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A FAMILY OF POLYNOMIALS AND RELATED CONGRUENCES AND SERIES

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Abstract. In this paper we study a family of polynomials

$$S_n^{(m)}(x) := \sum_{i,j=0}^{n} \binom{n}{i}^m \binom{n}{j}^m \binom{i+j}{i} x^{i+j} \quad (m, n = 0, 1, 2, \ldots).$$

For example, we show that

$$\sum_{k=0}^{p-1} S_k^{(0)}(x) \equiv \frac{x}{2x-1}\left(1 + \left(\frac{1-4x^2}{p}\right)\right) \pmod{p}$$

for any odd prime p and integer $x \not\equiv 1/2 \pmod{p}$, where $(\frac{\cdot}{p})$ denotes the Legendre symbol.

We also formulate some open conjectures on related congruences and series for $1/\pi$. For example, we conjecture that

$$\sum_{k=0}^{\infty} (7k+1) \frac{S_k^{(2)}(1/11)}{9^k} = \frac{5445}{104\sqrt{39}\,\pi}$$

and

$$\sum_{k=0}^{\infty} (1365k+181) \frac{S_k^{(2)}(1/18)}{16^k} = \frac{1377}{\sqrt{2}\,\pi}.$$

1. Introduction

The Apéry numbers given by

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad (n \in \mathbb{N} = \{0, 1, 2, \ldots\})$$

play important roles in R. Apéry's proof of the irrationality of $\zeta(3)$ (cf. [1, 11]). The author [13] conjectured that for any odd prime p we have

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

This was confirmed by C. Wang and the author [21] via the p-adic Gamma function.

The author [13] introduced the Apéry polynomials

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k \quad (n \in \mathbb{N}),$$

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and investigated their arithmetic properties. For example, Sun [13] proved that

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k(x)\in\mathbb{Z}[x]\quad\text{for all } n\in\mathbb{Z}^+=\{1,2,3,\ldots\}.$$

V.J.W. Guo and J. Zeng [6] confirmed a conjecture of Sun [13] which states that

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^kA_k(x)\in\mathbb{Z}[x]\quad\text{for all } n\in\mathbb{Z}^+.$$

For $m,n\in\mathbb{N}$, we introduce the polynomial

$$S_n^{(m)}(x)=\sum_{i=0}^n\sum_{j=0}^n\binom{n}{i}^m\binom{n}{j}^m\binom{i+j}{i}x^{i+j}. \tag{1.1}$$

This is motivated by A. Labelle's conjecture (cf. [7]) that

$$S_n^{(2)}(1)=A_n\ \text{ for all } n\in\mathbb{N}, \tag{1.2}$$

which has been confirmed by H. Rosengren, M. Alekseyev and A. Labelle in three different ways (see the answers in [7]).

Our first result is as follows.

Theorem 1.1. *Let $n\in\mathbb{N}$.*

(i) *For any $m\in\mathbb{Z}^+$, we have the identity*

$$S_n^{(m)}(x)=\sum_{k=0}^n\binom{n}{k}^2x^{2k}\bigg(\sum_{j=0}^{n-k}\binom{n}{j+k}^{m-1}\binom{n-k}{j}x^j\bigg)^2. \tag{1.3}$$

Also,

$$S_n^{(0)}(x)=\sum_{k=0}^n x^{2k}\bigg(\sum_{j=0}^{n-k}\binom{j+k}{k}x^j\bigg)^2. \tag{1.4}$$

(ii) *We have the identities*

$$S_n^{(1)}(x)=\sum_{k=0}^n\binom{n}{k}^2x^{2k}(1+x)^{2(n-k)}=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}x^{2k}(2x+1)^{n-k} \tag{1.5}$$

and the recurrence

$$(n+2)S_{n+2}^{(1)}(x)=(2n+3)(2x^2+2x+1)S_{n+1}^{(1)}(x)-(n+1)(2x+1)^2S_n^{(1)}(x). \tag{1.6}$$

Remark 1.1. By the Chu-Vandemonde identity (cf. [5, p. 22, (3.1)]) and Theorem 1.1(i), it is easy to see that

$$S_n^{(2)}(1)=A_n\ \text{ and }\ S_n^{(0)}(1)=\binom{2(n+1)}{n+1}-1.$$

A sequence $(P_n(q))_{n\geq0}$ of polynomials with integer coefficients is said to be *q-log-convex* if for each positive integer n all the coefficients of the polynomial

$$P_{n-1}(q)P_{n+1}(q)-P_n(q)^2\in\mathbb{Z}[q]$$

are nonnegative. In 2010, W.Y.C. Chen, R.L. Tang, L.X.W. Wang and A.L.B. Yang [2] proved that the sequence $(\sum_{k=0}^n\binom{n}{k}^2q^k)_{n\geq0}$ is q-log-convex, which was a previous conjecture of L.L. Liu and Y. Wang [8]. This, together with (1.5), implies that the sequence $(S_n^{(1)}(q))_{n\geq0}$ is q-log-convex. For $\beta_n(q)=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}q^k$, the author's conjecture (cf. [17, Conjecture 4.7]) on the q-log-convexity of the sequence $(\beta_n(q))_{n\geq 0}$ remains open.

Based on our computation, we formulate the following conjecture.

Conjecture 1.1. (i) *The sequence* $(\sum_{k=0}^n \binom{n}{k}^3 q^k)_{n\geq 8}$ *is q-log-convex. Also, for each integer $m\geq 2$, the sequence* $(\sum_{k=0}^n \binom{n+k}{k}^m q^k)_{n\geq 0}$ *is q-log-convex.*

(ii) *The sequences* $(S_n^{(2)}(q))_{n\geq 0}$ *and* $(S_n^{(3)}(q))_{n\geq 2}$ *are both q-log-convex.*

Now we state our second result.

Theorem 1.2. *Let p be any odd prime, and let $x\in\mathbb{Z}$. If $x\not\equiv 1/2 \pmod p$, then*

$$\sum_{k=0}^{p-1} S_k^{(0)}(x) \equiv \frac{x}{2x-1}\left(1+\left(\frac{1-4x^2}{p}\right)\right) \pmod p, \tag{1.7}$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol. When $x\equiv 1/2 \pmod p$, we have

$$\sum_{k=0}^{p-1} S_k^{(0)}(x) \equiv -\delta_{p,3} \pmod p. \tag{1.8}$$

Theorem 1.2 with $x\in\{(p-1)/2, 2\}$ yields the following corollary.

Corollary 1.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} S_k^{(0)}\left(-\frac{1}{2}\right) \equiv \frac{1}{4} \pmod p. \tag{1.9}$$

If $p>3$, then we have

$$\sum_{k=0}^{p-1} S_k^{(0)}(2) \equiv \frac{2}{3}\left(1+\left(\frac{p}{3}\right)\left(\frac{p}{5}\right)\right) \pmod p. \tag{1.10}$$

Now we give our last theorem.

Theorem 1.3. (i) *For any $n\in\mathbb{Z}^+$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1} S_k^{(1)}(x) \in \mathbb{Z}[x(x+1)] \tag{1.11}$$

and

$$\frac{(6,n)}{n}\sum_{k=0}^{n-1} kS_k^{(1)}(x) \in \mathbb{Z}[x(x+1)], \tag{1.12}$$

where $(6,n)$ is the greatest common divisor of 6 and n.

(ii) *Let p be any odd prime. Then*

$$\frac{1}{p}\sum_{k=0}^{p-1} S_k^{(1)}(x) \equiv 1-(x^{p-1}-1)((x+1)^{p-1}-1) \pmod{p\mathbb{Z}_p[x]}, \tag{1.13}$$

where $\mathbb{Z}_p$ is the ring of p-adic integers.

