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## NEW SERIES INVOLVING BINOMIAL COEFFICIENTS (II)

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ABSTRACT. In this paper, we evaluate some series of the form

$$\sum_{k=1}^{\infty} \frac{ak^2 + bk + c}{k(3k-1)(3k-2)m^k \binom{4k}{k}}.$$

For example, we prove that

$$\sum_{k=1}^{\infty} \frac{(5k^2 - 4k + 1)8^k}{k(3k-1)(3k-2)\binom{4k}{k}} = \frac{3}{2}\pi$$

and

$$\sum_{k=1}^{\infty} \frac{415k^2 - 343k + 62}{k(3k-1)(3k-2)(-8)^k \binom{4k}{k}} = -3 \log 2.$$

We also pose many new conjectural series identities involving binomial coefficients; for example, we conjecture that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3}{4096^k} \left( 9(42k+5) \sum_{0 \leq j < k} \frac{1}{(2j+1)^4} + \frac{25}{(2k+1)^3} \right) = \frac{5}{6}\pi^3.$$

### 1. INTRODUCTION

For a converging series  $\sum_{k=0}^{\infty} a_k$  with

$$\lim_{k \rightarrow +\infty} \frac{a_{k+1}}{a_k} = r \in (-1, 1)$$

we call  $r$  its geometric rate of convergence. As such series converge fast, one can easily check any identity for its value by numerical computations in *Mathematica*.

The classical Ramanujan-type series for  $1/\pi$  has the form:

$$\sum_{k=0}^{\infty} (ak + b) \frac{\binom{2k}{k} c_k}{m^k} = \frac{\sqrt{d}}{\pi},$$

where  $a, b, m$  are integers with  $am \neq 0$ ,  $d \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$  and  $c_k$  takes one of the forms

$$\binom{2k}{k}^2, \binom{2k}{k} \binom{3k}{k}, \binom{2k}{k} \binom{4k}{2k}, \binom{3k}{k} \binom{6k}{3k}.$$

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For example, S. Ramanujan [9] discovered that

$$\sum_{k=0}^{\infty} (42k + 5) \frac{\binom{2k}{k}^3}{4096^k} = \frac{16}{\pi} \quad (1.1)$$

with the series having  $1/64$  as its rate of convergence. For an excellent introduction to such series, one may consult S. Cooper [6, Chapter 14].

In 2014 W. Chu and W. Zhang [5] used known hypergeometric transformation formulas to deduce lots of hypergeometric series for special constants involving powers of  $\pi$  or the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\Re(s) > 1).$$

For example, Examples 24 and 21 of [5] give the identities

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} (7k-2)}{(2k-1)k^2 \binom{2k}{k} \binom{3k}{k}} = \frac{\pi^2}{12} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (56k^2 - 32k + 5)}{(2k-1)^2 k^3 \binom{2k}{k} \binom{3k}{k}} = 4\zeta(3).$$

Motivated by this, the present author [15, (4.12)] conjectured that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} (28k^2 - 18k + 3)}{(2k-1)^3 k^4 \binom{2k}{k} \binom{3k}{k}} = \frac{\pi^4}{45}.$$

In 2025, the present author [17] conjectured further that

$$\sum_{k=1}^{\infty} \frac{(-1)^k (560k^4 - 640k^3 + 408k^2 - 136k + 17)}{(2k-1)^4 k^5 \binom{2k}{k} \binom{3k}{k}} = 180\zeta(5) - \frac{56}{3}\pi^2\zeta(3), \quad (1.2)$$

which was later confirmed by K. C. Au (cf. his answer in [17] via the WZ method). As pointed out by J. Zuniga (cf. his answer in [17] or the website [http://www.numberworld.org/y-cruncher/news/2025.html#2025.1\\_25](http://www.numberworld.org/y-cruncher/news/2025.html#2025.1_25)), the identity (1.2) provides currently the fastest way to compute the important constant  $\zeta(5)$  whose irrationality still remains an open question.

In 1974 R. W. Gosper announced the curious identity

$$\sum_{k=0}^{\infty} \frac{25k-3}{2^k \binom{3k}{k}} = \frac{\pi}{2},$$

the proof of which can be found in [1]. In the recent papers [14, 15], the present author evaluated some series with summands involving a single binomial coefficient  $\binom{3k}{k}$  or  $\binom{4k}{2k}$ . In this paper, we mainly focus on series of the type

$$\sum_{k=1}^{\infty} \frac{ak^2 + bk + c}{k(3k-1)(3k-2)m^k \binom{4k}{k}},$$

where  $a, b, c$  and  $m \neq 0$  are rational numbers.

Now we present our main results.

**Theorem 1.1.** *We have*

$$\sum_{k=1}^{\infty} \frac{(95k^2 - 84k + 16)(9/8)^{k-1}}{k(3k-1)(3k-2)\binom{4k}{k}} = \frac{2\pi}{\sqrt{3}} \quad (1.3)$$

and

$$\sum_{k=1}^{\infty} \frac{(5k^2 - 4k + 1)8^k}{k(3k-1)(3k-2)\binom{4k}{k}} = \frac{3\pi}{2}. \quad (1.4)$$

**Remark 1.2.** For the series in (1.3) and (1.4), their rates of convergence are 243/2048 and 27/32, respectively. The identity (1.4) was previously conjectured by the present author [13, (2.3)]. A similar identity

$$\sum_{k=1}^{\infty} \frac{(35k^2 - 29k + 6)3^k}{k(3k-1)(3k-2)\binom{4k}{k}} = \sqrt{3}\pi$$

conjectured by the present author in 2023 was confirmed by Au [2] via the WZ method.

