\right)\right)-4\right)x\right).\end{aligned}\tag{4.12}$$

Conjecture 4.5. *Let $p>3$ be a prime. If $p\equiv 1\pmod 4$, then*

$$\begin{aligned}&\left|x+\left(\frac{j+k+1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-1)/2}\\ =&\left(\frac{2}{p}\right)2^{(p-1)/2}pb_p\left(x+\frac{p-2}{2}(yz-wx)\right)\\ &+\left(\frac{2}{p}\right)2^{(p-1)/2}a_p\left(w\left(x+\frac{p-2}{2}\right)-(y+1)(z+1)\right).\end{aligned}\tag{4.13}$$

When $p\equiv 3\pmod 4$, we have

$$\begin{aligned}&\left|x+\left(\frac{j+k+1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-1)/2}\\ =&\ 2^{(p-1)/2}\left(w\left(\frac{p-2}{2}-x\right)+(y+1)(z+1)\right).\end{aligned}\tag{4.14}$$

Remark 4.5. For any prime $p=2n+1>3$, by (1.3) we have

$$\begin{aligned}&\left|x+\left(\frac{j+k+1}{p}\right)\right|_{0\leqslant j,k\leqslant n}=\left|x+\left(\frac{(j+1)+(k+1)-1}{p}\right)\right|_{0\leqslant j,k\leqslant n}\\ =&\ \det C_p^*(x)=\begin{cases}(-1)^{n/2}2^n(pb_px-a_p) & \text{if } p\equiv 1\pmod 4,\\ 2^n & \text{if } p\equiv 3\pmod 4.\end{cases}\end{aligned}$$

Conjecture 4.6. *Let $p>3$ be a prime. If $p\equiv 1\pmod 4$, then*

$$\begin{aligned}&\left|x+\left(\frac{j+k+1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z\right|_{1\leqslant j,k\leqslant (p-1)/2}\\ =&\left(\frac{2}{p}\right)2^{(p-3)/2}\left((pb_p-2a_p)x+2(n_p+b_p-a_p)yz\right)\\ &+\left(\frac{2}{p}\right)2^{(p-3)/2}\left((2b_p-a_p-1)(y+z+1)+1\right)\end{aligned}\tag{4.15}$$

for some positive integer n_p. *When* $p \equiv 3 \pmod 4$, *we have*

$$\begin{aligned}&\left|x+\left(\frac{j+k+1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-1)/2}\\ &=2^{(p-3)/2}\left(1-(-1)^{(h(-p)-1)/2}\right)(y+z+2(yz-wx))\\ &\quad+2^{(p-3)/2}\left((p-3)\left(yz-wx+\frac{w}{2}\right)-2x+1\right).\end{aligned}\tag{4.16}$$

Remark 4.6. Our computation indicates that

$$n_5=1,\ n_{13}=11,\ n_{17}=39,\ n_{29}=68,\ n_{37}=230,\ n_{41}=1441,\ n_{53}=256.$$

For any odd prime $p=2n+1$, we clearly have

$$\begin{aligned}\left|x+\left(\frac{j+k+1}{p}\right)\right|_{1\leqslant j,k\leqslant n}&=\left|x+\left(\frac{(n-j)+(n-k)+1}{p}\right)\right|_{0\leqslant j,k\leqslant n-1}\\ &=(-1)^n\left|(-1)^nx+\left(\frac{j+k}{p}\right)\right|_{0\leqslant j,k\leqslant n-1}.\end{aligned}$$

Conjecture 4.7. *Let* $p>3$ *be a prime. If* $p\equiv 1 \pmod 4$, *then*

$$\begin{aligned}&\left|x+\left(\frac{j-k+1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-1)/2}\\ &=(pb_p'-a_p')(w(1-x)+(y+1)(z+1))\\ &\quad+p\left(wx-(y+1)z+\left(\frac{2}{p}\right)(b_p'-a_p')((1+w)x-yz)\right).\end{aligned}\tag{4.17}$$

When $p\equiv 3 \pmod 4$, *we have*

$$\begin{aligned}&\left|x+\left(\frac{j-k+1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-1)/2}\\ &=1-\left(\frac{2}{p}\right)px+w+y+\left(p\left(\frac{2}{p}\right)(-1)^{(h(-p)-1)/2}-1\right)z\\ &\quad+\left(p\left(\frac{2}{p}\right)\left(1+(-1)^{(h(-p)-1)/2}\right)-1\right)(yz-wx).\end{aligned}\tag{4.18}$$