The classical Ramanujan-type series for $1/\pi$ (cf. [12]) have the form

$$\sum_{k=0}^{\infty}(ak+b)\frac{c_k}{m^k} = \frac{\sqrt{d}}{\pi},$$

where a, b and $m \neq 0$ are integers, d is a positive rational number, and c_k (with $k \in \mathbb{N}$) is one of the following products:

$$\binom{2k}{k}^3,\ \binom{2k}{k}^2\binom{3k}{k},\ \binom{2k}{k}^2\binom{4k}{2k},\ \binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}.$$

One may consult S. Cooper [3] for an introduction to Ramanujan-type series. In 1997 van Hamme [20] realized that classical Ramanujan-type serie have corresponding p-adic congruences.

The Apéry numbers are related to series for $1/\pi$. In 2002 T. Sato announced the identity

$$\sum_{k=0}^{\infty}(20k+10-3\sqrt{5})\frac{A_k}{((\sqrt{5}+1)/2)^{12k}} = \frac{20\sqrt{3}+9\sqrt{15}}{6\pi}.$$

Motivated by this and the fact that $S_n^{(2)}(1) = A_n$, we seek new series for $1/\pi$ in the form

$$\sum_{k=0}^{\infty}(ak+b)\frac{S_k^{(2)}(c)}{m^k} = \frac{\sqrt{d}}{\pi},$$

where a, b, m are integers with $m \neq 0$, and c and $d > 0$ are rational numbers. For this purpose, we utilize the author's philosophy for series for $1/\pi$ stated in [14] and [18]. As a result, we make the following conjecture.

Conjecture 1.2. *We have*

$$\sum_{k=0}^{\infty}(7k+1)\frac{S_k^{(2)}(1/11)}{9^k} = \frac{5445}{104\sqrt{39}\,\pi} = \frac{1815\sqrt{39}}{1352\pi} \tag{1.14}$$

and

$$\sum_{k=0}^{\infty}(1365k+181)\frac{S_k^{(2)}(1/18)}{16^k} = \frac{1377}{\sqrt{2}\,\pi}. \tag{1.15}$$

Remark 1.2. The identities (1.14) and (1.15) are motivated by Conjectures 4.20 and 4.21 respectively, and we have checked them numerically via `Mathematica`.

We are going to present our proofs of Theorems 1.1-1.2 and Theorem 1.3 in Sections 2 and 3, respectively. In Section 4, we collect our conjectures on p-adic congruences involving the polynomials $S_n^{(2)}(x)$ ($n \in \mathbb{N}$).

2. Proofs of Theorems 1.1 and 1.2

As noted by Labelle [7], the Chu-Vandermonde identity yields that

$$\binom{i+j}{i} = \sum_{k=0}^{\min\{i,j\}}\binom{i}{k}\binom{j}{k} \quad \text{for all } \ i, j \in \mathbb{N}. \tag{2.1}$$

`Proof of Theorem 1.1`. (i) Let $m\in\mathbb{Z}^+$. In light of (2.1), we have

$$\begin{aligned} S_n^{(m)}(x) &= \sum_{i=0}^n\sum_{j=0}^n\binom{n}{i}^m\binom{n}{j}^m\sum_{k=0}^{\min\{i,j\}}\binom{i}{k}\binom{j}{k}x^{i+j} \\ &= \sum_{k=0}^n\sum_{i=k}^n\binom{n}{i}^m\binom{i}{k}x^i\sum_{j=k}^n\binom{n}{j}^m\binom{j}{k}x^j \\ &= \sum_{k=0}^n\bigg(\sum_{i=k}^n\binom{n}{i}^{m-1}\binom{n}{k}\binom{n-k}{i-k}x^i\bigg)^2 \\ &= \sum_{k=0}^n\binom{n}{k}^2x^{2k}\bigg(\sum_{j=0}^{n-k}\binom{n}{j+k}^{m-1}\binom{n-k}{j}x^j\bigg)^2. \end{aligned}$$

This proves (1.3). Similarly, by using (2.1) we get

$$\begin{aligned} S_n^{(0)}(x) &= \sum_{i=0}^n\sum_{j=0}^n\sum_{k=0}^{\min\{i,j\}}\binom{i}{k}\binom{j}{k}x^{i+j} \\ &= \sum_{k=0}^n\sum_{i=k}^n\binom{i}{k}x^i\sum_{j=k}^n\binom{j}{k}x^j \\ &= \sum_{k=0}^n\bigg(\sum_{j=0}^{n-k}\binom{j+k}{k}x^{j+k}\bigg)^2 \end{aligned}$$

and hence (1.4) holds.

(ii) Now we come to prove (1.5). Applying (1.3) with $m=1$, we get

$$S_n^{(m)}(x)=\sum_{k=0}^n\binom{n}{k}^2x^{2k}\bigg(\sum_{j=0}^{n-k}\binom{n-k}{j}x^j\bigg)^2=\sum_{k=0}^n\binom{n}{k}^2x^{2k}(1+x)^{2(n-k)}.$$

The Legendre polynomial of degree n is given by

$$P_n(z)=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}\left(\frac{z-1}{2}\right)^k.$$

It is well known (cf. [5, p. 38, (3.134)]) that

$$2^nP_n(z)=\sum_{k=0}^n\binom{n}{k}^2(z+1)^k(z-1)^{n-k}.$$

Thus

$$\begin{aligned} &(2x+1)^nP_n\left(\frac{2x^2+2x+1}{2x+1}\right) \\ =&\ \frac{(2x+1)^n}{2^n}\sum_{k=0}^n\binom{n}{k}^2\left(\frac{2(x+1)^2}{2x+1}\right)^k\left(\frac{2x^2}{2x+1}\right)^{n-k} \\ =&\ \sum_{k=0}^n\binom{n}{k}^2(x+1)^{2k}(x^2)^{n-k}=\sum_{k=0}^n\binom{n}{k}^2x^{2k}(1+x)^{2(n-k)}. \end{aligned}$$

Note that

$$(2x+1)^nP_n\left(\frac{2x^2+2x+1}{2x+1}\right)=(2x+1)^n\sum_{k=0}^n\binom nk\binom{n+k}k\left(\frac{x^2}{2x+1}\right)^k$$
$$=\sum_{k=0}^n\binom nk\binom{n+k}k x^{2k}(2x+1)^{n-k}.$$

Combining the last two paragraphs, we immediately obtain (1.5). Applying the Zeilberger algorithm (cf. [10]), we get the desired recursion formula (1.6) from (1.5). This ends our proof. □

For any odd prime p, we have

$$\binom{2k}{k}=(-4)^k\binom{-1/2}{k}\equiv(-4)^k\binom{(p-1)/2}{k}\pmod p \tag{2.2}$$

for every $k=0,\ldots,(p-1)/2$. Thus the following lemma follows easily from the binomial theorem.