**Theorem 1.3.** *We have the following identities:*

$$\sum_{k=1}^{\infty} \frac{77k^2 - 53k + 10}{k(3k-1)(3k-2)(-2)^k \binom{4k}{k}} = -3 \log 2, \quad (1.5)$$

$$\sum_{k=1}^{\infty} \frac{415k^2 - 343k + 62}{k(3k-1)(3k-2)(-8)^k \binom{4k}{k}} = -3 \log 2, \quad (1.6)$$

$$\sum_{k=1}^{\infty} \frac{187k^2 - 131k + 22}{k(3k-1)(3k-2)(-24)^k \binom{4k}{k}} = \log \frac{2}{3}, \quad (1.7)$$

$$\sum_{k=1}^{\infty} \frac{1261k^2 - 989k + 170}{k(3k-1)(3k-2)(-192)^k \binom{4k}{k}} = \log \frac{3}{4}. \quad (1.8)$$

Our proofs of Theorems 1.1 and 1.3 involve the Beta function

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx \quad (a > 0 \text{ and } b > 0).$$

A celebrated result of Euler states that

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

where the Gamma function is given by

$$\Gamma(x) = \int_0^{+\infty} \frac{t^{x-1}}{e^t} dt \quad \text{for } x > 0.$$

Now we state our third theorem.

**Theorem 1.4.** *We have the following identities:*

$$\sum_{k=0}^{\infty} \frac{(22k^2 + 17k - 2) \binom{4k}{k}}{(k+1)16^k} = 17, \quad (1.9)$$

$$\sum_{k=0}^{\infty} \frac{(11k^2 + 8k + 1) \binom{4k}{k}}{(3k+1)(3k+2)16^k} = 1, \quad (1.10)$$

$$\sum_{k=0}^{\infty} \frac{(22k^2 - 18k + 3) \binom{4k}{k}}{(2k-1)(4k-1)(4k-3)16^k} = -\frac{1}{3}. \quad (1.11)$$

**Remark 1.5.** The identity (1.11) was stated in [13, (3.1)] as a conjecture.

We are going to prove Theorem 1.1 and Theorems 1.3-1.4 in Sections 2 and 3 respectively.

Recall that the harmonic numbers are given by

$$H_n := \sum_{0 < k \leq n} \frac{1}{k} \quad (n = 0, 1, 2, \dots).$$

For each  $m = 2, 3, \dots$ , the (generalized) harmonic numbers of order  $m$  are defined by

$$H_n^{(m)} := \sum_{0 < k \leq n} \frac{1}{k^m} \quad (n = 0, 1, 2, \dots).$$

Inspired by [10, 8, 22, 23], the present author [15, 16] found that many hypergeometric series identities with summands involving binomial coefficients have certain variants with summands involving generalized harmonic numbers. For example, motivated by the Ramanujan series (1.1), the present author [16, (107)] conjectured the identity

$$\sum_{k=0}^{\infty} (42k + 5) \frac{\binom{2k}{k}^3}{4096^k} \left( H_{2k}^{(2)} - \frac{25}{92} H_k^{(2)} \right) = \frac{2\pi}{69}, \quad (1.12)$$

and this was later confirmed by C. Wei [21]. With this background in mind, in Sections 4 and 5 we shall pose many new series involving binomial coefficients and generalized harmonic numbers.

Now we introduce some basic notations in this paper. For a prime  $p$  and any integer  $a \not\equiv 0 \pmod{p}$ , we define the Fermat quotient  $q_p(a) = (a^{p-1} - 1)/p$ . For an odd prime  $p$ , we use  $\left(\frac{\cdot}{p}\right)$  to denote the Legendre symbol. The constant

$$G := L \left( 2, \left( \frac{-4}{\cdot} \right) \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \quad (1.13)$$

is called Catalan's constant. We also adopt the notation

$$K := L \left( 2, \left( \frac{\cdot}{3} \right) \right) = \sum_{k=1}^{\infty} \frac{\binom{k}{3}}{k^2} = \sum_{n=0}^{\infty} \left( \frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right). \quad (1.14)$$

This constant was recently proved to be irrational by F. Calegari, V. Dimitrov and Y. Tang [3]. As usual, we let  $B_0, B_1, \dots$  denote the Bernoulli

numbers, and  $B_n(x)$  with  $n \in \mathbb{N} = \{0, 1, 2, \dots\}$  stands for the Bernoulli polynomial of degree  $n$ . Also, we let  $E_0, E_1, \dots$  denote the Euler numbers, and  $E_n(x)$  with  $n \in \mathbb{N}$  stands for the Euler polynomial of degree  $n$ .

## 2. PROOF OF THEOREM 1.1

**Lemma 2.1.** *Let  $m$  be any nonzero complex number. Then, for any positive integer  $n$ , we have*

$$\begin{aligned} & \sum_{k=1}^n \frac{(256m - 27)k^3 - 3(128m + 9)k^2 + 2(88m - 3)k - 24m}{km^k \binom{4k}{k}} \\ &= 6 - \frac{3(3n + 1)(3n + 2)}{m^n \binom{4n}{n}}, \end{aligned}$$

$$\begin{aligned} & \sum_{k=1}^n \frac{(256m - 27)k^3 - 2(64m + 27)k^2 - (16m + 33)k + 8m - 6}{(4k + 1)m^k \binom{4k}{k}} \\ &= 6 - \frac{3(n + 1)(3n + 1)(3n + 2)}{(4n + 1)m^n \binom{4n}{n}}, \end{aligned}$$

$$\sum_{k=1}^n \frac{(256m - 27)k^3 - 384mk^2 + (176m + 3)k - 24m}{k(3k - 1)m^k \binom{4k}{k}} = 3 - \frac{3(3n + 1)}{m^n \binom{4n}{n}},$$

and

$$\sum_{k=1}^n \frac{(256m - 27)k^3 - 3(128m - 9)k^2 + 2(88m - 3)k - 24m}{k(3k - 1)(3k - 2)m^k \binom{4k}{k}} = 3 - \frac{3}{m^n \binom{4n}{n}}.$$

Consequently, if  $|m| > 27/256$  then we have

$$\sum_{k=1}^{\infty} \frac{(256m - 27)k^3 - 3(128m + 9)k^2 + 2(88m - 3)k - 24m}{km^k \binom{4k}{k}} = 6, \quad (2.1)$$

$$\sum_{k=1}^{\infty} \frac{(256m - 27)k^3 - 2(64m + 27)k^2 - (16m + 33)k + 8m - 6}{(4k + 1)m^k \binom{4k}{k}} = 6, \quad (2.2)$$

$$\sum_{k=1}^{\infty} \frac{(256m - 27)k^3 - 384mk^2 + (176m + 3)k - 24m}{k(3k - 1)m^k \binom{4k}{k}} = 3, \quad (2.3)$$

$$\sum_{k=1}^{\infty} \frac{(256m - 27)k^3 - 3(128m - 9)k^2 + 2(88m - 3)k - 24m}{k(3k - 1)(3k - 2)m^k \binom{4k}{k}} = 3. \quad (2.4)$$