Conjecture 4.8. *Let* $p>3$ *be a prime. If* $p\equiv 1 \pmod 4$, *then*

$$\begin{aligned}&\left|x+\left(\frac{j-k+1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-3)/2}\\ &=(pb_p'-a_p')((w+1)x-yz)+\left(\frac{2}{p}\right)(wx-(y+1)z)\\ &\quad+\left(\frac{2}{p}\right)(b_p'-a_p')(w(1-x)+(y+1)(z+1)).\end{aligned}\tag{4.19}$$

When $p\equiv 3\pmod 4$, *we have*

$$\begin{aligned}&\left|x+\left(\frac{j-k+1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{0\leqslant j,k\leqslant (p-3)/2}\\ =\ &x-\left(\frac{2}{p}\right)(w+y-z+1)-(-1)^{(h(-p)-1)/2}z\\ &+\left(1+(-1)^{(h(-p)-1)/2}-\left(\frac{2}{p}\right)\right)(wx-yz).\end{aligned}\tag{4.20}$$

Conjecture 4.9. *Let* $p>3$ *be a prime. If* $p\equiv 1\pmod 4$, *then*

$$\begin{aligned}&\left|x+\left(\frac{j-k+1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-1)/2}\\ =\ &\left(\frac{2}{p}\right)\frac{p-1}{2}((y+1)z-wx)+(pb_p'-a_p')((w+1)x-yz)\\ &+\left(\frac{2}{p}\right)(b_p'-a_p')(w(1-x)+(y+1)(z+1)).\end{aligned}\tag{4.21}$$

When $p\equiv 3\pmod 4$, *we have*

$$\begin{aligned}&\left|x+\left(\frac{j-k+1}{p}\right)+\left(\frac{j}{p}\right)y+\left(\frac{k}{p}\right)z+\left(\frac{jk}{p}\right)w\right|_{1\leqslant j,k\leqslant (p-1)/2}\\ =\ &(w+1)x-yz+\left(\frac{2}{p}\right)((y+1)(z-1)-w(x+1))\\ &+(-1)^{(h(-p)-1)/2}\frac{p+1}{2}(wx-(y+1)z).\end{aligned}\tag{4.22}$$

Conjecture 4.10. *Let* p *be an odd prime.*

(i) *When* $p\equiv 1\pmod 4$, *for any* $\delta_1,\delta_2\in\{\pm 1\}$ *the number*

$$2\left|\left(\frac{j+k}{p}\right)+\left(\frac{j-k}{p}\right)+\delta_1\left(\frac{j^2+\delta_2k^2}{p}\right)\right|_{0\leqslant j,k\leqslant (p-1)/2}$$

is a quadratic residue modulo p.

(ii) *If* $p\equiv 3\pmod 4$, *then the number*

$$2\left|\left(\frac{j+k}{p}\right)+\left(\frac{j^2+k^2}{p}\right)\right|_{0\leqslant j,k\leqslant (p-1)/2}$$

is a quadratic residue modulo p.

5. Conjectures on determinants of the form $\{c,d\}_n=|(\frac{j^2+cjk+dk^2}{n})|_{1<j,k<n-1}$

Let p be a prime with $p\equiv 1\pmod 4$. By a classical result conjectured by Fermat and confirmed by Euler, $p=a^2+b^2$ for some $a,b\in\mathbb{Z}$ with a odd.

In view of Jacobsthal's theorem (cf. Theorem 6.2.9 of [1, p. 195]),

$$p=\bigg(\sum_{x=1}^{(p-1)/2}\left(\frac{x(x^2+1)}{p}\right)\bigg)^2+\bigg(\sum_{x=1}^{(p-1)/2}\left(\frac{x(x^2+d)}{p}\right)\bigg)^2$$

for any $d\in\mathbb{Z}$ with $(\frac{d}{p})=-1$. As $x^2\equiv-1\pmod p$ for a unique number $x\in\{1,\ldots,(p-1)/2\}$, we have

$$\frac{1}{2}\sum_{x=1}^{p-1}\left(\frac{x(x^2+1)}{p}\right)=\sum_{x=1}^{(p-1)/2}\left(\frac{x(x^2+1)}{p}\right)=\pm a\neq 0.$$

Motivated by this, we obtain the following result.