Lemma 2.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}x^k\equiv(1-4x)^{(p-1)/2}\pmod p \tag{2.3}$$

and

$$\sum_{k=0}^{(p-1)/2}k\binom{2k}{k}x^k\equiv 2x(1-4x)^{(p-3)/2}\pmod p. \tag{2.4}$$

Lemma 2.2. *Let $k\in\mathbb{Z}^+$. Then*

$$\frac2k\sum_{k/2\leqslant i\leqslant k}i\binom ki=2^{k-1}+\binom{2\lfloor k/2\rfloor}{\lfloor k/2\rfloor}. \tag{2.5}$$

Proof. Observe that

$$\frac2k\sum_{k/2\leqslant i\leqslant k}i\binom ki=2\sum_{k/2\leqslant i\leqslant k}\binom{k-1}{i-1}=2\sum_{\lfloor(k-1)/2\rfloor\leqslant j\leqslant k-1}\binom{k-1}{j}$$
$$=\sum_{j=\lfloor(k-1)/2\rfloor}^{k-1}\left(\binom{k-1}{j}+\binom{k-1}{k-1-j}\right)$$
$$=\sum_{i=0}^{k-1}\binom{k-1}{i}+\binom{2\lfloor k/2\rfloor}{\lfloor k/2\rfloor}=2^{k-1}+\binom{2\lfloor k/2\rfloor}{\lfloor k/2\rfloor}.$$

This proves the desired (2.5). □

Proof of Theorem 1.2. Observe that

$$\begin{aligned}\sum_{n=0}^{p-1} S_n^{(0)}(x) &= \sum_{n=0}^{p-1}\sum_{i,j=0}^{n}\binom{i+j}{i}x^{i+j}\\ &= \sum_{i,j=0}^{p-1}\binom{i+j}{i}x^{i+j}\sum_{n=\max\{i,j\}}^{p-1}1 = \sum_{i,j=0}^{p-1}\frac{(i+j)!}{i!j!}x^{i+j}(p-\max\{i,j\})\\ &\equiv -\sum_{\substack{i,j=0\\ i+j\leqslant p-1}}^{p-1}\binom{i+j}{i}x^{i+j}\max\{i,j\} \pmod p.\end{aligned}$$

Thus

$$\sum_{k=0}^{p-1} S_k^{(0)}(x) \equiv -\sum_{k=1}^{p-1}x^k\sum_{i=0}^{k}\binom{k}{i}\max\{i,k-i\} \pmod p. \tag{2.6}$$

For any positive integer k, we have

$$\begin{aligned}\sum_{i=0}^{k}\binom{k}{i}\max\{i,k-i\} =& \sum_{k/2\leqslant i\leqslant k}\binom{k}{i}i+\sum_{k/2<j\leqslant k}\binom{k}{k-j}j\\ =&2\sum_{k/2\leqslant i\leqslant k}\binom{k}{i}i-\begin{cases}\frac{k}{2}\binom{k}{k/2} & \text{if } 2\mid k,\\ 0 & \text{if } 2\nmid k.\end{cases}\end{aligned}$$

Combining this with (2.6) and (2.5), we see that $\sum_{k=0}^{p-1}S_k^{(0)}(x)$ is congruent to

$$\begin{aligned}&-\sum_{k=1}^{p-1}x^k k\left(2^{k-1}+\binom{2\lfloor k/2\rfloor}{\lfloor k/2\rfloor}\right)+\sum_{\substack{k=1\\ 2\mid k}}^{p-1}x^k\frac{k}{2}\binom{k}{k/2}\\ =&\ -x\sum_{k=1}^{p-1}k(2x)^{k-1}-\sum_{k=1}^{p-1}kx^k\binom{2\lfloor k/2\rfloor}{\lfloor k/2\rfloor}+\sum_{k=1}^{(p-1)/2}k\binom{2k}{k}x^{2k}\end{aligned}$$

modulo p. Note that

$$\sum_{k=0}^{p}kx^k\binom{2\lfloor k/2\rfloor}{\lfloor k/2\rfloor} = \sum_{k=0}^{(p-1)/2}\binom{2k}{k}\left(2kx^{2k}+(2k+1)x^{2k+1}\right).$$

Therefore, with the aid of Lemma 2.1, we have

$$\begin{aligned}&\sum_{k=0}^{p-1}S_k^{(0)}(x)+x\sum_{k=1}^{p-1}k(2x)^{k-1}\\ \equiv&\ -(2x+1)\sum_{k=0}^{(p-1)/2}k\binom{2k}{k}x^{2k}-x\sum_{k=0}^{(p-1)/2}\binom{2k}{k}x^{2k}\\ \equiv&\ -(2x+1)2x^2(1-4x^2)^{(p-3)/2}-x(1-4x^2)^{(p-1)/2}\\ \equiv&\begin{cases}(\frac{2x^2}{2x-1}-x)(\frac{1-4x^2}{p})=\frac{x}{2x-1}(\frac{1-4x^2}{p}) \pmod p & \text{if } x\not\equiv\frac12 \pmod p,\\ -\delta_{p,3} \pmod p & \text{if } x\equiv\frac12 \pmod p.\end{cases}\end{aligned}$$

If $x \equiv 1/2 \pmod p$, then

$$\sum_{k=1}^{p-1} k(2x)^{k-1} \equiv \sum_{k=1}^{p-1} k = \frac{p(p-1)}{2} \equiv 0 \pmod p.$$

Since

$$\sum_{k=1}^{p-1} kt^{k-1} = \frac{d}{dt}\sum_{k=0}^{p-1} t^k = \frac{d}{dt}\left(\frac{t^p-1}{t-1}\right) = \frac{pt^{p-1}}{t-1} - \frac{t^p-1}{(t-1)^2},$$

when $x \not\equiv \frac{1}{2} \pmod p$ we obtain that

$$\sum_{k=1}^{p-1} k(2x)^{k-1} = \frac{p(2x)^{p-1}}{2x-1} - \frac{(2x)^p-1}{(2x-1)^2} \equiv -\frac{1}{2x-1} \pmod p$$

with the help of Fermat's little theorem. So, by the above we have the desired result. □

3. Proof of Theorem 1.3

For $n \in \mathbb{N}$, we define the generalized central trinomial coefficient

$$T_n(b,c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k \in \mathbb{Z}[b,c], \tag{3.1}$$

which is the coefficient of x^n in the expansion of $(x^2+bx+c)^n$. It is known (cf. [9]) that

$$(n+2)T_{n+2}(b,c) = (2n+3)bT_{n+1}(b,c) - (n+1)(n^2-4c)T_n(b,c)$$

for all $n \in \mathbb{N}$. For congruences involving $T_n(b,c)$, the reader may consult [15] The author [16, 18, 19] found nine types of series for $1/\pi$ involving generalized central trinomial coefficients.

Lemma 3.1. *For any $n \in \mathbb{N}$, we have*

$$S_n^{(1)}(x) = T_n\left(2x^2+2x+1, x^2(x+1)^2\right) \in \mathbb{Z}[x(x+1)]. \tag{3.2}$$

Proof. For a polynomial $P(z)$ in z, let $[z^n]P(z)$ denote the coefficient of z^n in the expansion of $P(z)$. Then

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}x^{2k}((1+x)^2)^{n-k} \\
&= [y^n](1+x^2y)^n(1+(1+x)^2y)^n \\
&= [y^n](1+(2x^2+2x+1)y+x^2(x+1)^2y^2)^n \\
&= [y^n](y^{-2}+(2x^2+2x+1)y^{-1}+x^2(x+1)^2)y^{2n} \\
&= [z^n](z^2+(2x^2+2x+1)z+x^2(x+1)^2) \\
&= T_n(2x^2+2x+1, x^2(x+1)^2 \\
&= \sum_{k=0}^{\lfloor n/2 \rfloor}\binom{n}{2k}\binom{2k}{k}(2x(x+1)+1)^{n-2k}(x(x+1))^{2k}.
\end{aligned}$$

Combining this with (1.5), we immediately obtain the desired (3.2). □

Lemma 3.2. *Let $n \in \mathbb{Z}^+$. Then*

$$\frac{2c}{n}\sum_{k=0}^{n-1} T_k(b,c^2)(b-2c)^{n-1-k} = -T_n(b,c^2) + (b+2c)T_{n-1}(b,c^2) \tag{3.3}$$

and

$$\begin{aligned}&\frac{12c^2}{n}\sum_{k=0}^{n-1} kT_k(b,c^2)(b-2c)^{n-1-k} - 4c^2\sum_{k=0}^{n-1} T_k(b,c^2)(b-2c)^{n-1-k}\\ &= (b+4c)T_n(b,c^2) - (b+2c)^2T_{n-1}(b,c).\end{aligned} \tag{3.4}$$

Proof. By Lemmas 3.1 and 3.2 of [15], both (3.3) and (3.4) hold for all $b, c \in \mathbb{Z}$. As $T_k(b,c^2)$ with $k \in \mathbb{N}$ is a polynomial in b and c, we do have (3.3) and (3.4) with b and c as variables. This ends the proof. □

