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SOME DETERMINANTS INVOLVING QUADRATIC RESIDUES MODULO PRIMES

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Abstract. In this paper we evaluate several determinants involving quadratic residues modulo primes. For example, for any prime $p > 3$ with $p \equiv 3 \pmod 4$ and $a, b \in \mathbb{Z}$ with $p \nmid ab$, we prove that

$$\det\left[1+\tan\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant\frac{p-1}{2}} = \begin{cases}-2^{(p-1)/2}p^{(p-3)/4} & \text{if } (\frac{ab}{p})=1,\\ p^{(p-3)/4} & \text{if } (\frac{ab}{p})=-1,\end{cases}$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol. We also pose some conjectures for further research.

1. Introduction

Let p be an odd prime, and let $(\frac{\cdot}{p})$ be the Legendre symbol. Let d be any integer. Sun [7] introduced the determinants

$$S(d,p)=\det\left[\left(\frac{j^2+dk^2}{p}\right)\right]_{1\leqslant j,k\leqslant(p-1)/2}$$

and

$$T(d,p)=\det\left[\left(\frac{j^2+dk^2}{p}\right)\right]_{0\leqslant j,k\leqslant(p-1)/2},$$

and determined the Legendre symbols

$$\left(\frac{S(d,p)}{p}\right) \text{ and } \left(\frac{T(d,p)}{p}\right).$$

Namely, the author [7, Theorem 1.2] showed that

$$\left(\frac{S(d,p)}{p}\right)=\begin{cases}(\frac{-1}{p}) & \text{if } (\frac{d}{p})=1,\\ 0 & \text{if } (\frac{d}{p})=-1,\end{cases}$$

and

$$\left(\frac{T(d,p)}{p}\right)=\begin{cases}(\frac{2}{p}) & \text{if } (\frac{d}{p})=1,\\ 1 & \text{if } (\frac{d}{p})=-1.\end{cases}$$

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D. Grinberg, the author and L. Zhao [2] proved that if $p>3$ then

$$\det\left[(j^2+dk^2)\left(\frac{j^2+dk^2}{p}\right)\right]_{0\leqslant j,k\leqslant(p-1)/2}\equiv 0 \pmod p.$$

For any positive integer n with $(p-1)/2\leqslant n\leqslant p-1$, we introduce the determinants

$$S_n(d,p)=\det\left[(j^2+dk^2)^n\right]_{1\leqslant j,k\leqslant(p-1)/2} \tag{1.1}$$

and

$$T_n(d,p)=\det\left[(j^2+dk^2)^n\right]_{0\leqslant j,k\leqslant(p-1)/2}. \tag{1.2}$$

Note that

$$S_{(p-1)/2}(d,p)\equiv S(d,p) \pmod p,\quad T_{(p-1)/2}(d,p)\equiv T(d,p) \pmod p,$$

and

$$T_{(p+1)/2}(d,p)\equiv \det\left[(j^2+dk^2)\left(\frac{j^2+dk^2}{p}\right)\right]_{0\leqslant j,k\leqslant(p-1)/2} \pmod p.$$

When $p>3$ and $p\nmid d$, the author [7, Conjecture 4.5(iii)] conjectured that

$$\left(\frac{S_{(p+1)/2}(d,p)}{p}\right)=\begin{cases}(\frac{d}{p})^{(p-1)/4} & \text{if } p\equiv 1 \pmod 4,\\ (\frac{d}{p})^{(p+1)/4}(-1)^{(h(-p)-1)/2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

where $h(-p)$ denotes the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$; this was confirmed by H.-L. Wu, Y.-F. She and L.-Y. Wang [12] in 2022.

Theorem 1.1. *Let $p>3$ be a prime, and let $d\in\mathbb{Z}$.*

(i) *Let $\bar S(d,p)$ be the determiant obtained from $\det[(\frac{j^2+dk^2}{p})]_{1\leqslant j,k\leqslant(p-1)/2}$ by replacing all the entries in the first row by 1. If $(\frac{d}{p})=1$, then*

$$\bar S(d,p)=-S(d,p).$$

When $(\frac{d}{p})=-1$, we have

$$\bar S(d,p)=\frac{2}{p-1}T(d,p)=\frac{p-1}{2}\det\left[\left(\frac{j^2+dk^2}{p}\right)\right]_{2\leqslant j,k\leqslant(p-1)/2}. \tag{1.3}$$

(ii) *For any integer n with $(p-1)/2<n<p-1$, we have*

$$T_n(d,p)\equiv 0 \pmod p. \tag{1.4}$$

Remark 1.1. Part (ii) of Theorem 1.1 extends [2, Theorem 1.1].

For any prime $p\equiv 3 \pmod 4$, Sun [7] proved that

$$S_{p-2}(1,p)\equiv\det\left[\frac{1}{j^2+k^2}\right]_{1\leqslant j,k\leqslant(p-1)/2}\equiv\left(\frac{2}{p}\right) \pmod p.$$

In contrast with this, we get the following result.

Theorem 1.2. *Let p be an odd prime, and let $d\in\mathbb{Z}$ with $(\frac{-d}{p})=-1$.*

(i) *We have*

$$\left(\frac{S_{p-2}(d,p)}{p}\right)=\left(\frac{2}{p}\right). \tag{1.5}$$

Moreover,

$$\det\left[\frac{1}{j^2+dk^2}\right]_{1\leqslant j,k\leqslant (p-1)/2}\equiv\begin{cases}d^{(p-1)/4}\pmod p & \text{if } p\equiv 1\pmod 4,\\ (-1)^{(p+1)/4}\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases} \tag{1.6}$$

(ii) *We have*

$$\left(\frac{S_{p-3}(d,p)}{p}\right)=\frac{1-(\frac{-1}{p})}{2}. \tag{1.7}$$

Moreover, when $p\equiv 3\pmod 4$ we have

$$\det\left[\frac{1}{(j^2+dk^2)^2}\right]_{1\leqslant j,k\leqslant (p-1)/2}\equiv\frac{1}{4}\prod_{r=1}^{(p-3)/4}\left(r+\frac14\right)^2\pmod p. \tag{1.8}$$

Let p be an odd prime, and let $a,b\in\mathbb{Z}$ with $p\nmid ab$. The author [10] introduced

$$T_p^{(0)}(a,b,x)=\det\left[x+\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant (p-1)/2} \tag{1.9}$$

and

$$T_p^{(1)}(a,b,x)=\det\left[x+\tan\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant (p-1)/2}, \tag{1.10}$$

and simply denote $T_p^{(0)}(a,b,0)$ and $T_p^{(1)}(a,b,0)$ by $T_p^{(0)}(a,b)$ and $T_p^{(1)}(a,b)$, respectively. When $p>3$ and $p\equiv 3\pmod 4$, the author [10, Theorem 1.1(ii)] proved that

$$T_p^{(0)}(a,b,x)=\begin{cases}2^{(p-1)/2}p^{(p+1)/4} & \text{if } (\frac{ab}{p})=1,\\ p^{(p+1)/4} & \text{if } (\frac{ab}{p})=-1.\end{cases} \tag{1.11}$$

When $p\equiv 1\pmod 4$, by [10, Theorem 1.1(i)] we have

$$\begin{aligned}T_p^{(1)}(a,b,x)&=T_p^{(1)}(a,b)\\ &=\begin{cases}\left(\frac{2c}{p}\right)p^{(p-3)/4}\varepsilon_p^{(\frac{a}{p})(2-(\frac{2}{p}))h(p)} & \text{if } p\mid b-ac^2 \text{ with } c\in\mathbb{Z},\\ \pm 2^{(p-1)/2}p^{(p-3)/4} & \text{if } (\frac{ab}{p})=-1,\end{cases}\end{aligned} \tag{1.12}$$

where ε_p and $h(p)$ are the fundamental unit and the class number of the real quadratic field $\mathbb{Q}(\sqrt{p})$, respectively. As a supplement to [10, Theorem 1.1], we obtain the following result.

Theorem 1.3. *Let $p>3$ be a prime, and let $a,b\in\mathbb{Z}$ with $p\nmid ab$.*

(i) *Assume that $p\equiv 1\pmod 4$. If $(\frac{ab}{p})=1$ and $ac^2\equiv b$ with $c\in\mathbb{Z}$, then*

$$T_p^{(0)}(a,b,x)=\left(\frac{2c}{p}\right)p^{(p+1)/4}\varepsilon_p^{(\frac{a}{p})((\frac{2}{p})-2)h(p)}x. \tag{1.13}$$

If $(\frac{ab}{p})=-1$, then

$$T_p^{(1)}(a,b)=-\delta(ab,p)2^{(p-1)/2}p^{(p-3)/4} \tag{1.14}$$

and

$$T_p^{(0)}(a,b,x)=pT_p^{(1)}(a,b)x=-\delta(ab,p)2^{(p-1)/2}p^{(p+1)/4}x, \tag{1.15}$$

where

$$\delta(c,p)=\begin{cases}1 & \text{if } c^{(p-1)/4}\equiv\frac{p-1}{2}!\pmod p,\\ -1 & \text{otherwise.}\end{cases} \tag{1.16}$$

(ii) *Suppose that $p\equiv 3\pmod 4$. Then*

$$T_p^{(1)}(a,b,x)=\begin{cases}-2^{(p-1)/2}p^{(p-3)/4}x & \text{if } (\frac{ab}{p})=1,\\ p^{(p-3)/4}x & \text{if } (\frac{ab}{p})=-1.\end{cases} \tag{1.17}$$

Remark 1.2. In light of Theorem 1.3 and [10, Theorem 1.1], for any prime $p>3$ and $a,b\in\mathbb{Z}$ with $p\nmid ab$, we have completely determined the exact values of $T_p^{(0)}(a,b,x)$ and $T_p^{(1)}(a,b,x)$.