*Proof.* The first four identities in Lemma 2.1 can be easily proved by induction on  $n$ . By Stirling's formula,  $n! \sim \sqrt{2\pi n}(n/e)^n$  as  $n \rightarrow +\infty$ . Thus

$$\binom{4n}{n} = \frac{(4n)!}{n!(3n)!} \sim \frac{\sqrt{2\pi(4n)}(4n/e)^{4n}}{\sqrt{2\pi n}(n/e)^n \sqrt{2\pi(3n)}(3n/e)^{3n}} = \frac{2}{\sqrt{6n\pi}} \left(\frac{256}{27}\right)^n$$

as  $n \rightarrow +\infty$ . When  $|m| > 27/256$ , by letting  $n \rightarrow +\infty$  in the first four identities in Lemma 2.1, we obtain the identities (2.1)-(2.4) since  $m^n \binom{4n}{n} \rightarrow +\infty$ . This concludes the proof.  $\square$

For  $|z| < 1$ , we clearly have

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \quad \sum_{k=1}^{\infty} k z^{k-1} = \frac{d}{dz}(1-z)^{-1} = \frac{1}{(1-z)^2}, \quad (2.5)$$

and

$$\sum_{k=2}^{\infty} k(k-1)z^{k-2} = \frac{d}{dz}(1-z)^{-2} = \frac{2}{(1-z)^3}. \quad (2.6)$$

We will make use of these basic facts several times.

**Lemma 2.2.** *We have the identity*

$$\sum_{k=1}^{\infty} \frac{12635k^2 - 5259k + 832}{k \binom{4k}{k}} \left(\frac{9}{8}\right)^{k-1} = 1944 + \frac{640}{\sqrt{3}}\pi. \quad (2.7)$$

*Proof.* For each  $k \in \mathbb{Z}^+$  we have

$$\frac{1}{k \binom{4k}{k}} = \frac{(3k)!(k-1)!}{(4k)!} = \frac{\Gamma(3k+1)\Gamma(k)}{\Gamma(4k+1)} = B(3k+1, k) = \int_0^1 x^{3k}(1-x)^{k-1} dx. \quad (2.8)$$

Let  $P(k) = 12635k^2 - 5259k + 832$ . By (2.8), we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{P(k)}{k \binom{4k}{k}} \left(\frac{9}{8}\right)^{k-1} &= \sum_{k=1}^{\infty} P(k) \int_0^1 x^{3k} \left(\frac{9}{8}(1-x)\right)^{k-1} dx \\ &= \int_0^1 \sum_{k=1}^{\infty} P(k) x^{3k} \left(\frac{9}{8}(1-x)\right)^{k-1} dx. \end{aligned}$$

In view of (2.5) and (2.6), we can easily verify that

$$\begin{aligned} &\sum_{k=1}^{\infty} P(k) x^{3k} \left(\frac{9}{8}(1-x)\right)^{k-1} \\ &= \frac{3456x^3(x^3(x-1)(156x^3(x-1) - 2705) + 1216)}{(9x^4 - 9x^3 + 8)^3} = \frac{64}{9} \cdot \frac{d}{dx} F(x), \end{aligned}$$

where

$$\begin{aligned} F(x) := &\frac{56(432x^3 - 108x^2 - 81x - 512)}{(9x^4 - 9x^3 + 8)^2} + \frac{108x^3 + 351x^2 + 27x + 4352}{9x^4 - 9x^3 + 8} \\ &+ 90\sqrt{3} \arctan(\sqrt{3}(x-1)) + 26 \log(9x^4 - 9x^3 + 8). \end{aligned}$$

(Note that  $9x^4 - 9x^3 + 8 = (3x^2 - 6x + 4)(3x^2 + 3x + 2)$  and the function  $F(x)$  can be found by **Mathematica** directly.) Therefore

$$\sum_{k=1}^{\infty} \frac{P(k)}{k \binom{4k}{k}} \left(\frac{9}{8}\right)^{k-1} = \frac{64}{9}(F(1) - F(0)) = 1944 + \frac{640}{\sqrt{3}}\pi$$

as desired.  $\square$

*Proof of (1.3).* Putting  $m = 8/9$  in the identities (2.3) and (2.4), we obtain

$$\sum_{k=1}^{\infty} \frac{1805k^3 - 3072k^2 + 1435k - 192}{k(3k-1)\binom{4k}{k}} \left(\frac{9}{8}\right)^{k-1} = 27 \times \frac{8}{9} = 24 \quad (2.9)$$

and

$$\sum_{k=1}^{\infty} \frac{1805k^3 - 2829k^2 + 1354k - 192}{k(3k-1)(3k-2)\binom{4k}{k}} \left(\frac{9}{8}\right)^{k-1} = 27 \times \frac{8}{9} = 24. \quad (2.10)$$

Since

$$\begin{aligned} & (3k-1)(12635k^2 - 5259k + 832) - 21(1805k^3 - 3072k^2 + 1435k - 192) \\ &= 20(1805k^2 - 1119k + 160), \end{aligned}$$

(2.7) minus  $21 \times$  (2.9) yields the identity

$$\sum_{k=1}^{\infty} \frac{1805k^2 - 1119k + 160}{k(3k-1)\binom{4k}{k}} \left(\frac{9}{8}\right)^{k-1} = 72 + \frac{32}{\sqrt{3}}\pi. \quad (2.11)$$

As

$$\begin{aligned} & (3k-2)(1805k^2 - 1119k + 160) - 3(1805k^3 - 2829k^2 + 1354k - 192) \\ &= 16(95k^2 - 84k + 16), \end{aligned}$$

(2.11) minus  $3 \times$  (2.10) yields the identity

$$\sum_{k=1}^{\infty} \frac{95k^2 - 84k + 16}{k(3k-1)(3k-2)\binom{4k}{k}} \left(\frac{9}{8}\right)^{k-1} = \frac{72 + 32\pi/\sqrt{3} - 3 \times 24}{16} = \frac{2\pi}{\sqrt{3}}.$$