`Proof of Theorem 1.3`. (i) Let $n \in \mathbb{Z}^+$. In view of Lemma 3.1 and (3.1), by taking $b = 2x^2 + 2x + 1$ and $c = x(x+1)$ in (3.3) we get

$$\begin{aligned}&\frac{2x(x+1)}{n}\sum_{k=0}^{n-1} S_k^{(1)}(x)\\ &= (2x+1)^2T_{n-1}(2x(x+1)+1, x^2(x+1)^2) - T_n(2x(x+1)+1, x^2(x+1)^2)\end{aligned}$$

and hence

$$\begin{aligned}&\frac{2x(x+1)}{n}\sum_{k=0}^{n-1} S_k^{(1)}(x)\\ &= (4x(x+1)+1)\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1}{2k}\binom{2k}{k}(2x(x+1)+1)^{n-1-2k}(x(x+1))^{2k}\\ &\quad - \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}(2x(x+1)+1)^{n-2k}(x(x+1))^{2k}.\end{aligned} \tag{3.5}$$

Thus (1.11) holds provided that

$$\frac{1}{2y}\left(\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1}{2k}\binom{2k}{k}y^{2k} - \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}y^{2k}\right) \in \mathbb{Z}[y]. \tag{3.6}$$

Since $\binom{2k}{k} = 2\binom{2k-1}{k-1}$ for all $k \in \mathbb{Z}^+$, we see that (3.6) does hold. So we have (1.11).

Similarly, by taking $b = 2x^2 + 2x + 1$ and $c = x(x+1)$ in (3.4) we obtain

$$\begin{aligned}&\frac{12x^2(x+1)^2}{n}\sum_{k=0}^{n-1} kS_k^{(1)}(x) - 4x^2(x+1)^2\sum_{k=0}^{n-1} S_k^{(1)}(x)\\ &= (6x(x+1)+1)T_n(2x(x+1)+1, x^2(x+1)^2)\\ &\quad - (2x+1)^4T_{n-1}(2x(x+1)+1, x^2(x+1)^2)\\ &= (6x(x+1)+1)\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}(2x(x+1)+1)^{n-2k}(x(x+1))^{2k}\\ &\quad - (2x+1)^4\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1}{2k}\binom{2k}{k}(2x(x+1)+1)^{n-1-2k}(x(x+1))^{2k}.\end{aligned}$$

Note that

$$(2x+1)^4 = (4x(x+1)+1)^2 = 16x^2(x+1)^2 + 8x(x+1) + 1$$

and

$$\frac{\binom{2k}{k}(x(x+1))^{2k}}{2x^2(x+1)^2} \in \mathbb{Z}[x(x+1)] \quad \text{for all } k \in \mathbb{Z}^+.$$

Also,

$$\begin{aligned}&(6x(x+1)+1)(2x(x+1)+1)^n - (8x(x+1)+1)(2x(x+1)+1)^{n-1}\\ =&\ 6x(x+1)(2x(x+1)+1)^n - 6x(x+1)(2x(x+1)+1)^{n-1}\\ =&\ 2x^2(x+1)^2P(x(x+1))\end{aligned}$$

for some polynomial $P(z) \in \mathbb{Z}[z]$. Thus, by the above, we see that

$$\frac{6}{n}\sum_{k=0}^{n-1} kS_k^{(1)}(x) - 2\sum_{k=0}^{n-1} S_k^{(1)}(x) \in \mathbb{Z}[x(x+1)].$$

Combining this with (1.11) we obtain

$$\frac{6}{n}\sum_{k=0}^{n-1} kS_k^{(1)}(x) \in \mathbb{Z}[x(x+1)].$$

By Lemma 3.1, $S_k^{(1)}(x) \in \mathbb{Z}[x(x+1)]$ for all $k \in \mathbb{N}$. Note also that $(6,n) = 6a + nb$ for some $a, b \in \mathbb{Z}$. Therefore we have the desired (1.12).

(ii) Taking $n = p$ in (3.5) we get the identity

$$\begin{aligned}&\frac{1}{p}\sum_{k=0}^{p-1} S_k^{(1)}(x)\\ =&\ \frac{1}{2x(x+1)}\sum_{k=0}^{(p-1)/2} f_p(k,x)\binom{2k}{k}((2x(x+1)+1)^{p-1-2k}(x(x+1))^{2k},\end{aligned}$$

where

$$f_p(k,x) = (4x(x+1)+1)\binom{p-1}{2k} - (2x(x+1)+1)\binom{p}{2k}.$$

Note that $f_p(0,x) = 4x(x+1)+1-(2x(x+1)+1) = 2x(x+1)$. So

$$\begin{aligned}&\frac{1}{p}\sum_{k=0}^{p-1} S_k^{(1)}(x) - (2x(x+1)+1)^{p-1}\\ =&\ \sum_{k=1}^{(p-1)/2} f_p(k,x)\frac{\binom{2k}{k}}{2}((2x(x+1)+1)^{p-1-2k}(x(x+1))^{2k-1}.\end{aligned}$$

If $1 \leqslant k \leqslant (p-1)/2$, then $\binom{p-1}{2k} \equiv (-1)^{2k} = 1 \pmod p$ and $\binom{p}{2k} \equiv 0 \pmod p$, hence $f_p(k,x) \equiv 4x(x+1)+1 \pmod{p\mathbb{Z}[x(x+1)]}$. Recall that (2.2) holds for each $k = 1, \ldots, (p-1)/2$. So

$$\frac{1}{p}\sum_{k=0}^{p-1} S_k^{(1)}(x) - (2x(x+1)+1)^{p-1} \equiv \frac{4x(x+1)+1}{2}\Sigma \pmod{p\mathbb{Z}[x(x+1)]},$$

where

$$\begin{aligned}\Sigma &:= \sum_{k=1}^{(p-1)/2}\binom{(p-1)/2}{k}(-4)^k((2x(x+1)+1)^{p-1-2k}(x(x+1))^{2k-1}\\&= \frac{1}{x(x+1)}\sum_{k=1}^{(p-1)/2}\binom{(p-1)/2}{k}\left((2x(x+1)+1)^2\right)^{(p-1)/2-k}(-4x^2(x+1)^2)^k\\&= \frac{1}{x(x+1)}\left(((2x(x+1)+1)^2-4x^2(x+1)^2)^{(p-1)/2}-(2x(x+1)+1)^{p-1}\right)\\&= \frac{1}{x(x+1)}\left(4x(x+1)+1)^{(p-1)/2}-(2x(x+1)+1)^{p-1}\right)\in 2\mathbb{Z}[x(x+1)].\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{p}\sum_{k=0}^{p-1}S_k^{(1)}(x) &\equiv (2x(x+1)+1)^{p-1}+\frac{(2x+1)^2}{2x(x+1)}\left((2x+1)^{p-1}-(2x(x+1)+1)^{p-1}\right)\\&= \frac{(2x+1)^{p+1}-(2x(x+1)+1)^p}{2x(x+1)}\\&\equiv \frac{(2x+1)^{p+1}-(2x(x+1))^p-1^p}{2x(x+1)}\\&\equiv \frac{(2x+1)^{p+1}-1}{2x(x+1)}-x^{p-1}(x+1)^{p-1} \pmod{p\mathbb{Z}[x(x+1)]}.\end{aligned}$$