Let $p>3$ be a prime, and let $a,b\in\mathbb{Z}$ with $(\frac{-ab}{p})=-1$. Define

$$C_p(a,b,x)=\det\left[x+\cot\pi\frac{aj^2+bk^2}{p}\right]_{1\leqslant j,k\leqslant(p-1)/2}. \tag{1.18}$$

By [10, Theorem 1.3],

$$C_p(a,b,x)=\begin{cases}T_p^{(1)}(a,b)/(-p)^{(p-1)/4}=\pm 2^{(p-1)/2}/\sqrt{p} & \text{if } p\equiv 1\pmod 4,\\ (-1)^{(h(-p)+1)/2}(\frac{a}{p})2^{(p-1)/2}/\sqrt{p} & \text{if } p\equiv 3\pmod 4.\end{cases} \tag{1.19}$$

In the case $p\equiv 1\pmod 4$, with the aid of (1.14) we have

$$C_p(a,b,x)=(-1)^{(p+3)/4}\delta(ab,p)\frac{2^{(p-1)/2}}{\sqrt{p}}. \tag{1.20}$$

Now we state our last two theorems.

Theorem 1.4. *Let $p>3$ be a prime, and let $a,b\in\mathbb{Z}$ with $p\nmid ab$. Let $\bar{T}_p(a,b,x)$ denote the determinant obtained from*

$$T_p^{(0)}(a,b,x)=\det\left[x+\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant(p-1)/2}$$

via replacing all the entries in the first row by 1.

(i) *Suppose that* $p\equiv 1\pmod 4$. *If* $(\frac{ab}{p})=1$ *and* $ac^2\equiv b\pmod p$ *with* $c\in\mathbb{Z}$, *then*

$$\bar T_p(a,b,x)=\left(\frac{2c}{p}\right)p^{(p-1)/4}. \tag{1.21}$$

If $(\frac{ab}{p})=-1$, *then*

$$\bar T_p(a,b,x)=-\delta(ab,p)2^{(p-1)/2}p^{(p-1)/4}\varepsilon_p^{(\frac{2a}{p})h(p)}. \tag{1.22}$$

(ii) *When* $p\equiv 3\pmod 4$, *we have*

$$\bar T_p(a,b,x)=(-1)^{\frac{p+1}{4}+\frac{h(-p)+1}{2}}\left(\frac{a}{p}\right)2^{(1+(\frac{ab}{p}))\frac{p-1}{4}}p^{(p-1)/4}. \tag{1.23}$$

Theorem 1.5. *Let* $p>3$ *be a prime, and let* $a,b\in\mathbb{Z}$ *with* $(\frac{-ab}{p})=-1$. *Let* $\bar C_p(a,b,x)$ *denote the determinant of the matrix* $[c_{jk}]_{0\leqslant j,k\leqslant (p-1)/2}$, *where*

$$c_{jk}=\begin{cases}1 & \text{if } j=0,\\ x+\cot\pi(aj^2+bk^2)/p & \text{if } j>0.\end{cases}$$

Then

$$\bar C_p(a,b,x)=\frac{2^{(p-1)/2}}{\sqrt p}\times\begin{cases}(-1)^{(p+3)/4}\delta(ab,p)\varepsilon_p^{(\frac{a}{p})2h(p)} & \text{if } p\equiv 1\pmod 4,\\ (-1)^{(h(-p)-1)/2}(\frac{a}{p}) & \text{if } p\equiv 3\pmod 4.\end{cases} \tag{1.24}$$

We are going to prove Theorems 1.1-1.2 in the next section. Based on two auxiliary theorems in Section 3, we will prove Theorem 1.3 in Section 4. Our proofs of Theorems 1.4-1.5 will be given in Section 5. In Section 6 we pose several conjectures on determinants for further research.

2. Proofs of Theorems 1.1-1.2

We need the following known lemma (cf. [1, p. 58]).

Lemma 2.1. *Let* p *be an odd prime, and let* $a,b,c\in\mathbb{Z}$ *with* $p\nmid a$. *Then*

$$\sum_{x=0}^{p-1}\left(\frac{ax^2+bx+c}{p}\right)=\begin{cases}(p-1)(\frac{a}{p}) & \text{if } p\mid b^2-4ac,\\ -(\frac{a}{p}) & \text{if } p\nmid b^2-4ac.\end{cases}$$

`Proof of Theorem 1.1(i).` By Lemma 2.1, for each $k=1,\ldots,(p-1)/2$ we have

$$\sum_{j=1}^{(p-1)/2}\left(\frac{j^2+dk^2}{p}\right)=\frac12\left(\sum_{j=0}^{p-1}\left(\frac{j^2+dk^2}{p}\right)-\left(\frac{dk^2}{p}\right)\right)=-\frac{1+(\frac{d}{p})}{2}. \tag{2.1}$$

Thus, for the determinant $T(d,p)=|(\frac{j^2+dk^2}{p})|_{0\leqslant j,k\leqslant (p-1)/2}$, if we add all the rows below the second row to the second row, then the second row becomes

$$\left(\frac{p-1}{2},-\frac{1+(\frac{d}{p})}{2},\ldots,-\frac{1+(\frac{d}{p})}{2}\right)$$

while the first row is

$$\left(0, \left(\frac{d}{p}\right), \dots, \left(\frac{d}{p}\right)\right).$$

Therefore, in the case $(\frac{d}{p}) = -1$, we have

$$T(d,p) = \frac{p-1}{2}\bar{S}(d,p).$$

Now we consider the case $(\frac{d}{p}) = 1$. If we add to the second row of $T(d,p)$ all the other rows, then the second row becomes $(\frac{p-1}{2}, 0, \dots, 0)$ by (2.1) while the first row is $(0, 1, \dots, 1)$. It follows that

$$T(d,p) = -\frac{p-1}{2}\bar{S}(d,p).$$

By [7, (1.20)],

$$T(d,p) = \frac{p-1}{2}S(d,p).$$

Combining the last two equalities, we get $\bar{S}(d,p) = -S(d,p)$.

By Lemma 2.1, for any $j = 1, \dots, (p-1)/2$ we have

$$\sum_{k=1}^{(p-1)/2}\left(\frac{j^2+dk^2}{p}\right) = \frac{1}{2}\left(\sum_{k=0}^{p-1}\left(\frac{j^2+dk^2}{p}\right) - 1\right) = -\frac{(\frac{d}{p})+1}{2}. \tag{2.2}$$

Suppose that $(\frac{d}{p}) = -1$. If we add to the first column of $\bar{S}(d,p)$ all the other columns, then the first column turns out to be $(\frac{p-1}{2}, 0, \dots, 0)^T$ by (2.2). Therefore,

$$\bar{S}(d,p) = \frac{p-1}{2}\det\left[\left(\frac{j^2+dk^2}{p}\right)\right]_{2\leqslant j,k\leqslant (p-1)/2}.$$

Combining the above, we have completed our proof of Theorem 1.1(i). □

`Proof of Theorem 1.1(ii)`. Let $k \in \{1, \dots, (p-1)/2\}$. In view of the binomial theorem, we have

$$\begin{aligned}\sum_{j=1}^{(p-1)/2}(j^2+dk^2)^n &= \sum_{j=1}^{(p-1)/2}\sum_{r=0}^{n}\binom{n}{r}j^{2r}(dk^2)^{n-r}\\ &\equiv \sum_{r=0}^{n}\binom{n}{r}(dk^2)^{n-r}\frac{1}{2}\sum_{j=1}^{(p-1)/2}\left(j^{2r}+(p-j)^{2r}\right)\\ &= \frac{1}{2}\sum_{r=0}^{n}\binom{n}{r}(dk^2)^{n-r}\sum_{j=1}^{p-1}j^{2r} \pmod{p}.\end{aligned}$$

By a well known result (cf. [3, Section 15.2, Lemma 2]),

$$\sum_{j=1}^{p-1}j^{2r} \equiv \begin{cases}-1 \pmod{p} & \text{if } p-1 \mid 2r,\\ 0 \pmod{p} & \text{otherwise.}\end{cases}$$

As $(p-1)/2 < n < p-1$, for $r \in \{0, \ldots, n\}$ we have

$$p-1 \mid 2r \iff \frac{p-1}{2} \mid r \iff r=0 \text{ or } r=\frac{p-1}{2}.$$

Thus

$$\begin{aligned} 2\sum_{j=1}^{(p-1)/2}(j^2+dk^2)^n &\equiv -\sum_{r\in\{0,(p-1)/2\}}\binom{n}{r}(dk^2)^{n-r} \\ &= -(dk^2)^n - \binom{n}{(p-1)/2}(dk^2)^{n-(p-1)/2} \pmod p \end{aligned}$$

and hence

$$\left(1+\left(\frac{d}{p}\right)\binom{n}{(p-1)/2}\right)(dk^2)^n + 2\sum_{j=1}^{(p-1)/2}(j^2+dk^2)^n \equiv 0 \pmod p.$$

As $p-1 \nmid 2n$, we also have

$$\left(1+\left(\frac{d}{p}\right)\binom{n}{(p-1)/2}\right)(d0^2)^n + 2\sum_{j=1}^{(p-1)/2}(j^2+d0^2)^n \equiv \sum_{j=1}^{p-1} j^{2n} \equiv 0 \pmod p.$$

Combining this with the last paragraph, we see that

$$\left(1+\left(\frac{d}{p}\right)\binom{n}{(p-1)/2}\right)t_{0k} + 2\sum_{j=1}^{(p-1)/2} t_{jk} \equiv 0 \pmod p$$

for all $k=0,\ldots,(p-1)/2$, where $t_{jk}=(j^2+dk^2)^n$. Therefore

$$T_n(d,p) = \det[t_{jk}]_{0\leqslant j,k\leqslant (p-1)/2} \equiv 0 \pmod p$$

as desired. □

The following well known result can be found in the survey [4, (5.5)].