This concludes our proof of (1.3).  $\square$

**Lemma 2.3.** *We have*

$$\sum_{k=1}^{\infty} \frac{(15k^2 - 124k - 40)8^k}{(4k+1)\binom{4k}{k}} = 30 - \frac{3}{2}\pi. \quad (2.12)$$

*Proof.* Set

$$P(k) = 15k^2 - 124k - 40 = 15k(k-1) - 109 - 40.$$

Then, for  $|z| < 1$ , with the aids of (2.5) and (2.6) we have

$$\begin{aligned} \sum_{k=0}^{\infty} P(k)z^k &= 15z^2 \sum_{k=2}^{\infty} k(k-1)z^{k-2} - 109z \sum_{k=1}^{\infty} kz^{k-1} - 40 \sum_{k=0}^{\infty} z^k \\ &= 15 \times \frac{2z^2}{(1-z)^3} - 109 \times \frac{z}{(1-z)^2} - \frac{40}{1-z} = \frac{99z^2 - 29z - 40}{(1-z)^3}. \end{aligned}$$

For each nonnegative integer  $k$ , clearly

$$\frac{1}{(4k+1)\binom{4k}{k}} = \frac{\Gamma(3k+1)\Gamma(k+1)}{\Gamma(4k+2)} = B(3k+1, k+1) = \int_0^1 x^{3k}(1-x)^k dx. \quad (2.13)$$

Thus, in view of the last paragraph, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{P(k)8^k}{(4k+1)\binom{4k}{k}} &= \int_0^1 \sum_{k=0}^{\infty} P(k) (8x^3(1-x))^k dx \\ &= \int_0^1 \frac{99 \times (8x^3(1-x))^2 - 29 \times 8x^3(1-x) - 40}{(1-8x^3(1-x))^3} dx \\ &= \int_0^1 \frac{f(x)}{(8x^4 - 8x^3 + 1)^3} dx, \end{aligned}$$

where

$$f(x) = 8(792x^8 - 1584x^7 + 792x^6 + 29x^4 - 29x^3 - 5).$$

It is easy to verify that  $f(x)/(8x^4 - 8x^3 + 1)^3$  is the derivation of the function

$$\begin{aligned} F(x) := & -\frac{96x^7 - 192x^6 + 640x^5 - 680x^4 + 76x^3 + 6x^2 + 74x - 5}{2(8x^4 - 8x^3 + 1)^2} \\ & - \frac{3}{2} \arctan \frac{2x(x-1)}{2x^2 - 1}. \end{aligned}$$

(We find  $F(x)$  by *Mathematica*.) Therefore

$$\sum_{k=0}^{\infty} \frac{P(k)8^k}{(4k+1)\binom{4k}{k}} = \int_0^1 F'(x) dx = F(1) - F(0) = -10 - \frac{3}{2}\pi,$$

which is equivalent to the desired identity (2.12).  $\square$

**Lemma 2.4.** *We have*

$$\sum_{k=1}^{\infty} \frac{(35k^2 - 316k + 43)8^k}{k\binom{4k}{k}} = 108 + \frac{15}{2}\pi. \quad (2.14)$$

*Proof.* Applying (2.1) and (2.2) with  $m = 1/8$ , we obtain the identities

$$\sum_{k=1}^{\infty} \frac{(5k^3 - 75k^2 + 16k - 3)8^k}{k\binom{4k}{k}} = 6 \quad (2.15)$$

and

$$\sum_{k=1}^{\infty} \frac{(k^3 - 14k^2 - 7k - 1)8^k}{(4k+1)\binom{4k}{k}} = \frac{6}{5}. \quad (2.16)$$

Since

$$\begin{aligned} & 172(k^3 - 14k^2 - 7k - 1) - 3(15k^2 - 124k - 40) \\ &= (4k+1)(43k^2 - 624k - 52), \end{aligned}$$

combining (2.16) and (2.12) we get the identity

$$\sum_{k=1}^{\infty} \frac{(43k^2 - 624k - 52)8^k}{\binom{4k}{k}} = \frac{582}{5} + \frac{9}{2}\pi. \quad (2.17)$$

As

$$5k(43k^2 - 624k - 52) - 43(5k^3 - 75k^2 + 16k - 3) = 3(35k^2 - 316k + 43),$$

from (2.16) and (2.15) we obtain

$$\sum_{k=1}^{\infty} \frac{(43k^2 - 624k - 52)8^k}{k \binom{4k}{k}} = \frac{582 + 45\pi/2 - 43 \times 6}{3} = 108 + \frac{15}{2}\pi.$$

This proves (2.14).  $\square$

*Proof of (1.4).* Applying (2.3) and (2.4) with  $m = 1/8$ , we obtain

$$\sum_{k=1}^{\infty} \frac{(5k^3 - 48k^2 + 25k - 3)8^k}{k(3k-1) \binom{4k}{k}} = 3 \quad (2.18)$$

and

$$\sum_{k=1}^{\infty} \frac{(5k^3 - 21k^2 + 16k - 3)8^k}{k(3k-1)(3k-2) \binom{4k}{k}} = 3. \quad (2.19)$$

Since

$$(3k-1)(35k^2 - 316k + 43) - 21(5k^3 - 48k^2 + 25k - 3) = 5(5k^2 - 16k + 4),$$

combining (2.14) and (2.18) we get

$$\sum_{k=1}^{\infty} \frac{(5k^2 - 16k + 4)8^k}{k(3k-1) \binom{4k}{k}} = 9 + \frac{3}{2}\pi. \quad (2.20)$$

As

$$(3k-2)(5k^2 - 16k + 4) - 3(5k^3 - 21k^2 + 16k - 3) = 5k^2 - 4k + 1,$$

from (2.20) and (2.19) we deduce the identity

$$\sum_{k=1}^{\infty} \frac{(5k^2 - 4k + 1)8^k}{k(3k-1)(3k-2) \binom{4k}{k}} = 9 + \frac{3}{2}\pi - 3 \times 3 = \frac{3}{2}\pi.$$