Observe that

$$\begin{aligned}\frac{(2x+1)^{p+1}-1}{x(x+1)} &= \left(\frac{1}{x}+\frac{1}{x+1}\right)(x+(x+1))^p+\frac{1}{x+1}-\frac{1}{x}\\&= \left(\frac{1}{x}+\frac{1}{x+1}\right)(x^p+(x+1)^p)\\&\quad+\left(\frac{1}{x}+\frac{1}{x+1}\right)\sum_{k=1}^{p-1}\binom{p}{k}x^k(x+1)^{p-k}+\frac{1}{x+1}-\frac{1}{x}\\&\equiv x^{p-1}+(x+1)^{p-1}+\frac{x^p+1}{x+1}+\frac{(x+1)^p-1}{x}\\&\equiv 2x^{p-1}+(x+1)^{p-1}+\frac{x^p+1}{x+1} \pmod{p\mathbb{Z}[x]}\end{aligned}$$

and

$$(x+1)^{p-1}=\sum_{k=0}^{p-1}\binom{p-1}{k}x^k\equiv\sum_{k=0}^{p-1}(-x)^k=\frac{x^p+1}{x+1} \pmod{p\mathbb{Z}[x]}.$$

Therefore

$$\begin{aligned}\frac{1}{p}\sum_{k=0}^{p-1}S_k^{(1)}(x) &\equiv x^{p-1}+(x+1)^{p-1}-x^{p-1}(x+1)^{p-1}\\&= 1-(x^{p-1}-1)((x+1)^{p-1}-1) \pmod{p\mathbb{Z}_p[x]}.\end{aligned}$$

This proves (1.13).

In view of the above, we have completed the proof of Theorem 1.3. □

Let $p>3$ be a prime. We can also determine $\frac{1}{p}\sum_{k=0}^{p-1}kS_k^{(1)}(x)$ modulo $p\mathbb{Z}_p[x]$. In fact, taking $n=p$ in the second paragraph of our proof of Theorem 1.3(i), we get

$$\begin{aligned}&\frac{12}{p}x^2(x+1)^2\sum_{k=0}^{p-1}kS_k^{(1)}(x)-4x^2(x+1)^2\sum_{k=0}^{p-1}S_k^{(1)}(x)\\=&\,((6x(x+1)+1)(2x(x+1)+1)-(2x+1)^4)(2x(x+1)+1)^{p-1}\\&+\sum_{k=1}^{(p-1)/2}\binom{2k}{k}(2x(x+1)+1)^{p-1-2k}(x(x+1))^{2k}g_p(k,x)\end{aligned}$$

and hence

$$\sigma_p:=\frac{3}{p}\sum_{k=0}^{p-1}kS_k^{(1)}(x)-\sum_{k=0}^{p-1}S_k^{(1)}(x)+(2x(x+1)+1)^{p-1}\tag{3.7}$$

coincides with

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k-1}{k-1}}{2}(2x(x+1)+1)^{p-1-2k}(x(x+1))^{2(k-1)}g_p(k,x),$$

where

$$\begin{aligned}g_p(k,x)&=(6x(x+1)+1)(2x(x+1)+1)\binom{p}{2k}-(2x+1)^4\binom{p-1}{2k}\\&\equiv\binom{p}{2k}-\binom{p-1}{2k}+(1-(2x+1)^4)\binom{p-1}{2k}\\&\equiv\binom{p-1}{2k-1}+1-(4x(x+1)+1)^2\\&\equiv p-(4x(x+1)+1)^2\ (\mathrm{mod}\ 2p\mathbb{Z}[x(x+1)])\end{aligned}$$

for each $k=1,\ldots,(p-1)/2$. (Note that $\binom{p-1}{2k-1}=\frac{p-1}{2k-1}\binom{p-2}{2k-2}$ is an even number congruent to $(-1)^{2k-1}=-1$ modulo p.) It follows that σ_p is congruent to

$$(p-(2x+1)^4)\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{4}(2x(x+1)+1)^{p-1-2k}(x(x+1))^{2(k-1)}$$

modulo $p\mathbb{Z}[x(x+1)]$. Recall that (2.2) holds for all $k=1,\ldots,(p-1)/2$. Thus σ_p is congruent to

$$\begin{aligned}&-\frac{(2x+1)^4}{4x^2(x+1)^2}\sum_{k=1}^{(p-1)/2}\binom{(p-1)/2}{k}\left((2x(x+1)+1)^2\right)^{(p-1)/2-k}(-4x^2(x+1)^2)^k\\=&-\frac{(2x+1)^4}{4x^2(x+1)^2}\left(((2x(x+1)+1)^2-4x^2(x+1)^2)^{(p-1)/2}-(2x(x+1)+1)^{p-1}\right)\\=&-\frac{(2x+1)^4}{4x^2(x+1)^2}\left((2x+1)^{p-1}-(2x(x+1)+1)^{p-1}\right)\end{aligned}$$

modulo $p\mathbb{Z}_p[x(x+1)]$. Combining this with (1.13), we get

$$\frac{3}{p}\sum_{k=0}^{p-1}kS_k^{(1)}(x)\equiv-\frac{(2x+1)^{p+3}}{4x^2(x+1)^2}+\frac{6x(x+1)+1}{4x^2(x+1)^2}(2x(x+1)+1)^p\ (\mathrm{mod}\ p\mathbb{Z}_p[x]).\tag{3.8}$$

4. Conjectures on congruences involving $S_n^{(2)}(x)$

Conjecture 4.1. *Let p be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1} 2^k S_k^{(2)}\left(-\frac{1}{2}\right) \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2\ (x,y\in\mathbb{Z}) \text{ with } 2\nmid x, \\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4. \end{cases}$$

(ii) *We have*

$$\sum_{k=0}^{p-1}(3k+2)2^k S_k^{(2)}\left(-\frac{1}{2}\right) \equiv 2p\left(2-\left(\frac{2}{p}\right)\right) \pmod{p^2}.$$

Moreover, when $p\equiv \pm 1 \pmod 8$ we have

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(3k+2)2^k S_k^{(2)}\left(-\frac{1}{2}\right) - p\sum_{k=0}^{n-1}(3k+2)2^k S_k^{(2)}\left(-\frac{1}{2}\right)\right) \in \mathbb{Z}_p.$$

Remark 4.1. By elementary number theory, each prime $p\equiv 1 \pmod 4$ can be written uniquely as x^2+y^2 with $x,y\in\mathbb{Z}^+$, $2\nmid x$ and $2\mid y$.

Conjecture 4.2. *Let $p>3$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{S_k^{(2)}(-1)}{(-3)^k} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2 \text{ with } 3\nmid x \text{ and } 3\mid y, \\ 4xy \pmod{p^2} & \text{if } p=x^2+y^2 \text{ with } x\equiv y\not\equiv 0 \pmod 3, \\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4, \end{cases}$$

where x and y are integers.

(ii) *We have the congruence*

$$\sum_{k=0}^{p-1}(28k+17)\frac{S_k^{(2)}(-1)}{(-3)^k} \equiv p\left(11+6\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Moreover, when $p\equiv 1 \pmod 3$, for any positive integer n we have

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(28k+17)\frac{S_k^{(2)}(-1)}{(-3)^k} - p\sum_{k=0}^{n-1}(28k+17)\frac{S_k^{(2)}(-1)}{(-3)^k}\right) \in \mathbb{Z}_p.$$

Remark 4.2. $S_0^{(2)}(-1),\ldots,S_9^{(2)}(-1)$ take the values 1, 1, 9, 73, 361, 5001, 35001, 348489, 3693033, 31360681, respectively.

Conjecture 4.3. *Let p be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1} 9^k S_k^{(2)}\left(\frac{2}{3}\right) \equiv \sum_{k=0}^{p-1} 9^k S_k^{(2)}\left(\frac{4}{3}\right)$$

$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2 \text{ with } 3\nmid x \text{ and } 3\mid y, \\ 4xy \pmod{p^2} & \text{if } p=x^2+y^2 \text{ with } x\equiv y\not\equiv 0 \pmod 3, \\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4, \end{cases}$$

where x and y are integers.