Lemma 2.2 (Cauchy)**.** *We have*

$$\det\left[\frac{1}{x_j+y_k}\right]_{1\leqslant j,k\leqslant n} = \frac{\prod_{1\leqslant j<k\leqslant n}(x_j-x_k)(y_j-y_k)}{\prod_{j=1}^n\prod_{k=1}^n(x_j+y_k)}. \tag{2.3}$$

Let p be an odd prime. In view of Wilson's theorem,

$$\prod_{k=1}^{(p-1)/2} k(p-k) = (p-1)! \equiv -1 \pmod p$$

and hence

$$\left(\frac{p-1}{2}!\right)^2 \equiv (-1)^{(p+1)/2} \pmod p. \tag{2.4}$$

By [7, (1.5)], we have

$$\prod_{1\leqslant j<k\leqslant (p-1)/2}(k^2-j^2) \equiv \begin{cases} -\frac{p-1}{2}! \pmod p & \text{if } p\equiv 1 \pmod 4, \\ 1 \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{2.5}$$

Therefore

$$\prod_{1\leqslant j<k\leqslant (p-1)/2}(k^2-j^2)^2\equiv(-1)^{(p+1)/2}\pmod p. \tag{2.6}$$

`Proof of Theorem 1.2(i)`. Let $n=(p-1)/2$. By Lemma 2.2 and (2.6), we have

$$\begin{aligned}\det\left[\frac{1}{j^2+dk^2}\right]_{1\leqslant j,k\leqslant n}&=\frac{\prod_{1\leqslant j<k\leqslant n}(k^2-j^2)(dk^2-dj^2)}{\prod_{j=1}^n\prod_{k=1}^n(j^2+dk^2)}\\&=\frac{d^{n(n-1)/2}}{\Pi}\prod_{1\leqslant j<k\leqslant n}(k^2-j^2)^2\\&\equiv(-1)^{n+1}\frac{d^{n(n-1)/2}}{\Pi}\pmod p,\end{aligned}$$

where

$$\Pi:=\prod_{k=1}^n\left(k^{2n}\prod_{j=1}^n\left(\frac{j^2}{k^2}+d\right)\right)\equiv\prod_{k=1}^n\prod_{x=1}^n(x^2+d)\pmod p.$$

Note that

$$\prod_{x=1}^n(x^2+d)\equiv(-1)^{n+1}2\pmod p$$

by [7, Lemma 3.1]. Thus

$$\Pi\equiv((-1)^{n+1}2)^n=2^n\equiv\left(\frac{2}{p}\right)=(-1)^{(p^2-1)/8}\pmod p.$$

If $p\equiv1\pmod 4$, then $2\mid n$ and hence

$$d^{n(n-1)/2}=(d^n)^{n/2-1}d^{n/2}\equiv\left(\frac{d}{p}\right)^{n/2-1}d^{n/2}=(-1)^{n/2-1}d^{(p-1)/4}\pmod p.$$

If $p\equiv3\pmod 4$, then $2\nmid n$ and hence

$$d^{n(n-1)/2}=(d^n)^{(n-1)/2}\equiv\left(\frac{d}{p}\right)^{(n-1)/2}=1\pmod p.$$

Therefore

$$\frac{d^{n(n-1)/2}}{\Pi}\equiv\begin{cases}-d^{(p-1)/4}\pmod p&\text{if }p\equiv1\pmod 4,\\(-1)^{(p+1)/4}\pmod p&\text{if }p\equiv3\pmod 4.\end{cases}$$

Combining this with the first paragraph in the proof, we immediately obtain the congruence (1.6), which clearly implies (1.5). This concludes the proof. □

Recall that the permanent of an $n\times n$ matrix $A=[a_{j,k}]_{1\leqslant j,k\leqslant n}$ over a field is given by

$$\mathrm{per}(A)=\mathrm{per}[a_{j,k}]_{1\leqslant j,k\leqslant n}=\sum_{\sigma\in S_n}\prod_{j=1}^n a_{j,\sigma(j)}.$$

Lemma 2.3. *Let p be an odd prime, and let $d \in \mathbb{Z}$ with $(\frac{-d}{p}) = -1$. Then*

$$\begin{aligned} &\mathrm{per}\left[\frac{1}{j^2+dk^2}\right]_{1\leqslant j,k\leqslant (p-1)/2} \\ &\equiv \begin{cases} 0 \pmod p & \text{if } p\equiv 1 \pmod 4, \\ \frac{(-1)^{(p+1)/4}}{4}\prod_{r=1}^{(p-3)/4}(r+\frac14)^2 \pmod p & \text{if } p\equiv 3 \pmod 4. \end{cases} \end{aligned} \tag{2.7}$$

Proof. Let g be a primitive root modulo p, and set $n = (p-1)/2$. Then those g^{2k} $(k = 1, \ldots, n)$ are incongruent quadratic residues modulo p. Thus

$$\begin{aligned} \mathrm{per}\left[\frac{1}{j^2+dk^2}\right]_{1\leqslant j,k\leqslant n} &= \frac{1}{\prod_{k=1}^n k^2}\mathrm{per}\left[\frac{1}{1+dk^2/j^2}\right]_{1\leqslant j,k\leqslant n} \\ &\equiv \frac{1}{(n!)^2}\mathrm{per}\left[\frac{1}{1+dg^{2(j-k)}}\right]_{1\leqslant j,k\leqslant n} \\ &\equiv (-1)^{n-1}\prod_{r=1}^n\left(\frac{n(-d)^n}{1-(-d)^n}+r\right) \pmod p \end{aligned}$$

by (2.4) and [9, Theorem 1.3(i)]. As $(-d)^n \equiv (\frac{-d}{p}) = -1 \pmod p$, from the above we get

$$\mathrm{per}\left[\frac{1}{j^2+dk^2}\right]_{1\leqslant j,k\leqslant n} \equiv (-1)^{n-1}\prod_{r=1}^n\left(r+\frac14\right) \pmod p. \tag{2.8}$$

If $p \equiv 1 \pmod 4$, then $r + 1/4 \equiv 0 \pmod p$ for $r = (p-1)/4$. When $p \equiv 3 \pmod 4$, we have

$$\begin{aligned} \prod_{r=1}^n\left(r+\frac14\right) &= \left(\frac{p-1}{2}+\frac14\right)\prod_{r=1}^{(p-3)/4}\left(r+\frac14\right)\left(\frac{p-1}{2}-r+\frac14\right) \\ &\equiv \frac{(-1)^{(p+1)/4}}{4}\prod_{r=1}^{(p-3)/4}\left(r+\frac14\right)^2 \pmod p. \end{aligned}$$

Therefore (2.8) implies the desired congruence (2.7). □

The following result due to Borchardt can be found in [5].

Lemma 2.4. *We have*

$$\det\left[\frac{1}{(x_j+y_k)^2}\right]_{1\leqslant j,k\leqslant n} = \det\left[\frac{1}{x_j+y_k}\right]_{1\leqslant j,k\leqslant n}\mathrm{per}\left[\frac{1}{x_j+y_k}\right]_{1\leqslant j,k\leqslant n}. \tag{2.9}$$

`Proof of Theorem` 1.2(ii). Combining (1.6), and Lemmas 2.3 and 2.4, we immediately get the desired results. □

3. Two Auxiliary Theorems

Our first auxiliary theorem is as follows.

Theorem 3.1. *Let p be an odd prime, and let $k,m\in\mathbb{Z}^+=\{1,2,3,\ldots\}$ with $km=p-1$. Let G be the multiplicative group $\{r+p\mathbb{Z}:\ r=1,\ldots,p-1\}$ and let H be its subgroup $\{x^m+p\mathbb{Z}:\ x=1,\ldots,p-1\}$ of order k. Suppose that all the m distinct cosets of H in G are*

$$\{a_{1j}+p\mathbb{Z}:\ j=1,\ldots,k\},\ \ldots,\ \{a_{mj}+p\mathbb{Z}:\ j=1,\ldots,k\}$$

with $1\leqslant a_{i1}<\ldots<a_{ik}\leqslant p-1$ for all $i=1,\ldots,m$. Then

$$\begin{aligned}&\prod_{i=1}^{m}\prod_{1\leqslant s<t\leqslant k}(a_{it}-a_{is})\\&\equiv\begin{cases}(-1)^{\frac{p+1}{2}\cdot\frac{p-1}{2m}+\lfloor\frac{p-3}{4}\rfloor}\frac{p-1}{2}!\pmod p & \text{if } p\equiv 1\pmod{2m},\\(-1)^{\frac{p+1}{2}\cdot\frac{p-1-m}{2m}}\pmod p & \text{if } p\equiv 1+m\pmod{2m}.\end{cases}\end{aligned}\tag{3.1}$$

Proof. Set

$$R_m=\{1\leqslant r\leqslant p-1:\ x^m\equiv r\pmod p\text{ for some }x=1,\ldots,p-1\}.$$