This proves (1.4).  $\square$

### 3. PROOFS OF THEOREMS 1.3-1.4

**Lemma 3.1.** *We have*

$$\sum_{k=1}^{\infty} \frac{18675k^2 + 7627k + 670}{(4k+1)(-8)^k \binom{4k}{k}} = -30 - 192 \log 2. \quad (3.1)$$

*Proof.* Set

$$P(k) = 18675k^2 + 7627k + 670 = 18675k(k-1) + 26302k + 670.$$

Then, for  $|z| < 1$ , by using (2.5) and (2.6) we get

$$\begin{aligned} \sum_{k=0}^{\infty} P(k)z^k &= 18675z^2 \sum_{k=2}^{\infty} k(k-1)z^{k-2} + 26302z \sum_{k=1}^{\infty} kz^{k-1} + 670 \sum_{k=0}^{\infty} z^k \\ &= 18675 \times \frac{2z^2}{(1-z)^3} + 26302 \times \frac{z}{(1-z)^2} + \frac{670}{1-z} \\ &= \frac{2(5859z^2 + 12481z + 335)}{(1-z)^3}. \end{aligned}$$

As (2.13) holds for all  $k \in \mathbb{N}$ , in view of the last paragraph we have

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{P(k)}{(4k+1)(-8)^k \binom{4k}{k}} \\
&= \int_0^1 \sum_{k=0}^{\infty} P(k) \left( \frac{x^3(1-x)}{-8} \right)^k dx \\
&= 2 \int_0^1 \frac{5859x^6(1-x)^2/64 - 12481x^3(1-x)/8 + 335}{(1+x^3(1-x)/8)^3} dx \\
&= -16 \int_0^1 \frac{5859x^6(x-1)^2 + 99848x^3(x-1) + 21440}{(x^3(x-1) - 8)^3} dx.
\end{aligned}$$

Though the last definite integration can be evaluated via *Mathematica*, we prefer to avoid using any software. Observe that

$$x^3(x-1) - 8 = (x-2)(x^3 + x^2 + 2x + 4)$$

and

$$\begin{aligned}
& \frac{5859x^6(x-1)^2 + 99848x^3(x-1) + 21440}{(x^3(x-1) - 8)^3} \\
&= \frac{747}{5(x-2)^3} + \frac{403}{5(x-2)^2} - \frac{9}{x-2} + \frac{Q(x)}{5(x^3 + x^2 + 2x + 4)^3},
\end{aligned}$$

where

$$\begin{aligned}
Q(x) &= 45x^8 - 178x^7 - 2713x^6 + 12108x^5 + 34656x^4 \\
&\quad + 54960x^3 - 16506x^2 - 21172x - 15312.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{P(k)}{(4k+1)(-8)^k \binom{4k}{k}} + \frac{16}{5} \int_0^1 \frac{Q(x)}{(x^3 + x^2 + 2x + 4)^3} dx \\
&= -\frac{16}{5} \int_0^1 \left( \frac{747}{(x-2)^3} + \frac{403}{(x-2)^2} - \frac{45}{x-2} \right) dx \\
&= -\frac{16}{5} \left( \frac{747}{-2} (x-2)^{-2} \Big|_0^1 + \frac{403}{-1} (x-2)^{-1} \Big|_0^1 - 45 \log(2-x) \Big|_0^1 \right) \\
&= -\frac{16}{5} \left( \frac{747}{-2} \left( 1 - \frac{1}{4} \right) - 403 \left( -1 + \frac{1}{2} \right) - 45(0 - \log 2) \right) \\
&= \frac{1258}{5} - 144 \log 2.
\end{aligned}$$

So we have reduced (3.1) to the identity

$$\int_0^1 \frac{Q(x)}{(x^3 + x^2 + 2x + 4)^3} dx = 15 \log 2 - \frac{971}{8}. \quad (3.2)$$

Note that

$$x^3 + x^2 + 2x + 4 = (x - \alpha)(x^2 + (\alpha + 1)x + \alpha^2 + \alpha + 2),$$

where

$$\alpha = \frac{1}{3} \left( \sqrt[3]{3\sqrt{249} - 46} - \sqrt[3]{3\sqrt{249} + 46} - 1 \right) \approx -1.48.$$

However, it is rather complicated to prove (3.2) via writing the rational function  $Q(x)/(x^3 + x^2 + 2x + 4)^3$  as a sum of partial fractions. Instead, we define  $G(x)$  as

$$15 \log(x^3 + x^2 + 2x + 4) + \frac{298x^5 + 1663x^4 - 3190x^3 - 4451x^2 - 4878x - 930}{(x^3 + x^2 + 2x + 4)^2}.$$

It is easy to verify that

$$\frac{d}{dx} G(x) = \frac{Q(x)}{(x^3 + x^2 + 2x + 4)^3}.$$

Therefore

$$\int_0^1 \frac{Q(x)}{(x^3 + x^2 + 2x + 4)^3} dx = G(1) - G(0) = 15 \log 8 - \frac{11488}{8^2} - 15 \log 4 + \frac{930}{4^2}$$

and hence (3.2) holds.

In view of the above, we have proved the desired identity (3.1).  $\square$

With the help of Lemma 3.1, we can deduce the following lemma.

**Lemma 3.2.** *We have*

$$\sum_{k=1}^{\infty} \frac{26975k^2 - 17111k + 2968}{k(-8)^k \binom{4k}{k}} = -297 - 120 \log 2. \quad (3.3)$$

*Proof.* Applying (2.1) and (2.2) with  $m = -8$  we obtain the identities

$$\sum_{k=1}^{\infty} \frac{2075k^3 - 3045k^2 + 1414k - 192}{k(-8)^k \binom{4k}{k}} = -6 \quad (3.4)$$

and

$$\sum_{k=1}^{\infty} \frac{415k^3 - 194k^2 - 19k + 14}{(4k+1)(-8)^k \binom{4k}{k}} = \frac{64}{5} - 14 = -\frac{6}{5}. \quad (3.5)$$

As

$$\begin{aligned} & 3(18675k^2 + 7627k + 670) + 1484(415k^3 - 194k^2 - 19k + 14) \\ &= (4k+1)(153965k^2 - 96459k + 22786), \end{aligned}$$

from (3.1) and (3.5) we get

$$\sum_{k=1}^{\infty} \frac{153965k^2 - 96459k + 22786}{(-8)^k \binom{4k}{k}} = -\frac{9354}{5} - 576 \log 2. \quad (3.6)$$