Theorem 5.1. *Let $n>1$ be an integer with $n\equiv 1\pmod 4$. Then n is not a sum of two squares if and only if*

$$\sum_{x=0}^{n-1}\left(\frac{x(x^2+1)}{n}\right)=0.$$

Proof. Write $n=p_1^{a_1}\cdots p_k^{a_k}$, where $p_1,\ldots,p_k$ are distinct odd primes and $a_1,\ldots,a_k\in\mathbb{N}$. Applying the Chinese Remainder Theorem, we obtain

$$\begin{aligned}\sum_{x=0}^{n-1}\left(\frac{x(x^2+1)}{n}\right)&=\sum_{x_1=0}^{p_1^{a_1}-1}\cdots\sum_{x_k=0}^{p_k^{a_k}-1}\prod_{s=1}^{k}\left(\frac{x_s(x_s^2+1)}{p_s^{a_s}}\right)\\&=\prod_{s=1}^{k}\sum_{x_s=0}^{p_s^{a_s}-1}\left(\frac{x_s(x_s^2+1)}{p_s}\right)^{a_s}\\&=\prod_{s=1}^{k}\sum_{q_s=0}^{p_s^{a_s-1}-1}\sum_{r_s=0}^{p_s-1}\left(\frac{(p_sq_s+r_s)((p_sq_s+r_s)^2+1)}{p_s}\right)^{a_s}\\&=\prod_{s=1}^{k}p_s^{a_s-1}\sum_{x=1}^{p_s-1}\left(\frac{x(x^2+1)}{p_s}\right)^{a_s}.\end{aligned}$$

If $p_s\equiv 1\pmod 4$, then $|\{1\leqslant x\leqslant p-1:\ x^2\equiv-1\pmod{p_s}\}|=2$ and hence

$$\sum_{x=1}^{p_s-1}\left(\frac{x(x^2+1)}{p_s}\right)^{a_s}\equiv p_s-3\not\equiv 0\pmod 4.$$

When $p_s\equiv 3\pmod 4$ and $2\mid a_s$, we have

$$\sum_{x=1}^{p_s-1}\left(\frac{x(x^2+1)}{p}\right)^{a_s}=p_s-1\neq 0.$$

When $p_s\equiv 3\pmod 4$ and $2\nmid a_s$, we have

$$\sum_{x=1}^{p_s-1}\left(\frac{x(x^2+1)}{p}\right)^{a_s}=\sum_{x=1}^{(p_s-1)/2}\left(\left(\frac{x(x^2+1)}{p_s}\right)+\left(\frac{-x(x^2+1)}{p_s}\right)\right)=0.$$

In view of the above, we see that $\sum_{x=0}^{n-1}(\frac{x(x^2+1)}{n})=0$ if and only if $p_s\equiv 3\pmod 4$ and $2\nmid a_s$ for some $s=1,\ldots,k$. By a known result (cf. [3, p. 279]), n is not a sum of two squares if and only if for some prime divisor p of n we have $p\equiv 3\pmod 4$ and $\mathrm{ord}_p(n)\equiv 1\pmod 2$, where $\mathrm{ord}_p(n)$ is the p-adic order of n at the prime p. So the desired result follows. □

Let $n>1$ be an odd integer, and let $c,d\in\mathbb{Z}$. The author [8] investigated the new kinds of determinants

$$[c,d]_n=\left|\left(\frac{j^2+cjk+dk^2}{n}\right)\right|_{0\leqslant j,k\leqslant n-1}$$

and

$$(c,d)_n=\left|\left(\frac{j^2+cjk+dk^2}{n}\right)\right|_{1\leqslant j,k\leqslant n-1},$$

where $(\frac{\cdot}{n})$ denotes the Jacobi symbol. Some conjectures on such determinants were later confirmed by D. Krachun, F. Petrov, Z.-W. Sun and M. Vsemirnov [4]. Now we introduce the new determinant

$$\{c,d\}_n=\left|\left(\frac{j^2+cjk+dk^2}{n}\right)\right|_{1<j,k<n-1}. \tag{5.1}$$

This is motivated by the standard proof of Wilson's theorem; in fact, for any prime $p>3$ we have $\prod_{1<k<p-1}k\equiv 1\pmod p$ since $\{2,\ldots,p-2\}$ can be partitioned as $\{x_1,y_1\}\cup\cdots\cup\{x_{(p-3)/2},y_{(p-3)/2}\}$ with $x_ky_k\equiv 1\pmod p$ for all $k=1,\ldots,(p-3)/2$.

Conjecture 5.1. (i) *For any positive integer* $n\equiv 1\pmod 4$ *which is not a sum of two squares, we have* $\{3,2\}_n=0$.

(ii) *For any positive integer* $n\equiv 3\pmod 4$, *we have* $\frac{\varphi(n)}{2}\mid\{3,2\}_n$, *where* φ *is Euler's totient function.*

(iii) *For any positive integer* $n\equiv 3\pmod 8$, *we have*

$$\{3,2\}_n=\frac{\varphi(n)}{2}x^2$$

for some $x\in\mathbb{Z}$.