Then $H=\{r+p\mathbb{Z}:\ r\in R_m\}$ and $|H|=|R_m|=(p-1)/m=k$. Note that

$$\prod_{i=1}^{m}\prod_{1\leqslant s<t\leqslant k}(a_{it}-a_{is})=\prod_{d=1}^{p-1}d^{e_d},$$

where

$$\begin{aligned}e_d:&=|\{1\leqslant x<p-d:\ \{x,x+d\}\subseteq\{a_{i1},\ldots,a_{ik}\}\text{ for some }i=1,\ldots,m\}|\\&=\left|\left\{1\leqslant x<p-d:\ \frac{x+d}{x}\equiv r\pmod p\text{ for some }r\in R_m\right\}\right|.\end{aligned}$$

Clearly,

$$\prod_{d=1}^{p-1}d^{e_d}=\prod_{d=1}^{(p-1)/2}d^{e_d}(p-d)^{e_{p-d}}\equiv(-1)^{\sum_{d=1}^{(p-1)/2}e_{p-d}}\prod_{d=1}^{(p-1)/2}d^{e_d+e_{p-d}}\pmod p.$$

For any $d\in\{1,\ldots,p-1\}$, obviously

$$\begin{aligned}e_{p-d}&=\left|\left\{1\leqslant x<d:\ \frac{x+p-d}{x}\equiv r\pmod p\text{ for some }r\in R_m\right\}\right|\\&=\left|\left\{p-d<y<p:\ 1+\frac{p-d}{p-y}\equiv r\pmod p\text{ for some }r\in R_m\right\}\right|\\&=\left|\left\{p-d\leqslant y<p:\ \frac{y+d}{y}\equiv r\pmod p\text{ for some }r\in R_m\right\}\right|\end{aligned}$$

and hence

$$e_d+e_{p-d}=\left|\left\{1\leqslant x<p:\ 1+\frac{d}{x}\equiv r\pmod p\text{ for some }r\in R_m\right\}\right|$$

$$= |\{1 < y < p : \ y \equiv r \pmod p \text{ for some } r \in R_m\}|$$
$$= |R_m| - 1 = k - 1.$$

Observe that

$$\sum_{d=1}^{(p-1)/2} e_{p-d} = \sum_{d=1}^{(p-1)/2} \left|\left\{1 \leqslant x < d : \ \frac{d-x}{x} \equiv -r \pmod p \text{ for some } r \in R_m\right\}\right|$$

coincides with

$$\left|\left\{(x,y) \in (\mathbb{Z}^+)^2 : \ x + y \leqslant \frac{p-1}{2} \text{ and } \frac{y}{x} \equiv -r \pmod p \ \text{ for some } r \in R_m\right\}\right|.$$

As H is a multiplicative group, given $x, y \in \{1, \ldots, p-1\}$ we have

$$\frac{y}{x} \equiv -r \pmod p \ \text{ for some } r \in R_m \iff \frac{x}{y} \equiv -r \pmod p \ \text{ for some } r \in R_m.$$

Therefore, $\sum_{d=1}^{(p-1)/2} e_{p-d}$ has the same parity with

$$\left|\left\{x \in \mathbb{Z}^+ : \ x + x \leqslant \frac{p-1}{2} \text{ and } \frac{x}{x} \equiv -r \pmod p \ \text{ for some } r \in R_m\right\}\right|$$
$$= \left|\left\{1 \leqslant x < \frac{p}{4} : \ p - 1 \in R_m\right\}\right| = \left|\left\{1 \leqslant x < \frac{p}{4} : \ (-1)^{(p-1)/m} = 1\right\}\right|$$
$$= \begin{cases} \lfloor (p-1)/4 \rfloor & \text{if } 2 \mid k, \\ 0 & \text{if } 2 \nmid k, \end{cases}$$

and hence

$$(-1)^{\sum_{d=1}^{(p-1)/2} e_{p-d}} = (-1)^{(k-1)\lfloor \frac{p-1}{4} \rfloor}.$$

Combining the above, we see that

$$\prod_{i=1}^{m} \prod_{1 \leqslant s < t \leqslant k} (a_{it} - a_{is}) \equiv (-1)^{(k-1)\lfloor \frac{p-1}{4} \rfloor} \prod_{d=1}^{(p-1)/2} d^{k-1} \pmod p.$$

Recall that

$$\left(\frac{p-1}{2}!\right)^2 \equiv (-1)^{(p+1)/2} \pmod p$$

by Wilson's theorem. So, by the last two congruences we immediately obtain the desired congruence (3.1). □

Theorem 3.1 in the case $m = 2$ yields the following result.

Corollary 3.1. *Let $p = 2n + 1$ be an odd prime, and write*

$$\{1, \ldots, p-1\} = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\}$$

with $a_1 < \ldots < a_n$ and $b_1 < \ldots < b_n$ such that $a_1, \ldots, a_n$ are quadratic residues modulo p, and $b_1, \ldots, b_n$ are quadratic nonresidues modulo p. Then

$$\prod_{1 \leqslant j < k \leqslant n} (a_k - a_j)(b_k - b_j) \equiv \begin{cases} -n! \pmod p & \text{if } p \equiv 1 \pmod 4, \\ 1 \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{3.2}$$

For any odd prime p and integer $a \not\equiv 0 \pmod p$, we define

$$s_p(a) = (-1)^{|\{\{j,k\}:\ 1\leqslant j<k\leqslant (p-1)/2 \text{ and } \{aj^2\}_p>\{ak^2\}_p\}|},$$

where $\{m\}_p$ denotes the least nonnegative residue of an integer m modulo p. The author [8, Theorem 1.4(i)] determined $s_p(1)$ in the case $p \equiv 3 \pmod 4$. When $p \equiv 1 \pmod 4$, H.-L. Wu [11] deduced a complicated formula for $s_p(1)$ modulo p, which involves the fundamental unit ε_p and the class numbers of the quadratic fields $\mathbb{Q}(\sqrt{\pm p})$.

Based on Corollary 3.1, we get the following result.

Lemma 3.1. *Let p be an odd prime, and let $a, b \in \mathbb{Z}$ with $(\frac{a}{p}) = 1$ and $(\frac{b}{p}) = -1$. Then*

$$s_p(a)s_p(b) = \begin{cases} (-1)^{(p+3)/4}\delta(ab,p) & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{(p-3)/4} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{3.3}$$

Proof. Let $n = (p-1)/2$, and write $\{1, \ldots, p-1\} = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\}$ with $a_1 < \ldots < a_n$ and $b_1 < \ldots < b_n$ such that $a_1, \ldots, a_n$ are quadratic residues modulo p and $b_1, \ldots, b_n$ are quadratic nonresidues modulo p. As

$$\{\{aj^2\}_p :\ j = 1, \ldots, n\} = \{a_1, \ldots, a_n\}$$

and

$$\{\{bj^2\}_p :\ j = 1, \ldots, n\} = \{b_1, \ldots, b_n\},$$

we have

$$s_p(a)s_p(b) = \prod_{1\leqslant j<k\leqslant n} \frac{\{ak^2\}_p - \{aj^2\}_p}{a_k - a_j} \times \prod_{1\leqslant j<k\leqslant n} \frac{\{bk^2\}_p - \{aj^2\}_p}{b_k - b_j}$$

and hence

$$s_p(a)s_p(b) \prod_{1\leqslant j<k\leqslant n} (a_k - a_j)(b_k - b_j)$$
$$\equiv \prod_{1\leqslant j<k\leqslant n} (ak^2 - aj^2)(bk^2 - bj^2) = (ab)^{n(n-1)/2} \prod_{1\leqslant j<k\leqslant n} (k^2 - j^2)^2 \pmod p.$$

Note that

$$(ab)^n \equiv \left(\frac{ab}{p}\right) = -1 \pmod p.$$

By (2.6) we have

$$\prod_{1\leqslant j<k\leqslant n} (k^2 - j^2)^2 \equiv (-1)^{(p+1)/2} \pmod p.$$

Therefore

$$s_p(a)s_p(b) \prod_{1\leqslant j<k\leqslant n} (a_k - a_j)(b_k - b_j)$$
$$\equiv \begin{cases} (-1)^{n/2-1}(ab)^{n/2} \times (-1) \pmod p & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{(n-1)/2} \times 1 \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Combining this with (3.2), we obtain that

$$s_p(a)s_p(b) \equiv \begin{cases} (-1)^{n/2}(ab)^{n/2}/(-n!) \pmod p & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{(n-1)/2} \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

In the case $p \equiv 1 \pmod 4$, we have

$$(ab)^n \equiv -1 \equiv (n!)^2 \pmod p$$

and hence

$$(ab)^{n/2} \equiv \pm n! \pmod p,$$

therefore

$$s_p(a)s_p(b) = (-1)^{n/2+1}\delta(ab,p) = (-1)^{(p+3)/4}\delta(ab,p).$$

This concludes our proof. □

Now we are ready to present another auxiliary theorem.