Since

$$\begin{aligned} & 5k(153965k^2 - 96459k + 22786) - 371(2075k^3 - 3045k^2 + 1414k - 192) \\ &= 24(26975k^2 - 17111k + 2968), \end{aligned}$$

combining (3.6) and (3.4) we obtain the desired (3.3).  $\square$

**Remark 3.3.** By similar arguments, we can also deduce the identities

$$\sum_{k=0}^{\infty} \frac{524975k^2 + 195959k - 32986}{(2k+1)(-8)^k \binom{4k}{k}} = -64(592 + 45 \log 2)$$

and

$$\sum_{k=0}^{\infty} \frac{1819775k^2 + 669431k - 521898}{(4k+3)(-8)^k \binom{4k}{k}} = -64(2818 + 45 \log 2).$$

*Proof of (1.6).* Applying (2.3) and (2.4) with  $m = -8$ , we obtain the identities

$$\sum_{k=1}^{\infty} \frac{2075k^3 - 3072k^2 + 1405k - 192}{k(3k-1)(-8)^k \binom{4k}{k}} = -3 \quad (3.7)$$

and

$$\sum_{k=1}^{\infty} \frac{2075k^3 - 3099k^2 + 1414k - 192}{k(3k-1)(3k-2)(-8)^k \binom{4k}{k}} = -3. \quad (3.8)$$

Since

$$\begin{aligned} & (3k-1)(26975k^2 - 17111k + 2968) \\ & - 39(2075k^3 - 3072k^2 + 1405k - 192) \\ & = 20(2075k^2 - 1439k + 226), \end{aligned}$$

combining (3.3) and (3.7) we get

$$\sum_{k=1}^{\infty} \frac{2075k^2 - 1439k + 226}{k(3k-1)(-8)^k \binom{4k}{k}} = -9 - 6 \log 2. \quad (3.9)$$

As

$$\begin{aligned} & (3k-2)(2075k^2 - 1439k + 226) - 3(2075k^3 - 3099k^2 + 1414k - 192) \\ & = 2(415k^2 - 343k + 62), \end{aligned}$$

from (3.9) and (3.8) we obtain

$$\sum_{k=1}^{\infty} \frac{415k^2 - 343k + 62}{k(3k-1)(3k-2)(-8)^k \binom{4k}{k}} = \frac{-9 - 6 \log 2 - 3(-3)}{2} = -3 \log 2.$$

This ends our proof of (1.6).  $\square$

The following result similar to Lemma 3.1 can be proved in the way we prove Lemma 3.1.

**Lemma 3.4.** *We have*

$$\sum_{k=1}^{\infty} \frac{11319k^2 - 497k - 746}{(4k+1)(-2)^k \binom{4k}{k}} = 48 \log 2 - 246, \quad (3.10)$$

$$\sum_{k=1}^{\infty} \frac{39083k^2 - 2829k - 3106}{(4k+1)(-24)^k \binom{4k}{k}} = -94 - 64 \log \frac{2}{3}, \quad (3.11)$$

$$\sum_{k=1}^{\infty} \frac{442611k^2 + 41347k - 17434}{(4k+1)(-192)^k \binom{4k}{k}} = 26 + 512 \log \frac{3}{4}. \quad (3.12)$$

The following result similar to Lemma 3.2 can be proved by using Lemma 3.4, in the way we proved Lemma 3.2.

**Lemma 3.5.** *We have*

$$\sum_{k=1}^{\infty} \frac{5929k^2 - 4675k + 914}{k(-2)^k \binom{4k}{k}} = -189 - 30 \log 2, \quad (3.13)$$

$$\sum_{k=1}^{\infty} \frac{39083k^2 - 31627k + 5624}{k(-24)^k \binom{4k}{k}} = 40 \log \frac{2}{3} - 117, \quad (3.14)$$

$$\sum_{k=1}^{\infty} \frac{475397k^2 - 335665k + 55072}{k(-192)^k \binom{4k}{k}} = 160 \log \frac{3}{4} - 207. \quad (3.15)$$

*Proof of (1.5), (1.7) and (1.8).* Just make use of Lemma 3.5 in the way we proved (1.6).  $\square$

**Lemma 3.6.** *We have the identity*

$$\sum_{k=0}^{\infty} (22k^2 - 92k + 11) \frac{\binom{4k}{k}}{16^k} = -5. \quad (3.16)$$

**Remark 3.7.** The present author [20] conjectured the identity (3.16) in Feb. 2025, and this was confirmed by Max Alekseyev (cf. his answer in [20]) who made use of the fact that the generating function  $f(x) = \sum_{k=0}^{\infty} \binom{4k}{k} x^k$  ( $|x| < 27/256$ ) satisfies the functional equation

$$(f(x) - 1)(3f(x) + 1)^3 = x(4f(x))^4.$$

**Lemma 3.8.** *Let  $m$  be any nonzero complex number. Then, for any  $n \in \mathbb{N}$  we have the identities*

$$\begin{aligned} & \sum_{k=0}^n \frac{((256 - 27m)k^3 + 384k^2 + (176 + 21m)k - 6m + 24) \binom{4k}{k}}{(k+1)m^k} \\ &= -6m + \frac{8(2n+1)(4n+1)(4n+3) \binom{4n}{n}}{(n+1)m^n}, \end{aligned}$$

$$\begin{aligned} & \sum_{k=0}^n \frac{((256 - 27m)k^3 + 384k^2 + (176 + 3m)k + 24) \binom{4k}{k}}{(3k + 1)m^k} \\ &= \frac{8(2n + 1)(4n + 1)(4n + 3) \binom{4n}{n}}{(3n + 1)m^n}, \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=0}^n \frac{((256 - 27m)k^3 + 3(128 - 9m)k^2 + 2(88 - 3m)k + 24) \binom{4k}{k}}{(3k + 1)(3k + 2)m^k} \\ &= \frac{8(2n + 1)(4n + 1)(4n + 3) \binom{4n}{n}}{(3n + 1)(3n + 2)m^n}. \end{aligned}$$