(ii) *For any* $n\in\mathbb{Z}^+$ *we have*

$$\frac{2}{n}\sum_{k=0}^{n-1}(3k+2)9^kS_k^{(2)}\left(\frac{2}{3}\right)\in\mathbb{Z}$$

and

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}(3k+2)9^kS_k^{(2)}\left(\frac{2}{3}\right)-p\sum_{k=0}^{n-1}(3k+2)9^kS_k^{(2)}\left(\frac{2}{3}\right)\bigg)\in\mathbb{Z}_p.$$

When $p\equiv 1\pmod 4$, *we also have*

$$\sum_{k=0}^{p-1}(24k+19)9^kS_k^{(2)}\left(\frac{4}{3}\right)\equiv p\pmod{p^2}.$$

Remark 4.3. For $a\in\mathbb{Z}$ and $n\in\mathbb{N}$, we clearly have

$$9^nS_n^{(2)}\left(\frac{a}{3}\right)=\sum_{i,j=0}^{n}\binom{n}{i}^2\binom{n}{j}^2a^{i+j}3^{2n-i-j}\in\mathbb{Z}.$$

Conjecture 4.4. *Let* $p>5$ *be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}25^kS_k^{(2)}\left(\frac{4}{15}\right)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+y^2\ (x,y\in\mathbb{Z},\ 5\nmid x\ \&\ 5\mid y),\\ 4xy\pmod{p^2} & \text{if } p=x^2+y^2\ (x,y\in\mathbb{Z}\ \&\ x\equiv y\not\equiv 0\pmod 5),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

(ii) *For any* $n\in\mathbb{Z}^+$ *we have*

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}(168k+125)25^kS_k^{(2)}\left(\frac{4}{15}\right)-p\sum_{k=0}^{n-1}(168k+125)25^kS_k^{(2)}\left(\frac{4}{15}\right)\bigg)\in\mathbb{Z}_p.$$

Remark 4.4. For any prime $p=x^2+y^2>5$ with $x,y\in\mathbb{Z}$ and $5\nmid xy$, either $x^2\equiv y^2\equiv 1\pmod 5$ or $x^2\equiv y^2\equiv 4\pmod 5$, and hence x is congruent to y or $-y$ modulo 5.

Conjecture 4.5. *Let* $p>7$ *be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{S_k^{(2)}(1/7)}{5^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+y^2\ (x,y\in\mathbb{Z},\ 5\nmid x\ \&\ 5\mid y),\\ 4xy\pmod{p^2} & \text{if } p=x^2+y^2\ (x,y\in\mathbb{Z}\ \&\ x\equiv y\not\equiv 0\pmod 5),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

(ii) *We have the congruence*

$$\sum_{k=0}^{p-1}(124k+43)\frac{S_k^{(2)}(1/7)}{5^k}\equiv p\left(53-10\left(\frac{p}{5}\right)\right)\pmod{p^2}.$$

Moreover, when $p\equiv 1,4 \pmod 5$, for any $n\in\mathbb{Z}^+$ we have

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}(124k+43)\frac{S_k^{(2)}(1/7)}{5^k}-p\sum_{k=0}^{n-1}(124k+43)\frac{S_k^{(2)}(1/7)}{5^k}\bigg)\in\mathbb{Z}_p.$$

Remark 4.5. This is similar to Conjecture 4.4.

Conjecture 4.6. *Let p be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}16^kS_k^{(2)}\left(\frac{3}{8}\right)\equiv\sum_{k=0}^{p-1}81^kS_k^{(2)}\left(\frac{8}{9}\right)$$
$$\equiv\begin{cases}(\frac{p}{3})(4x^2-2p)\pmod{p^2}&\text{if } p=x^2+2y^2 \text{ with } x,y\in\mathbb{Z},\\0\pmod{p^2}&\text{if } p\equiv 5,7\pmod 8.\end{cases}$$

If $p\neq 5$, then

$$\sum_{k=0}^{p-1}S_k^{(2)}\left(\frac{12}{5}\right)\equiv\begin{cases}(\frac{p}{3})(4x^2-2p)\pmod{p^2}&\text{if } p=x^2+2y^2 \text{ with } x,y\in\mathbb{Z},\\0\pmod{p^2}&\text{if } p\equiv 5,7\pmod 8.\end{cases}$$

(ii) *When $p\equiv 1,3\pmod 8$, we have*

$$\sum_{k=0}^{p-1}(480k+377)81^kS_k^{(2)}\left(\frac{8}{9}\right)\equiv 122p\pmod{p^2}.$$

If $p>3$ and $n\in\mathbb{Z}^+$, then

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}(32k+23)16^kS_k^{(2)}\left(\frac{3}{8}\right)-p\sum_{k=0}^{n-1}(32k+23)16^kS_k^{(2)}\left(\frac{3}{8}\right)\bigg)\in\mathbb{Z}_p.$$

If $p\neq 5$ and $n\in\mathbb{Z}^+$, then

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}(104k+69)S_k^{(2)}\left(\frac{12}{5}\right)-p\sum_{k=0}^{n-1}(104k+69)S_k^{(2)}\left(\frac{12}{5}\right)\bigg)\in\mathbb{Z}_p.$$

Remark 4.6. It is well known (cf. [4]) that any prime $p\equiv 1,3\pmod 8$ can be written uniquely as x^2+2y^2 with $x,y\in\mathbb{Z}^+$.

Conjecture 4.7. *Let p be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}S_k^{(2)}(2)\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p\equiv 1,9\pmod{20}\ \&\ p=x^2+5y^2,\\2x^2-2p\pmod{p^2}&\text{if } p\equiv 3,7\pmod{20}\ \&\ 2p=x^2+5y^2,\\0\pmod{p^2}&\text{if } p\equiv 11,13,17,19\pmod{20},\end{cases}$$

where x and y are integers. Also,

$$\sum_{k=0}^{p-1}81^kS_k^{(2)}\left(\frac{10}{9}\right)$$
$$\equiv\begin{cases}(\frac{p}{3})(4x^2-2p)\pmod{p^2}&\text{if } p\equiv 1,9\pmod{20}\ \&\ p=x^2+5y^2,\\(\frac{p}{3})(2p-2x^2)\pmod{p^2}&\text{if } p\equiv 3,7\pmod{20}\ \&\ 2p=x^2+5y^2,\\0\pmod{p^2}&\text{if } p\equiv 11,13,17,19\pmod{20},\end{cases}$$

where x and y are integers.

(ii) *If* $(\frac{-5}{p})=1$ *(i.e.,* $p\equiv 1,3,7,9 \pmod{20}$*), then*

$$\sum_{k=0}^{p-1}(8k+5)S_k^{(2)}(2)\equiv \frac{14}{5}p \pmod{p^2}.$$

Also, we have

$$\sum_{k=0}^{p-1}(15k+13)81^kS_k^{(2)}\left(\frac{10}{9}\right)\equiv \frac{p}{16}\left(43+165\left(\frac{-15}{p}\right)\right) \pmod{p^2}.$$

Moreover, when $(\frac{-15}{p})=1$*, for any* $n\in\mathbb{Z}^+$ *we have*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(15k+13)81^kS_k^{(2)}\left(\frac{10}{9}\right)-p\sum_{k=0}^{n-1}(15k+13)81^kS_k^{(2)}\left(\frac{10}{9}\right)\right)\in\mathbb{Z}_p.$$

Remark 4.7. It is known (cf. [4]) that any prime $p\equiv 1,9 \pmod{20}$ can be written uniquely as x^2+5y^2 with $x,y\in\mathbb{Z}^+$, and for any prime $p\equiv 3,7 \pmod{20}$ we can write $2p$ as x^2+5y^2 with $x,y\in\mathbb{Z}^+$ in a unique way.

Conjecture 4.8. *Let* $p>3$ *be a prime with* $p\neq 11$.