Theorem 3.2. *Let p be a prime with $p \equiv 1 \pmod 4$, and let $\zeta = e^{2\pi i/p}$. Let $a, b \in \mathbb{Z}$ with $(\frac{ab}{p}) = -1$. Then, we have*

$$\prod_{1 \leqslant j<k \leqslant (p-1)/2} (\zeta^{aj^2} - \zeta^{ak^2})(\zeta^{bj^2} - \zeta^{bk^2}) = -\delta(ab,p)p^{(p-3)/4} \tag{3.4}$$

and

$$\begin{aligned}\prod_{1 \leqslant j<k \leqslant (p-1)/2} &\left(\cot\pi\frac{aj^2}{p} - \cot\pi\frac{ak^2}{p}\right)\left(\cot\pi\frac{bj^2}{p} - \cot\pi\frac{bk^2}{p}\right) \\ &= \delta(ab,p)(-1)^{(p+3)/4}\left(\frac{2^{p-1}}{p}\right)^{(p-3)/4}.\end{aligned} \tag{3.5}$$

Remark 3.1. For any prime $p > 3$ with $p \equiv 3 \pmod 4$ and integer $a \not\equiv 0 \pmod p$, the author [7, part (ii) of Theorems 1.3-1.4] obtained closed forms for the products

$$\prod_{1 \leqslant j<k \leqslant (p-1)/2} \left(e^{2\pi i a j^2/p} - e^{2\pi i a k^2/p}\right) \text{ and } \prod_{1 \leqslant j<k \leqslant (p-1)/2} \left(\cot\pi\frac{aj^2}{p} - \cot\pi\frac{ak^2}{p}\right).$$

`Proof of Theorem` 3.2. Set $n = (p-1)/2$ and $\zeta = e^{2\pi i/p}$. By [8, (4.2) and (4.3)], we have

$$\prod_{1 \leqslant j<k \leqslant n} \sin\pi\frac{a(k^2-j^2)}{p} = (-1)^{a(n+1)n/2}\left(\frac{i}{2}\right)^{n(n-1)/2} \prod_{1 \leqslant j<k \leqslant n} (\zeta^{aj^2} - \zeta^{ak^2})$$

and

$$\frac{\prod_{1 \leqslant j<k \leqslant n} \sin\pi\frac{a(k^2-j^2)}{p}}{\prod_{1 \leqslant j<k \leqslant n}(\cot\pi\frac{aj^2}{p} - \cot\pi\frac{ak^2}{p})} = \left(\frac{p}{2^{p-1}}\right)^{(n-1)/2}(-1)^{(a-1)n/2}\varepsilon_p^{(\frac{a}{p})(1-n)h(p)}.$$

Therefore

$$\prod_{1\leqslant j<k\leqslant n}\frac{\cot\pi\frac{aj^2}{p}-\cot\pi\frac{ak^2}{p}}{\zeta^{aj^2}-\zeta^{ak^2}}=\left(\frac{2^n}{p}\right)^{(n-1)/2}i^{n(n+1)/2}\varepsilon_p^{(\frac{a}{p})(1-n)h(p)}. \tag{3.6}$$

Similarly,

$$\prod_{1\leqslant j<k\leqslant n}\frac{\cot\pi\frac{bj^2}{p}-\cot\pi\frac{bk^2}{p}}{\zeta^{bj^2}-\zeta^{bk^2}}=\left(\frac{2^n}{p}\right)^{(n-1)/2}i^{n(n+1)/2}\varepsilon_p^{(\frac{b}{p})(1-n)h(p)}. \tag{3.7}$$

Combining (3.6) with (3.7), and noting $(\frac{a}{p})+(\frac{b}{p})=0$, we deduce that

$$\prod_{1\leqslant j<k\leqslant n}\frac{(\cot\pi\frac{aj^2}{p}-\cot\pi\frac{ak^2}{p})(\cot\pi\frac{bj^2}{p}-\cot\pi\frac{bk^2}{p})}{(\zeta^{aj^2}-\zeta^{ak^2})(\zeta^{bj^2}-\zeta^{bk^2})}=(-1)^{n/2}\left(\frac{2^n}{p}\right)^{n-1}. \tag{3.8}$$

So (3.4) and (3.5) are equivalent.

By [8, Theorem 1.3(i)],

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}-\zeta^{ak^2})=t_p(a)i^{n/2}p^{(n-1)/4}\varepsilon_p^{(\frac{a}{p})\frac{h(p)}{2}}$$

for some $t_p(a)\in\{\pm1\}$. Combining this with (3.6) we see that $t_p(a)$ coincides with the sign of the product

$$\prod_{1\leqslant j<k\leqslant n}\left(\cot\pi\frac{aj^2}{p}-\cot\pi\frac{ak^2}{p}\right)$$

which should be

$$(-1)^{|\{1\leqslant j<k\leqslant n:\ \{aj^2\}_p>\{ak^2\}_p\}|}=s_p(a).$$

Thus

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}-\zeta^{ak^2})=s_p(a)i^{n/2}p^{(n-1)/4}\varepsilon_p^{(\frac{a}{p})\frac{h(p)}{2}}.$$

Similarly,

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{bj^2}-\zeta^{bk^2})=s_p(b)i^{n/2}p^{(n-1)/4}\varepsilon_p^{(\frac{b}{p})\frac{h(p)}{2}}.$$

Therefore

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}-\zeta^{ak^2})(\zeta^{bj^2}-\zeta^{bk^2})$$
$$=s_p(a)s_p(b)(-1)^{n/2}p^{(n-1)/2}=-\delta(ab,p)p^{(p-3)/4}.$$

This proves (3.4).

In view of the above, we have completed our proof of Theorem 3.2. □

4. Proof of Theorem 1.3

The following lemma is a known result (see, e.g., [8, (1.12)]).

Lemma 4.1. *For any prime $p \equiv 1 \pmod 4$ and integer $a \not\equiv 0 \pmod p$, we have*

$$\prod_{k=1}^{(p-1)/2}\left(1-e^{2\pi iak^2/p}\right)=\sqrt{p}\,\varepsilon_p^{-(\frac{a}{p})h(p)}. \tag{4.1}$$

Lemma 4.2. *Let $m,n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$ with $2\nmid n$. Let $a_k,b_k\in\mathbb{Z}$ for $k=0,1,\ldots,m$. with $a_0+b_0=0$. Then*

$$\det\left[x+\tan\pi\frac{a_j+b_k}{n}\right]_{0\leqslant j,k\leqslant m}-\det\left[\tan\pi\frac{a_j+b_k}{n}\right]_{0\leqslant j,k\leqslant m}$$
$$=x\det\left[\tan\pi\frac{a_j+b_k}{n}\right]_{1\leqslant j,k\leqslant m}\times\prod_{k=1}^{m}\left(\tan\pi\frac{a_k+b_0}{n}\times\tan\pi\frac{a_0+b_k}{n}\right). \tag{4.2}$$

Proof. Let $a_{jk}=\tan\pi(a_j+b_k)/n$ for $j,k=0,\ldots,m$. By [10, Lemma 2.1], we have

$$\det[x+a_{jk}]_{0\leqslant j,k\leqslant m}-\det[a_{jk}]_{0\leqslant j,k\leqslant m}=x\det[b_{jk}]_{1\leqslant j,k\leqslant m}, \tag{4.3}$$

where $b_{jk}=a_{jk}-a_{j0}-a_{0k}+a_{00}$. Note that $a_{00}=\tan 0=0$ and recall the known identity

$$(1-\tan x_1\times\tan x_2)\tan(x_1+x_2)=\tan x_1+\tan x_2.$$

Then we have

$$\begin{aligned}b_{jk}&=\tan\pi\frac{a_j+b_k}{n}-\tan\pi\frac{a_j+b_0}{n}-\tan\pi\frac{a_0+b_k}{n}\\&=\tan\pi\frac{a_j+b_0}{n}\times\tan\pi\frac{a_0+b_k}{n}\times\tan\pi\frac{a_j+b_k}{n}.\end{aligned}$$

Thus

$$\det[b_{jk}]_{1\leqslant j,k\leqslant m}=\det\left[\tan\pi\frac{a_j+b_k}{n}\right]_{1\leqslant j,k\leqslant m}\prod_{k=1}^{m}\left(\tan\pi\frac{a_k+b_0}{n}\times\tan\pi\frac{a_0+b_k}{n}\right).$$

Combining this with (4.3), we immediately obtain the desired identity (4.2). □

`Proof of Theorem` 1.3(i). Let $n=(p-1)/2$, and let $a_{jk}=\tan\pi(aj^2+bk^2)/p$ for $j,k=0,\ldots,n$. Set $q=n!$. By (2.4) we have $q^2\equiv-1 \pmod p$. Thus

$$\begin{aligned}T_p^{(0)}(a,b)&=\det\left[\tan\pi\frac{a(qj)^2+b(qk)^2}{p}\right]_{0\leqslant j,k\leqslant n}\\&=\det\left[-\tan\pi\frac{aj^2+bk^2}{p}\right]_{0\leqslant j,k\leqslant n}=-T_p^{(0)}(a,b)\end{aligned}$$

and hence $T_p^{(0)}(a,b)=0$ (which also follows from [10, (1.3)]).