Remark 5.1. We have verified this for all positive odd integers $n<2000$. By [4, Corollary 1.1], $(3,2)_n=[3,2]_n=0$ for any positive integer $n\equiv 3\pmod 4$.

Conjecture 5.2. (i) *We have* $\{2,2\}_p=0$ *for any prime* $p\equiv 13,19\pmod{24}$.

(ii) *We have* $\{2,2\}_p\equiv 0\pmod p$ *for any prime* $p\equiv 17,23\pmod{24}$.

Remark 5.2. We have verified this conjecture for odd primes $p<2000$.

Conjecture 5.3. (i) *We have* $\{4,2\}_n=\{8,8\}_n=0$ *for any positive integer* $n\equiv 5\pmod 8$.

(ii) *We have* $\{3,3\}_n=0$ *for any positive integer* $n\equiv 5\pmod{12}$.

Remark 5.3. We have verified this conjecture for positive odd integers $n<2000$. By [4, Corollary 1.1], $(4,2)_n=(8,8)_n=(3,3)_n=0$ for any positive integer $n\equiv 3\pmod 4$.

Conjecture 5.4. *We have* $\{42,-7\}_n = \{21,112\}_n = 0$ *for any positive integer* $n \equiv 1 \pmod 4$ *with* $(\frac{n}{7}) = -1$.

Remark 5.4. We have verified this conjecture for positive odd integers $n < 2000$. By [4, Theorem 1.1(iv)], $(42,-7)_n = (21,112)_n = 0$ for any positive integer n with $(\frac{n}{7}) = -1$.

Conjecture 5.5. (i) *Let* $n > 3$ *be an odd integer. Then* $\{2,3\}_n \equiv 0 \pmod n$. *Moreover,* $\{2,3\}_n \equiv 0 \pmod{n^2}$ *if* $n \not\equiv \pm 1 \pmod{12}$.

(ii) *For any odd integer* $n > 7$, *we have* $\{6,15\}_n \equiv 0 \pmod n$.

Remark 5.5. We have verified this conjecture for positive odd integers $n < 2000$. The author [8, Conjecture 4.8] conjectured that $(2,3)_n \equiv 0 \pmod{n^2}$ for each odd integer $n > 3$, and that $(6,15)_n \equiv 0 \pmod{n^2}$ for any odd integer $n > 5$.

Conjecture 5.6. (i) *For any positive integer* $n \equiv 13, 17 \pmod{20}$ *which is a sum of two squares, we have* $\{5,5\}_n = 0$.

(ii) *Let* $n > 1$ *be an odd integer. We have* $(\frac{\{5,5\}_n}{n}) = 0$ *if* $n \equiv 11, 19 \pmod{20}$, *or* $n \equiv 9 \pmod{60}$ *and* $n > 69$.

Remark 5.6. We have verified this conjecture for positive odd integers $n < 2000$. By [4, Theorem 1.4], $[5,5]_p = 0$ for any prime $p \equiv 13, 17 \pmod{20}$.

Conjecture 5.7. (i) *For any positive integer* $n \equiv 5 \pmod{12}$ *which is a sum of two squares, we have* $\{10,9\}_n = 0$.

(ii) *We have* $\{10,9\}_p \equiv 0 \pmod p$ *for any prime* $p \equiv 11 \pmod{12}$.

Remark 5.7. We have verified this conjecture for $n, p < 2000$. By [4, Theorem 1.4], $(10,9)_p = 0$ for any prime $p \equiv 5 \pmod{12}$.

Conjecture 5.8. (i) *For any positive integer* $n \equiv 13, 17 \pmod{24}$ *which is a sum of two squares, we have* $\{8,18\}_n = 0$.

(ii) *We have* $\{8,18\}_p \equiv 0 \pmod{p^2}$ *for any prime* $p \equiv 19 \pmod{24}$.

(iii) *We have* $\{8,18\}_p \equiv 0 \pmod p$ *for any prime* $p \equiv 23 \pmod{24}$.

Remark 5.8. We have verified this conjecture for $n, p < 2000$. In 2018, the author [7] conjectured that $[8,18]_p = 0$ for any prime $p \equiv 13, 17 \pmod{24}$, which was confirmed by Michael Stoll (cf. [7]) by using advanced tools such as elliptic curves with complex multiplication by $\mathbb{Z}[\sqrt{-6}]$ and ℓ-adic Tate modules.

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