Consequently, if  $|m| > 256/27$  then

$$\sum_{k=0}^{\infty} \frac{((256 - 27m)k^3 + 384k^2 + (176 + 21m)k - 6m + 24) \binom{4k}{k}}{(k + 1)m^k} = -6m, \quad (3.17)$$

$$\sum_{k=0}^{\infty} \frac{((256 - 27m)k^3 + 384k^2 + (176 + 3m)k + 24) \binom{4k}{k}}{(3k + 1)m^k} = 0 \quad (3.18)$$

and

$$\sum_{k=0}^{\infty} \frac{((256 - 27m)k^3 + 384k^2 + (176 + 3m)k + 24) \binom{4k}{k}}{(3k + 1)m^k} = 0. \quad (3.19)$$

*Proof.* The first three identities in Lemma 3.8 can be easily proved by induction on  $n$ . Letting  $n \rightarrow +\infty$  we then obtain the identities (3.17), (3.18) and (3.19). This ends the proof.  $\square$

*Proof of Theorem 1.4.* Putting  $m = 16$  in (3.17), we obtain the identity

$$\sum_{k=0}^{\infty} \frac{(22k^3 - 48k^2 - 64k + 9) \binom{4k}{k}}{(k + 1)16^k} = 12. \quad (3.20)$$

As

$$22k^3 - 48k^2 - 64k + 9 - (k + 1)(22k^2 - 92k + 11) = 22k^2 + 17k - 2,$$

we obtain (1.9) via (3.20) minus (3.16).

Putting  $m = 16$  in (3.18) and (3.19), we get the identities

$$\sum_{k=0}^{\infty} \frac{(22k^3 - 48k^2 - 28k - 3) \binom{4k}{k}}{(3k + 1)16^k} = 0 \quad (3.21)$$

and

$$\sum_{k=0}^{\infty} \frac{(22k^3 + 6k^2 - 10k - 3) \binom{4k}{k}}{(3k + 1)(3k + 2)16^k} = 0. \quad (3.22)$$

Since

$$3(22k^3 - 48k^2 - 28k - 3) - (3k + 1)(22k^2 - 92k + 11) = 5(22k^2 - 5k - 4),$$

via  $3 \times (3.21) - (3.16)$  we get the identity

$$\sum_{k=0}^{\infty} \frac{(22k^2 - 5k - 4) \binom{4k}{k}}{(3k+1)16^k} = 1. \quad (3.23)$$

As

$$(3k+2)(22k^2 - 5k - 4) - 3(22k^3 + 6k^2 - 10k - 3) = 11k^2 + 8k + 1,$$

via (3.23)– $3 \times (3.22)$  we obtain the identity (1.10).

By induction,

$$\sum_{k=0}^n \frac{\binom{4k}{k} (2k(11k^2 - 14k + 4) + 22k^2 - 18k + 3)}{(2k-1)(4k-1)(4k-3)16^k} = -\frac{\binom{4n}{n}}{16^n}$$

for any  $n \in \mathbb{N}$ . Letting  $n \rightarrow +\infty$ , we then get

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k} (2k(11k^2 - 14k + 4) + 22k^2 - 18k + 3)}{(2k-1)(4k-1)(4k-3)16^k} = 0.$$

Thus (1.11) has the equivalent form

$$\sum_{k=1}^{\infty} \frac{3k \binom{4k}{k} (11(k-1)^2 + 8(k-1) + 1)}{(2k-1)(4k-1)(4k-3)16^k} = \frac{1}{2}. \quad (3.24)$$

Note that

$$\frac{3k \binom{4k}{k}}{(2k-1)(4k-1)(4k-3)} = \frac{8 \binom{4(k-1)}{k-1}}{(3(k-1)+1)(3(k-1)+2)},$$

for any  $k \in \mathbb{Z}^+$ . So, from (1.10) we obtain the identity (3.24).

In view of the above, we have completed our proof of Theorem 1.4.  $\square$

#### 4. CONJECTURAL SERIES WITH SUMMANDS INVOLVING $\binom{4k}{k}$

**Conjecture 4.1** (2025-02-18). *Let  $P(k) = 22k^2 - 92k + 11$ .*

(i) *We have*

$$\sum_{k=1}^{\infty} \frac{\binom{4k}{k}}{16^k} \left( P(k)H_k - 54k + 108 - \frac{10}{3k} \right) = -\frac{20}{3} \log 2, \quad (4.1)$$

$$\sum_{k=1}^{\infty} \frac{\binom{4k}{k}}{16^k} \left( P(k)H_{2k} + 287k - 115 - \frac{25}{6k} \right) = 214 - \frac{40}{3} \log 2, \quad (4.2)$$

$$\sum_{k=1}^{\infty} \frac{\binom{4k}{k}}{16^k} \left( P(k)H_{3k} - 296k + 178 - \frac{25}{3k} \right) = -196 - \frac{80}{3} \log 2, \quad (4.3)$$

$$\sum_{k=1}^{\infty} \frac{\binom{4k}{k}}{16^k} \left( P(k)H_{4k} - \frac{449k - 275}{2} - \frac{85}{12k} \right) = -151 - \frac{80}{3} \log 2. \quad (4.4)$$

(ii) For any prime  $p > 3$ , we have the congruence

$$\begin{aligned} & \sum_{k=1}^{p-1} \frac{\binom{4k}{k}}{16^k} \left( P(k)H_k - 54k + 108 - \frac{10}{3k} \right) \\ & \equiv -10q_p(2) + 5p q_p(2)^2 - \frac{10}{3} p^2 q_p(2)^3 \pmod{p^3}. \end{aligned}$$

**Remark 4.2.** This is motivated by the identity (3.16).

**Conjecture 4.3** (2025-02-16). (i) We have

$$\sum_{k=1}^{\infty} \frac{\binom{4k}{k} ((11k^2 + 8k + 1)H_k + 6k + 6 + 4/(3k))}{(3k+1)(3k+2)16^k} = \frac{4}{3} \log 2, \quad (4.5)$$

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k} ((11k^2 + 8k + 1)(H_{2k} - \frac{5}{4}H_k) + 4k + 1)}{(3k+1)(3k+2)16^k} = \log 2, \quad (4.6)$$