In view of the above and Lemma 4.2, we have

$$T_p^{(0)}(a,b,x)=xT_p^{(1)}(a,b)\prod_{k=1}^n\left(\tan\pi\frac{ak^2}{p}\times\tan\pi\frac{bk^2}{p}\right).$$

For any $x\in\mathbb{Q}$ with odd denominator, clearly

$$\tan\pi x=\frac{2\sin\pi x}{2\cos\pi x}=\frac{(e^{i\pi x}-e^{-i\pi x})/i}{e^{i\pi x}+e^{-i\pi x}}=i\frac{1-e^{2\pi ix}}{1+e^{2\pi ix}}=i\frac{(1-e^{2\pi ix})^2}{1-e^{2\pi i(2x)}}.$$

In view of this and Lemma 4.1, we deduce that

$$\begin{aligned}\prod_{k=1}^n\tan\pi\frac{ak^2}{p}&=i^n\frac{\prod_{k=1}^n(1-e^{2\pi iak^2/p})^2}{\prod_{k=1}^n(1-e^{2\pi i(2a)k^2/p})}\\&==(i^2)^{n/2}\frac{(\sqrt{p}\,\varepsilon_p^{-(\frac{a}{p})h(p)})^2}{\sqrt{p}\,\varepsilon_p^{-(\frac{2a}{p})h(p)}}=(-1)^{(p-1)/4}\sqrt{p}\,\varepsilon_p^{((\frac{2}{p})-2)(\frac{a}{p})h(p)}.\end{aligned}$$

Similarly,

$$\prod_{k=1}^n\tan\pi\frac{bk^2}{p}=(-1)^{(p-1)/4}\sqrt{p}\,\varepsilon_p^{((\frac{2}{p})-2)(\frac{b}{p})h(p)}.$$

If $(\frac{ab}{p})=-1$, then

$$\prod_{k=1}^n\left(\tan\pi\frac{ak^2}{p}\times\tan\pi\frac{bk^2}{p}\right)=\sqrt{p}^2=p.$$

When $(\frac{ab}{p})=1$, we have

$$\prod_{k=1}^n\left(\tan\pi\frac{ak^2}{p}\times\tan\pi\frac{bk^2}{p}\right)=p\varepsilon_p^{2((\frac{2}{p})-2)(\frac{a}{p})h(p)}.$$

Combining the above with (1.12), we see that it suffices to prove (1.14) in the case $(\frac{ab}{p})=-1$.

Now assume $(\frac{ab}{p})=-1$ and set $\zeta=e^{2\pi i/p}$. By the proof of [10, Theorem 1.1(i)], $T_p^{(1)}(a,b)$ is the real part of

$$D_p(a,b):=\det\left[\frac{2i}{\zeta^{aj^2+bk^2}+1}\right]_{1\leqslant j,k\leqslant n},$$

and

$$D_p(a,b)=(-1)^{n/2}2^n\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}-\zeta^{ak^2})(\zeta^{-bj^2}-\zeta^{-bk^2}).$$

Since

$$\left(\frac{a(-b)}{p}\right)=\left(\frac{ab}{p}\right)=-1,$$

by Theorem 3.2 we have

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}-\zeta^{ak^2})(\zeta^{-bj^2}-\zeta^{-bk^2})=-\delta(-ab,p)p^{(p-3)/4}$$

and hence

$D_p(a,b)=(-1)^{n/2}2^n\times(-1)^{n/2+1}\delta(ab,p)p^{(p-3)/4}=-\delta(ab,p)2^{(p-1)/2}p^{(p-3)/4}.$

Therefore

$$T_p^{(1)}(a,b)=\Re(D_p(a,b))=-\delta(ab,p)2^{(p-1)/2}p^{(p-3)/4}.$$

This proves the desired (1.14).

By the above, we have completed our proof of Theorem 1.3(i). □

Lemma 4.3 (Sun [8]). *Let $p>3$ be a prime with $p\equiv 3 \pmod 4$. Let $\zeta=e^{2\pi i/p}$, and $a\in\mathbb{Z}$ with $p\nmid a$. Then*

$$\prod_{k=1}^{(p-1)/2}(1-\zeta^{ak^2})=(-1)^{(h(-p)+1)/2}\left(\frac{a}{p}\right)\sqrt{p}\,i, \tag{4.4}$$

and

$$\prod_{1\leqslant j<k\leqslant (p-1)/2}(\zeta^{aj^2}-\zeta^{ak^2}) =\begin{cases}(-p)^{(p-3)/8} & \text{if } p\equiv 3 \pmod 8,\\ (-1)^{(p+1)/8+(h(-p)-1)/2}(\frac{a}{p})p^{(p-3)/8}i & \text{if } p\equiv 7 \pmod 8,\end{cases} \tag{4.5}$$

where $h(-p)$ denotes the class number of the quadratic field $\mathbb{Q}(\sqrt{-p})$. Also,

$$\prod_{1\leqslant j<k\leqslant (p-1)/2}(\zeta^{aj^2}+\zeta^{ak^2})=1, \tag{4.6}$$

The following result can be found in [10, Lemma 2.5].

Lemma 4.4 (Sun [10]). *Let $p>3$ be a prime with $p\equiv 3 \pmod 4$. Let $\zeta=e^{2\pi i/p}$, and $a,b\in\mathbb{Z}$ with $(\frac{ab}{p})=1$. Then*

$$\prod_{j=1}^{(p-1)/2}\prod_{k=1}^{(p-1)/2}(1-\zeta^{aj^2+bk^2})=(-1)^{(h(-p)-1)/2}\left(\frac{a}{p}\right)p^{(p-1)/4}i. \tag{4.7}$$

`Proof of Theorem` 1.3(ii). By [10, Lemma 2.1],

$$T_p^{(1)}(a,b,x)=c+dx$$

for some real numbers c and d not depending on x. So, it suffices to determine the value of $T_p^{(1)}(a,b,i)$.

Let $n=(p-1)/2$ and $\zeta=e^{2\pi i/p}$. Then $\prod_{k=1}^n\zeta^{k^2}=1$ since

$$\sum_{k=0}^n k^2=\frac{n(n+1)(2n+1)}{6}=\frac{p^2-1}{24}p\equiv 0 \pmod p.$$

For any integer r, clearly

$$i+\tan\pi\frac{r}{p}=i+\frac{(e^{i\pi r/p}-e^{-i\pi r/p})/(2i)}{(e^{i\pi r/p}+e^{-i\pi r/p})/2}=i-i\frac{\zeta^r-1}{\zeta^r+1}=\frac{2i}{\zeta^r+1}.$$

Thus, with the aid of Lemma 2.2, we have

$$\begin{aligned}T_p^{(1)}(a,b,i)&=\det\left[\frac{2i}{\zeta^{aj^2+bk^2}+1}\right]_{1\leqslant j,k\leqslant n}\\&=\prod_{k=1}^n\frac{2i}{\zeta^{bk^2}}\times\det\left[\frac{1}{\zeta^{aj^2}+\zeta^{-bk^2}}\right]_{1\leqslant j,k\leqslant n}\\&=\frac{2^ni(i^2)^{(n-1)/2}}{\zeta^{b\sum_{k=1}^nk^2}}\times\frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}-\zeta^{ak^2})(\zeta^{-bj^2}-\zeta^{-bk^2})}{\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2}+\zeta^{-bk^2})}\end{aligned}$$

and hence

$$T_p^{(1)}(a,b,i)=i(-1)^{(p-3)/4}2^{(p-1)/2}\times\frac{\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}-\zeta^{ak^2})(\zeta^{-bj^2}-\zeta^{-bk^2})}{\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2+bk^2}+1)}. \tag{4.8}$$

By Lemma 4.3,

$$\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}-\zeta^{ak^2})(\zeta^{-bj^2}-\zeta^{-bk^2})=\begin{cases}p^{(p-3)/4}&\text{if }p\equiv3\pmod 8,\\(\frac{ab}{p})p^{(p-3)/4}&\text{if }p\equiv7\pmod 8.\end{cases}$$

If $(\frac{ab}{p})=-1$, then $(\frac{b}{p})=(\frac{-a}{p})$ and hence

$$\begin{aligned}\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2+bk^2}+1)&=\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2-ak^2}+1)=\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2}+\zeta^{ak^2})\\&=\prod_{k=1}^n(2\zeta^{ak^2})\times\prod_{1\leqslant j<k\leqslant n}(\zeta^{aj^2}+\zeta^{ak^2})^2=2^{(p-1)/2}\end{aligned}$$

by (4.6). If $(\frac{ab}{p})=1$, then by Lemma 4.4 we have

$$\prod_{j=1}^n\prod_{k=1}^n(\zeta^{aj^2+bk^2}+1)=\prod_{j=1}^n\prod_{k=1}^n\frac{1-\zeta^{2aj^2+2bk^2}}{1-\zeta^{aj^2+bk^2}}=\frac{(\frac{2a}{p})}{(\frac{a}{p})}=\left(\frac{2}{p}\right)=(-1)^{(p+1)/4}.$$

Combining (4.8) with the last paragraph, we see that if $(\frac{ab}{p})=-1$ then

$$c+di=T_p^{(1)}(a,b,i)=i(-1)^{(p-3)/4}2^{(p-1)/2}\times\frac{(-p)^{(p-3)/4}}{2^{(p-1)/2}}=ip^{(p-3)/4}$$

and hence

$$T_p^{(1)}(a,b,x)=c+dx=p^{(p-3)/4}x.$$

Similarly, when $(\frac{ab}{p})=1$ we have

$$c+di=T_p^{(1)}(a,b,i)=i(-1)^{(p-3)/4}2^{(p-1)/2}\times\frac{p^{(p-3)/4}}{(-1)^{(p+1)/4}}=-i2^{(p-1)/2}p^{(p-3)/4}$$

and hence

$$T_p^{(1)}(a,b,x)=c+dx=-2^{(p-1)/2}p^{(p-3)/4}x.$$

This concludes our proof of Theorem 1.3(ii). □

5. Proofs of Theorems 1.4 and 1.5

`Proof of Theorem 1.4`. Note that $\bar{T}_p(a,b,x)=\det[t_{jk}]_{0\leqslant j,k\leqslant n}$, where $n=(p-1)/2$ and

$$t_{jk}=\begin{cases}1 & \text{if } j=0,\\ x+\tan\pi\frac{aj^2+bk^2}{p} & \text{if } j>0.\end{cases}$$