(i) *We have*

$$\sum_{k=0}^{p-1}S_k^{(2)}\left(-\frac{2}{11}\right)\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,9 \pmod{20} \ \&\ p=x^2+5y^2,\\ 2x^2-2p \pmod{p^2} & \text{if } p\equiv 3,7 \pmod{20} \ \&\ 2p=x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 11,13,17,19 \pmod{20},\end{cases}$$

where x *and* y *are integers.*

(ii) *We have the congruence*

$$\sum_{k=0}^{p-1}(100k+61)S_k^{(2)}\left(-\frac{2}{11}\right)\equiv\frac{p}{11}\left(826\left(\frac{3}{p}\right)-155\right) \pmod{p^2}.$$

Moreover, when $(\frac{3}{p})=1$ *(i.e.,* $p\equiv\pm1 \pmod{12}$*), for any* $n\in\mathbb{Z}^+$ *we have*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(100k+61)S_k^{(2)}\left(-\frac{2}{11}\right)-p\sum_{k=0}^{n-1}(100k+61)S_k^{(2)}\left(-\frac{2}{11}\right)\right)\in\mathbb{Z}_p.$$

Remark 4.8. This is similar to Conjecture 4.7.

Conjecture 4.9. *Let* $p>3$ *be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}S_k^{(2)}\left(\frac{1}{3}\right)\equiv\sum_{k=0}^{p-1}S_k^{(2)}\left(\frac{4}{3}\right)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{5})=1 \ \&\ p=x^2+10y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{5})=-1 \ \&\ p=2x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-10}{p})=-1,\end{cases}$$

where x *and* y *are integers.*

(ii) *We have the congruence*

$$\sum_{k=0}^{p-1}(60k+31)S_k^{(2)}\left(\frac{1}{3}\right)\equiv p\left(41-10\left(\frac{p}{5}\right)\right) \pmod{p^2}.$$

Moreover, when $p\equiv 1,4 \pmod 5$, for any $n\in\mathbb{Z}^+$ we have

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}(60k+31)S_k^{(2)}\left(\frac13\right)-p\sum_{k=0}^{n-1}(60k+31)S_k^{(2)}\left(\frac13\right)\bigg)\in\mathbb{Z}_p.$$

If $(\frac{-10}{p})=1$, then

$$\sum_{k=0}^{p-1}(25k+13)S_k^{(2)}\left(\frac43\right)\equiv\frac{245}{24}p \pmod{p^2}.$$

Remark 4.9. The imaginary quadratic field $\mathbb{Q}(\sqrt{-10})$ has class number 2.

Conjecture 4.10. *Let $p>3$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}S_k^{(2)}\left(-\frac23\right)\equiv\sum_{k=0}^{p-1}9^kS_k^{(2)}\left(-\frac23\right)\equiv\sum_{k=0}^{p-1}\frac{S_k^{(2)}(-1/2)}{(-4)^k}\equiv\sum_{k=0}^{p-1}16^kS_k^{(2)}\left(\frac98\right)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{13})=1\ \&\ p=x^2+13y^2,\\ 2x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{13})=-1\ \&\ 2p=x^2+13y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-13}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *Let $n\in\mathbb{Z}^+$. Then*

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}(39k+25)S_k^{(2)}\left(-\frac23\right)-p\sum_{k=0}^{n-1}(39k+25)S_k^{(2)}\left(-\frac23\right)\bigg)\in\mathbb{Z}_p$$

and

$$\frac1n\sum_{k=0}^{n-1}(6k+5)9^kS_k^{(2)}\left(-\frac23\right)\in\mathbb{Z}^+.$$

When $(\frac{-13}{p})=1$, we also have

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(1872k+1387)16^kS_k^{(2)}\left(\frac98\right)\in\mathbb{Z}_p.$$

Remark 4.10. The imaginary quadratic field $\mathbb{Q}(\sqrt{-13})$ has class number 2.

Conjecture 4.11. *Let $p>3$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{S_k^{(2)}(2)}{9^k}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=x^2+21y^2,\\ 2x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1,\ (\frac{-1}{p})=(\frac{p}{3})=-1\ \&\ 2p=x^2+21y^2,\\ 12x^2-2p \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-1}{p})=(\frac{p}{7})=-1\ \&\ p=3x^2+7y^2,\\ 6x^2-2p \pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ 2p=3x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-21}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *For any positive integer n, we have*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(7k+3)\frac{S_k^{(2)}(2)}{9^k}-p\sum_{k=0}^{n-1}(7k+3)\frac{S_k^{(2)}(2)}{9^k}\right)\in\mathbb{Z}_p.$$

Remark 4.11. The imaginary quadratic field $\mathbb{Q}(\sqrt{-21})$ has class number 4.

Conjecture 4.12. *Let $p>3$ be a prime.*

(i) *We have*

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{S_k^{(2)}(-4)}{81^k}\equiv&\sum_{k=0}^{p-1}\frac{S_k^{(2)}(1/6)}{4^k}\\ \equiv&\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=1\ \&\ p=x^2+22y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=-1\ \&\ p=2x^2+11y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-22}{p})=-1,\end{cases}\end{aligned}$$

where x and y are integers.

(ii) *For any positive integer n, we have*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(11k+6)\frac{S_k^{(2)}(-4)}{81^k}-p\sum_{k=0}^{n-1}(11k+6)\frac{S_k^{(2)}(-4)}{81^k}\right)\in\mathbb{Z}_p.$$

Remark 4.12. The imaginary quadratic field $\mathbb{Q}(\sqrt{-22})$ has class number 2.

Conjecture 4.13. *Let $p>5$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}S_k^{(2)}\left(\frac{8}{15}\right)\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=1\ \&\ p=x^2+22y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{11})=-1\ \&\ p=2x^2+11y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-22}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *For any positive integer n, we have*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(2244k+1147)S_k^{(2)}\left(\frac{8}{15}\right)-p\sum_{k=0}^{n-1}(2244k+1147)S_k^{(2)}\left(\frac{8}{15}\right)\right)\in\mathbb{Z}_p.$$

Remark 4.13. This is similar to Conjecture 4.12.

Conjecture 4.14. *Let $p>3$ be a prime.*

(i) *We have*

$$\begin{aligned}&\sum_{k=0}^{p-1}S_k^{(2)}\left(-\frac{4}{3}\right)\\ \equiv&\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})-(\frac{p}{5})=1\ \&\ p=x^2+30y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+15y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=3x^2+10y^2,\\ 2p-20x^2\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=5x^2+6y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases}\end{aligned}$$

where x and y are integers.

(ii) *We have the congruence*

$$\sum_{k=0}^{p-1}(120k+79)S_k^{(2)}\left(-\frac{4}{3}\right)\equiv p\left(47+32\left(\frac{2}{p}\right)\right)\pmod{p^2}.$$

Moreover, when $p\equiv\pm1\pmod 8$, for any $n\in\mathbb{Z}^+$ we have

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(120k+79)S_k^{(2)}\left(-\frac{4}{3}\right)-p\sum_{k=0}^{n-1}(120k+79)S_k^{(2)}\left(-\frac{4}{3}\right)\right)\in\mathbb{Z}_p.$$

Remark 4.14. The imaginary quadratic field $\mathbb{Q}(\sqrt{-30})$ has class number 4.

Conjecture 4.15. *Let $p>3$ be a prime with $p\neq 11$.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{S_k^{(2)}(-4)}{9^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{11})=1\ \&\ p=x^2+33y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{11})=-1\ \&\ 2p=x^2+33y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{p}{11})=1,\ (\frac{-1}{p})=(\frac{p}{3})=-1\ \&\ p=3x^2+11y^2,\\ 6x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-1}{p})=(\frac{p}{11})=-1\ \&\ 2p=3x^2+11y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-33}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *For any positive integer n, we have*

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(8k+5)\frac{S_k^{(2)}(-4)}{9^k}-p\left(\frac{33}{p}\right)\sum_{k=0}^{n-1}(8k+5)\frac{S_k^{(2)}(-4)}{9^k}\right)\in\mathbb{Z}_p.$$

Remark 4.15. The imaginary quadratic field $\mathbb{Q}(\sqrt{-33})$ has class number 4.