Let $k\in\{1,\ldots,n\}$. Clearly, $t_{0k}-t_{00}=0$. Let $\zeta=e^{2\pi i/p}$. As

$$\tan\pi y=\frac{2\sin\pi y}{2\cos\pi y}=\frac{(e^{i\pi y}-e^{-i\pi y})/i}{e^{i\pi y}+e^{-i\pi y}}=\frac{2i}{e^{2\pi i y}+1}-i$$

for all $y\in\mathbb{R}$ with $2y\not\in\{2m+1:\ m\in\mathbb{Z}\}$, for each $j=1,\ldots,n$ we have

$$\begin{aligned}t_{jk}-t_{j0}&=\frac{2i}{\zeta^{aj^2+bk^2}+1}-\frac{2i}{\zeta^{aj^2}+1}=\frac{1-\zeta^{bk^2}}{1+\zeta^{-aj^2}}\times\frac{2i}{\zeta^{aj^2+bk^2}+1}\\&=\frac{(1-\zeta^{aj^2})(1-\zeta^{bk^2})}{1-\zeta^{-2aj^2}}\times\left(i+\tan\pi\frac{aj^2+bk^2}{p}\right).\end{aligned}$$

In view of the last paragraph, via all the columns (except for the first column) of $\bar{T}_p(a,b,x)$ minus the first column, we see that

$$\bar{T}_p(a,b,x)=\det[t_{jk}-t_{j0}]_{1\leqslant j,k\leqslant n}=\frac{\prod_{k=1}^n(1-\zeta^{-ak^2})(1-\zeta^{bk^2})}{\prod_{j=1}^n(1-\zeta^{-2aj^2})}\times T_p^{(1)}(a,b,i). \tag{5.1}$$

Case 1. $p\equiv 1\pmod 4$.

In this case, by Lemma 4.1 we have

$$\begin{aligned}\frac{\prod_{k=1}^n(1-\zeta^{-ak^2})(1-\zeta^{bk^2})}{\prod_{j=1}^n(1-\zeta^{-2aj^2})}&=\frac{\sqrt{p}\,\varepsilon_p^{-(\frac{-a}{p})h(p)}\sqrt{p}\,\varepsilon_p^{-(\frac{b}{p})h(p)}}{\sqrt{p}\,\varepsilon_p^{-(\frac{-2a}{p})h(p)}}\\&=\sqrt{p}\,\varepsilon_p^{((\frac{2a}{p})-(\frac{a}{p})-(\frac{b}{p}))h(p)}\\&=\begin{cases}\sqrt{p}\,\varepsilon_p^{(\frac{a}{p})((\frac{2}{p})-2)h(p)} & \text{if } (\frac{ab}{p})=1,\\ \sqrt{p}\,\varepsilon_p^{(\frac{2a}{p})h(p)} & \text{if } (\frac{ab}{p})=-1.\end{cases}\end{aligned}$$

Combining this with (5.1), (1.12) and Theorem 1.3(i), we obtain the desired result concerning the exact value of $\bar{T}_p(a,b,x)$.

Case 2. $p\equiv 3\pmod 4$.

In this case, by Lemma 4.3 we have

$$\begin{aligned}&\frac{\prod_{k=1}^n(1-\zeta^{-ak^2})(1-\zeta^{bk^2})}{\prod_{j=1}^n(1-\zeta^{-2aj^2})}\\&=(-1)^{\frac{h(-p)+1}{2}}\left(\frac{b}{p}\right)\sqrt{p}\,i\times\frac{(\frac{-a}{p})}{(\frac{-2a}{p})}=(-1)^{\frac{h(-p)+1}{2}}\left(\frac{2b}{p}\right)\sqrt{p}\,i.\end{aligned}$$

Combining this with (5.1) and (1.17), we obtain the desired (1.23).

In view of the above, we have completed the proof of Theorem 1.4. □

Proof of Theorem 1.5. Set $n=(p-1)/2$. Let $k\in\{1,\ldots,n\}$. Clearly, $c_{0k}-c_{00}=0$. Let $\zeta=e^{2\pi i/p}$. As

$$\cot\pi y=\frac{2\cos\pi y}{2\sin\pi y}=\frac{e^{i\pi y}+e^{-i\pi y}}{(e^{i\pi y}-e^{-i\pi y})/i}=i+\frac{2i}{e^{2\pi iy}-1}\quad\text{for all } y\in\mathbb{R}\setminus\mathbb{Z},$$

for each $j=1,\ldots,n$ we have

$$\begin{aligned}c_{jk}-c_{j0}&=\frac{2i}{\zeta^{aj^2+bk^2}-1}-\frac{2i}{\zeta^{aj^2}-1}=\frac{1-\zeta^{bk^2}}{1-\zeta^{-aj^2}}\times\frac{2i}{\zeta^{aj^2+bk^2}-1}\\&=\frac{1-\zeta^{bk^2}}{1-\zeta^{-aj^2}}\times\left(-i+\cot\pi\frac{aj^2+bk^2}{p}\right).\end{aligned}$$

In view of the last paragraph, via all the columns (except for the first column) of $\bar{C}_p(a,b,x)$ minus the first column, we see that

$$\bar{C}_p(a,b,x)=\det[c_{jk}-c_{j0}]_{1\leqslant j,k\leqslant n}=\frac{\prod_{k=1}^n(1-\zeta^{bk^2})}{\prod_{j=1}^n(1-\zeta^{-aj^2})}\times C_p(a,b,-i).\quad(5.2)$$

Case 1. $p\equiv1\pmod 4$.

In this case, by Lemma 4.1 we have

$$\frac{\prod_{k=1}^n(1-\zeta^{bk^2})}{\prod_{j=1}^n(1-\zeta^{-aj^2})}=\frac{\sqrt{p}\,\varepsilon_p^{-(\frac{b}{p})h(p)}}{\sqrt{p}\,\varepsilon_p^{-(\frac{-a}{p})h(p)}}=\varepsilon_p^{2(\frac{a}{p})h(p)}.$$

Combining this with (5.2) and (1.20), we obtain

$$\bar{C}_p(a,b,x)=(-1)^{(p+3)/4}\delta(ab,p)\frac{2^{(p-1)/2}}{\sqrt{p}}\varepsilon_p^{2(\frac{a}{p})h(p)}.$$

Case 2. $p\equiv3\pmod 4$.

In this case, by Lemma 4.3 we have

$$\frac{\prod_{k=1}^n(1-\zeta^{bk^2})}{\prod_{j=1}^n(1-\zeta^{-aj^2})}=\frac{(\frac{b}{p})}{(\frac{-a}{p})}=\left(\frac{-ab}{p}\right)=-1.$$

Combining this with (5.2) and (1.19), we obtain

$$\bar{C}_p(a,b,x)=(-1)^{\frac{h(-p)-1}{2}}\left(\frac{a}{p}\right)\frac{2^{(p-1)/2}}{\sqrt{p}}.$$

In view of the above, we have completed the proof of Theorem 1.5. □

6. Some conjectures

Let p be an odd prime, and let $d\in\mathbb{Z}$ with $p\nmid d$. We first show that the determiants

$$\det\left[x+\left(\frac{j^2+dk^2}{p}\right)\right]_{1\leqslant j,k\leqslant(p-1)/2}\quad\text{and}\quad\det\left[x+\left(\frac{j^2+dk^2}{p}\right)\right]_{0\leqslant j,k\leqslant(p-1)/2}$$

can be expressed in terms of x, $S(d,p)$ and $T(d,p)$.

Suppose that $(\frac{d}{p})=1$. For any $k=1,\ldots,(p-1)/2$, we have

$$\sum_{j=1}^{(p-1)/2}\left(\left(\frac{j^2+dk^2}{p}\right)+\frac{2}{p-1}\right)=-1+\frac{p-1}{2}\times\frac{2}{p-1}=0.$$

with the aid of (2.1). Thus

$$\det\left[\left(\frac{j^2+dk^2}{p}\right)+\frac{2}{p-1}\right]_{1\leqslant j,k\leqslant(p-1)/2}=0,$$

and hence

$$\det\left[x+\left(\frac{j^2+dk^2}{p}\right)\right]_{1\leqslant j,k\leqslant(p-1)/2}=\left(1-\frac{p-1}{2}x\right)S(d,p). \tag{6.1}$$

by [10, Lemma 2.1]. Recall that $T(d,p)=\frac{p-1}{2}S(d,p)$ by [7, (1.20)]. Thus, by applying [10, Lemma 2.1] we get that

$$\begin{aligned}&\det\left[x+\left(\frac{j^2+dk^2}{p}\right)\right]_{0\leqslant j,k\leqslant(p-1)/2}\\ =&\ T(d,p)+x\det\left[\left(\frac{j^2+dk^2}{p}\right)-2\right]_{1\leqslant j,k\leqslant(p-1)/2}\\ =&\ \frac{p-1}{2}S(d,p)+x\left(1-2\times\frac{p-1}{2}\right)S(d,p).\end{aligned}$$

Therefore

$$\begin{aligned}&\det\left[x+\left(\frac{j^2+dk^2}{p}\right)\right]_{0\leqslant j,k\leqslant(p-1)/2}\\ =&\ \left(px+\frac{p-1}{2}\right)S(d,p)=\left(1+\frac{2px}{p-1}\right)T(d,p).\end{aligned} \tag{6.2}$$

Now we assume that $(\frac{d}{p})=-1$. Then $S(d,p)=0$ by [7, (1.15)], and hence

$$\det\left[x+\left(\frac{j^2+dk^2}{p}\right)\right]_{0\leqslant j,k\leqslant(p-1)/2}=T(d,p)+S(d,p)x=T(d,p) \tag{6.3}$$

with the aid of [10, Lemma 2.1]. Note that

$$1+\left(\frac{0^2+d0^2}{p}\right)=1\ \text{and}\ \ 1+\left(\frac{0^2+dk^2}{p}\right)=0\ \text{for all}\ k=1,\ldots,\frac{p-1}{2}.$$

Thus

$$\det\left[1+\left(\frac{j^2+dk^2}{p}\right)\right]_{0\leqslant j,k\leqslant(p-1)/2}=\det\left[1+\left(\frac{j^2+dk^2}{p}\right)\right]_{1\leqslant j,k\leqslant(p-1)/2},$$

and hence

$$\begin{aligned}&\det\left[x+\left(\frac{j^2+dk^2}{p}\right)\right]_{1\leqslant j,k\leqslant (p-1)/2}\\&=x\det\left[1+\left(\frac{j^2+dk^2}{p}\right)\right]_{1\leqslant j,k\leqslant (p-1)/2}=xT(d,p)\end{aligned}\tag{6.4}$$

in light of [10, Lemma 2.1] and (6.3).

Let $p>3$ be a prime, and let $d\in\mathbb{Z}$ with $(\frac{d}{p})=-1$. By (1.3),

$$T(d,p)=\left(\frac{p-1}{2}\right)^2\det\left[\left(\frac{j^2+dk^2}{p}\right)\right]_{2\leqslant j,k\leqslant (p-1)/2}.\tag{6.5}$$

If $p\equiv 3\pmod 4$, then $T(d,p)=T(-1,p)$ by [7, (1.14)], and $T(-1,p)$ is an integer square by Cayley's theorem (cf. [6, Prop. 2.2]) since it is skew-symmetric and of even order.

Conjecture 6.1. *Let p be a prime with $p\equiv 1\pmod 4$. Then, there is a positive integer t_p with $(\frac{t_p}{p})=1$ such that for any $d\in\mathbb{Z}$ with $(\frac{d}{p})=-1$, we have*

$$T(d,p)=2^{(p-3)/2}\left(\frac{p-1}{4}t_p\right)^2\sum_{x=1}^{(p-1)/2}\left(\frac{x(x^2+d)}{p}\right),\tag{6.6}$$

which has the equivalent form

$$\det\left[\left(\frac{j^2+dk^2}{p}\right)\right]_{2\leqslant j,k\leqslant (p-1)/2}=2^{(p-7)/2}t_p^2\sum_{x=1}^{(p-1)/2}\left(\frac{x(x^2+d)}{p}\right).\tag{6.7}$$

Remark 6.1. For any prime $p\equiv 1\pmod 4$ and $d\in\mathbb{Z}$ with $(\frac{d}{p})=-1$, by Jacobsthal's theorem (cf. Theorem 6.2.9 of [1, p. 195]) we have

$$p=\left(\sum_{x=1}^{(p-1)/2}\left(\frac{x(x^2+1)}{p}\right)\right)^2+\left(\sum_{x=1}^{(p-1)/2}\left(\frac{x(x^2+d)}{p}\right)\right)^2.$$

So Conjecture 6.1 is a refinement of [7, Conjecture 4.2(ii)]. We have verified Conjecture 6.1 for all primes $p<1000$ with $p\equiv 1\pmod 4$, and found that

$$\begin{gathered}t_5=t_{13}=t_{17}=1,\ t_{29}=13,\ t_{37}=3^2,\ t_{41}=2\times 3^2,\\t_{53}=131,\ t_{61}=2^4\times 3\times 11^2,\ t_{73}=2^4\times 3^3\times 19\times 109,\\t_{89}=109\times 199\times 8273\text{ and }t_{97}=2^9\times 3^2\times 47^2\times 79.\end{gathered}$$

Let p be an odd prime, and let $d\in\mathbb{Z}$ with $(\frac{-d}{p})=-1$. For the matrix $A_p=[a_{jk}]_{0\leqslant j,k\leqslant (p-1)/2}$ with

$$a_{jk}=\begin{cases}1&\text{if } j=0,\\1/(j^2+dk^2)&\text{if } j>0,\end{cases}$$

we have

$$\begin{aligned}\det A_p &= (-d)^{(p-1)/2}\det\left[\frac{1}{j^2+dk^2}\right]_{1\leqslant j,k\leqslant (p-1)/2}\\ &\equiv -\det\left[\frac{1}{j^2+dk^2}\right]_{1\leqslant j,k\leqslant (p-1)/2} \pmod p;\end{aligned}$$

this can be seen by considering each column (except the first column) minus the first column and noting that

$$\frac{1}{j^2+dk^2}-\frac{1}{j^2+d0^2}=\frac{-dk^2}{j^2(j^2+dk^2)}\quad\text{for all } j,k=1,\ldots,\frac{p-1}{2}.$$

Thus, with the aid of (1.6), we get

$$\det A_p \equiv \begin{cases} -d^{(p-1)/4} \pmod p & \text{if } p\equiv 1 \pmod 4,\\ (-1)^{(p-3)/4} \pmod p & \text{if } p\equiv 3 \pmod 4,\end{cases} \tag{6.8}$$

and hence

$$\left(\frac{\det A_p}{p}\right) = (-1)^{\lfloor (p-3)/4\rfloor} = \left(\frac{-2}{p}\right). \tag{6.9}$$

Conjecture 6.2. *Let p be a prime with $p\equiv 1 \pmod 4$, and let $d\in\mathbb{Z}$ with $(\frac{d}{p})=-1$. Then*

$$3\bar{S}_{p-2}(1,p)\equiv S_{p-2}(1,p)\equiv 2\delta(d,p)\sum_{x=1}^{(p-1)/2}\left(\frac{x(x^2+d)}{p}\right) \pmod p, \tag{6.10}$$

where $\bar{S}_{p-2}(1,p)=\det[s_{jk}]_{0\leqslant j,k\leqslant (p-1)/2}$ with

$$s_{jk}=\begin{cases}1 & \text{if } j=0,\\ (j^2+k^2)^{p-2} & \text{if } j>0.\end{cases}$$

Remark 6.2. Let $p\equiv 1 \pmod 4$ be a prime, and write $p=x^2+y^2$ with $x,y\in\mathbb{Z}^+$ and $2\mid y$. Then, for any $d\in\mathbb{Z}$ with $(\frac{d}{p})=-1$, we have $\sum_{x=1}^{(p-1)/2}\left(\frac{x(x^2+d)}{p}\right)=\pm y$ by Jacobsthal's theorem. Let $q=\frac{p-1}{2}!$. Then $(y/x)^2\equiv -1\equiv q^2 \pmod p$ and hence

$$\left(\frac{y}{p}\right)=\left(\frac{qx}{p}\right)=\left(\frac{q}{p}\right)\left(\frac{p}{x}\right)=\left(\frac{2}{p}\right)$$

with the aid of [7, Lemma 2.3]. Thus Conjecture 6.2 implies that

$$\left(\frac{S_{p-2}(1,p)}{p}\right)=\left(\frac{3\bar{S}_{p-2}(1,p)}{p}\right)=1. \tag{6.11}$$

Let $m,n\in\mathbb{Z}^+$ with n odd. For the determinant

$$D_n^{(m)} := \det\left[(j^2-k^2)^m\left(\frac{j^2-k^2}{n}\right)\right]_{1\leqslant j,k\leqslant (n-1)/2}, \tag{6.12}$$

clearly

$$\begin{aligned} D_n^{(m)} &= \det\left[(k^2-j^2)^m\left(\frac{k^2-j^2}{n}\right)\right]_{1\leqslant j,k\leqslant (n-1)/2} \\ &= \left((-1)^m\left(\frac{-1}{n}\right)\right)^{(n-1)/2} D_n^{(m)} = (-1)^{(m-1)(n-1)/2}D_n^{(m)}, \end{aligned}$$

and hence $D_n^{(m)} = 0$ when $2 \mid m$ and $4 \mid n-3$. If $2 \nmid m$ and $4 \mid n-1$, then $D_n^{(m)}$ is skew-symmetric and of even order, hence it is an integer square by Cayley's theorem.

Conjecture 6.3. *For any prime* $p \equiv 1 \pmod 4$*, we have*

$$\left(\frac{\sqrt{D_p^{(1)}}}{p}\right) = (-1)^{|\{0<k<\frac{p}{4}:\ (\frac{k}{p})=-1\}|}\left(\frac{p}{3}\right). \tag{6.13}$$

Remark 6.3. We have verified (6.13) for all primes $p < 1000$ with $p \equiv 1 \pmod 4$.

Conjecture 6.4. *For any prime* $p \equiv 1 \pmod 4$*, we have*

$$\left(\frac{\sqrt{D_p^{(3)}}}{p}\right) = (-1)^{|\{0<k<\frac{p}{4}:\ (\frac{k}{p})=-1\}|}\left(\frac{p}{4+(-1)^{(p-1)/4}}\right). \tag{6.14}$$

Remark 6.4. We have verified (6.14) for all primes $p < 1000$ with $p \equiv 1 \pmod 4$.

Conjecture 6.5. *For any positive odd integer* m*, the set*

$$E(m) = \left\{p:\ p \text{ is a prime with } 4 \mid p-1 \text{ and } p \mid D_p^{(m)}\right\}$$

is finite. In particular,

$$E(5) = \{29\},\ E(7) = \{13,\, 53\},\ E(9) = \{13,\, 17,\, 29\},\ E(11) = \{17,\, 29\}.$$

Remark 6.5. This is based on our computation. For $m = 5, 7, 9, 11$, we find those primes $p < 1000$ in $E(m)$ via `Mathematica`. We also note that $\{p \in E(13):\ p < 1000\} = \{17,\, 109,\, 401\}$.

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