Conjecture 4.16. *Let $p>5$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{S_k^{(2)}(18)}{625^k}\equiv\sum_{k=0}^{p-1}\frac{S_k^{(2)}(16/5)}{81^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{37})=1\ \&\ p=x^2+37y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{37})=-1\ \&\ 2p=x^2+37y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-37}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *If $(\frac{-37}{p})=1$, then*

$$\sum_{k=0}^{p-1}(263736k+173561)\frac{S_k^{(2)}(18)}{625^k}\equiv\frac{386918}{5}p\pmod{p^2}.$$

For any positive integer n, we have

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(53872k+20157)\frac{S_k^{(2)}(16/5)}{81^k}-p\sum_{k=0}^{n-1}(53872k+20157)\frac{S_k^{(2)}(16/5)}{81^k}\right)\in\mathbb{Z}_p.$$

Remark 4.16. The imaginary quadratic field $\mathbb{Q}(\sqrt{-37})$ has class number 2.

Conjecture 4.17. *Let p be an odd prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}81^kS_k^{(2)}\left(\frac{4}{9}\right)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=x^2+42y^2,\\ 8x^2-2p\pmod{p^2}&\text{if }(\frac{p}{7})=1,\ (\frac{-2}{p})=(\frac{p}{3})=-1\ \&\ p=2x^2+21y^2,\\ 12x^2-2p\pmod{p^2}&\text{if }(\frac{-2}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ p=3x^2+14y^2,\\ 24x^2-2p\pmod{p^2}&\text{if }(\frac{p}{3})=1,\ (\frac{-2}{p})=(\frac{p}{7})=-1\ \&\ p=6x^2+7y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-42}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *We have the congruence*

$$\sum_{k=0}^{p-1}(252k+185)81^kS_k^{(2)}\left(\frac{4}{9}\right)\equiv p\left(17+168\left(\frac{6}{p}\right)\right)\pmod{p^2}.$$

Moreover, when $(\frac{6}{p})=1$, for any $n\in\mathbb{Z}^+$ we have

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(252k+185)81^kS_k^{(2)}\left(\frac{4}{9}\right)-p\sum_{k=0}^{n-1}(252k+185)81^kS_k^{(2)}\left(\frac{4}{9}\right)\right)\in\mathbb{Z}_p.$$

Remark 4.17. The imaginary quadratic field $\mathbb{Q}(\sqrt{-42})$ has class number 4.

Conjecture 4.18. *Let $p>3$ be a prime with $p\neq 7$.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{S_k^{(2)}(2/7)}{9^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{19})=1\ \&\ p=x^2+57y^2,\\ 2x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=1,\ (\frac{p}{5})=(\frac{p}{19})=-1\ \&\ 2p=x^2+57y^2,\\ 12x^2-2p\pmod{p^2}&\text{if }(\frac{p}{3})=1,\ (\frac{-1}{p})=(\frac{p}{19})=-1\ \&\ p=3x^2+19y^2,\\ 6x^2-2p\pmod{p^2}&\text{if }(\frac{p}{19})=1,\ (\frac{-1}{p})=(\frac{p}{3})=-1\ \&\ 2p=3x^2+19y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-57}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *We have the congruence*

$$\sum_{k=0}^{p-1}(133k+48)\frac{S_k^{(2)}(2/7)}{9^k}\equiv\frac{p}{14}\left(1185-513\left(\frac{57}{p}\right)\right)\pmod{p^2}.$$

Moreover, when $(\frac{57}{p})=1$, for any $n\in\mathbb{Z}^+$ we have

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(133k+48)\frac{S_k^{(2)}(2/7)}{9^k}-p\sum_{k=0}^{n-1}(133k+48)\frac{S_k^{(2)}(2/7)}{9^k}\right)\in\mathbb{Z}_p.$$

Remark 4.18. The imaginary quadratic field $\mathbb{Q}(\sqrt{-57})$ has class number 4.

Conjecture 4.19. *Let $p>3$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\left(\frac{16}{81}\right)^k S_k^{(2)}\left(-\frac{5}{8}\right) \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{5})=(\frac{p}{7})=1\ \&\ p=x^2+70y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=2x^2+35y^2,\\ 20x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{7})=-1\ \&\ p=5x^2+14y^2,\\ 2p-28x^2 \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1\ \&\ p=7x^2+10y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-70}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *We have the congruence*

$$\sum_{k=0}^{p-1}(560k+303)\left(\frac{16}{81}\right)^k S_k^{(2)}\left(-\frac{5}{8}\right)\equiv 303p \pmod{p^2}.$$

Remark 4.19. The imaginary quadratic field $\mathbb{Q}(\sqrt{-70})$ has class number 4.

Conjecture 4.20. *Let $p>3$ be a prime with $p\neq 11$.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{S_k^{(2)}(1/11)}{9^k} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{13})=1\ \&\ p=x^2+78y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{13})=-1\ \&\ p=2x^2+39y^2,\\ 12x^2-2p \pmod{p^2} & \text{if } (\frac{p}{13})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=3x^2+26y^2,\\ 24x^2-2p \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{13})=-1\ \&\ p=6x^2+13y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-78}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *We have the congruence*

$$\sum_{k=0}^{p-1}(7k+1)\frac{S_k^{(2)}(1/11)}{9^k}\equiv \frac{p}{26}\left(35\left(\frac{-39}{p}\right)-9\left(\frac{13}{p}\right)\right) \pmod{p^2}.$$

Moreover, when $p\equiv 1 \pmod 3$, for any $n\in\mathbb{Z}^+$ we have

$$\frac{1}{(pn)^2}\bigg(\sum_{k=0}^{pn-1}(7k+1)\frac{S_k^{(2)}(1/11)}{9^k}-p\left(\frac{p}{13}\right)\sum_{k=0}^{n-1}(7k+1)\frac{S_k^{(2)}(1/11)}{9^k}\bigg)\in\mathbb{Z}_p.$$

Remark 4.20. The imaginary quadratic field $\mathbb{Q}(\sqrt{-78})$ has class number 4.

Conjecture 4.21. *Let $p>3$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{S_k^{(2)}(1/18)}{16^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{5})=(\frac{p}{13})=1\ \&\ p=x^2+130y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=1,\ (\frac{p}{5})=(\frac{p}{13})=-1\ \&\ p=2x^2+65y^2,\\ 20x^2-2p\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{-2}{p})=(\frac{p}{13})=-1\ \&\ p=5x^2+26y^2,\\ 40x^2-2p\pmod{p^2} & \text{if } (\frac{p}{13})=1,\ (\frac{-2}{p})=(\frac{p}{5})=-1\ \&\ p=10x^2+13y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-130}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) *We have the congruence*

$$\sum_{k=0}^{p-1}(1365k+181)\frac{S_k^{(2)}(1/18)}{16^k}\equiv p\left(221\left(\frac{-2}{p}\right)-40\left(\frac{5}{p}\right)\right)\pmod{p^2}.$$

Moreover, when $(\frac{-10}{p})=1$, for any $n\in\mathbb{Z}^+$ the number

$$\frac{1}{(pn)^2}\left(\sum_{k=0}^{pn-1}(1365k+181)\frac{S_k^{(2)}(1/18)}{16^k}-p\left(\frac{-2}{p}\right)\sum_{k=0}^{n-1}(1365k+181)\frac{S_k^{(2)}(1/18)}{16^k}\right)$$

is a p-adic integer.

Remark 4.21. The imaginary quadratic field $\mathbb{Q}(\sqrt{-130})$ has class number 4.

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