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k} ((11k^2 + 8k + 1)(10H_{4k} - 17H_{2k}) + 2k + 18)}{(3k+1)(3k+2)16^k} = 8 \log 2, \quad (4.7)$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k} ((11k^2 + 8k + 1)(H_{2k}^{(2)} + \frac{3}{4}H_k^{(2)}) + \frac{4k+1}{2k+1})}{(3k+1)(3k+2)16^k} = \frac{\pi^2}{6}. \quad (4.8)$$

(ii) Let  $p$  be an odd prime. Then

$$\sum_{k=1}^{p-1} \frac{(11k^2 + 8k + 1)\binom{4k}{k}}{(3k+1)(3k+2)16^k} \equiv \frac{21}{4} p H_{p-1} \pmod{p^4}.$$

Moreover, for any  $n \in \mathbb{Z}^+$  we have

$$\frac{1}{(pn)^3} \sum_{k=n}^{pn-1} \frac{(11k^2 + 8k + 1)\binom{4k}{k}}{(3k+1)(3k+2)16^k} \in \mathbb{Z}_p,$$

where  $\mathbb{Z}_p$  is the ring of  $p$ -adic integers.

**Remark 4.4.** This is motivated by (1.10).

For convenience, we define

$$H(k) := 2H_{4k} - 3H_{2k} + H_k = 2 \sum_{k \leq j < 2k} \frac{1}{2j+1} \quad \text{for all } k \in \mathbb{N}.$$

**Conjecture 4.5** (2025-02-01). We have

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k} ((7k^2 + 10k + 3)H(k) - 2k - 4)}{(3k+1)(3k+2)24^k} = -\sqrt{3} \log 3 \quad (4.9)$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k}}{24^k} ((49k^2 - 146k + 21)H(k) - 3038k + 1160) = \sqrt{3} (216 - 5 \log 3). \quad (4.10)$$

**Remark 4.6.** This is motivated by our following observations:

$$\sum_{k=0}^{\infty} \frac{(7k^2 + 10k + 3) \binom{4k}{k}}{(3k+1)(3k+2)24^k} = \sqrt{3},$$

$$\sum_{k=0}^{\infty} \frac{(49k^2 - 146k + 21) \binom{4k}{k}}{24^k} = 5\sqrt{3}.$$

The last two identities are provable by the method we deduced Theorem 1.4.

**Conjecture 4.7** (2025-02-21). *We have*

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k} ((133k^2 + 131k + 26)H(k) + 76k + 40)}{(3k+1)(3k+2)(-25)^k} = 5\sqrt{5} \log 5 \quad (4.11)$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k}}{(-25)^k} \left( (21413k^2 - 1409k + 1036)H(k) + \frac{4}{23}(118237k + 17320) \right)$$

$$= \sqrt{5} \left( \frac{1440}{23} - 100 \log 5 \right). \quad (4.12)$$

**Remark 4.8.** This is motivated by our following observations:

$$\sum_{k=0}^{\infty} \frac{(133k^2 + 131k + 26) \binom{4k}{k}}{(3k+1)(3k+2)(-25)^k} = 5\sqrt{5},$$

$$\sum_{k=0}^{\infty} \frac{(21413k^2 - 1409k + 1036) \binom{4k}{k}}{(-25)^k} = -100\sqrt{5}.$$

The last two identities are provable by the method we deduced Theorem 1.4.

**Conjecture 4.9** (2025-02-01). *We have*

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k} ((55k^2 + 54k + 11)H(k) + 22k + 12)}{(3k+1)(3k+2)(-72)^k} = 3\sqrt{3} \log 3 \quad (4.13)$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k}}{(-72)^k} \left( (3575k^2 - 1026k + 67)H(k) + \frac{242}{13}(175k + 12) \right)$$

$$= \sqrt{3} \left( \frac{216}{13} - 15 \log 3 \right). \quad (4.14)$$

**Remark 4.10.** This is motivated by our following observations:

$$\sum_{k=0}^{\infty} \frac{(55k^2 + 54k + 11) \binom{4k}{k}}{(3k+1)(3k+2)(-72)^k} = 3\sqrt{3},$$

$$\sum_{k=0}^{\infty} \frac{(3575k^2 - 1026k + 67) \binom{4k}{k}}{(-72)^k} = -15\sqrt{3}.$$

The last two identities are provable by the method we deduced Theorem 1.4.

**Conjecture 4.11** (2025-02-16). *We have*

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k}}{128^k} ((200k^2 + 76k - 17)H(k) - 8(725k - 49)) = \sqrt{2}(144 + 5 \log 2), \quad (4.15)$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k} ((40k^2 + 44k + 11)H(k) - 8(k + 1))}{(3k + 1)(3k + 2)128^k} = -4\sqrt{2} \log 2. \quad (4.16)$$

**Remark 4.12.** In the spirit of Theorem 1.4 and its proof, the identity

$$\sum_{k=0}^{\infty} \frac{(24k^2 - 4k - 3)\binom{4k}{k}}{(2k - 1)(4k - 1)(4k - 3)128^k} = \frac{5}{6}\sqrt{2}$$

observed in [13, (3.2)] is equivalent to any of the following three identities:

$$\begin{aligned} \sum_{k=0}^{\infty} (200k^2 + 76k - 17) \frac{\binom{4k}{k}}{128^k} &= -5\sqrt{2}, \\ \sum_{k=0}^{\infty} \frac{(600k^2 + 476k - 127)\binom{4k}{k}}{(k + 1)128^k} &= -96 - 10\sqrt{2}, \\ \sum_{k=0}^{\infty} \frac{(40k^2 + 44k + 11)\binom{4k}{k}}{(3k + 1)(3k + 2)128^k} &= 4\sqrt{2}. \end{aligned}$$

**Conjecture 4.13** (2025-02-16). *We have*

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k} ((112k^2 + 110k + 23)H(k) + 28k + 16)}{(3k + 1)(3k + 2)(-256)^k} = 8\sqrt{2} \log 2 \quad (4.17)$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{4k}{k}}{(-256)^k} ((224k^2 - 86k + 1)H(k) + 182k + 5) = \frac{9 - 5 \log 2}{4\sqrt{2}}. \quad (4.18)$$

**Remark 4.14.** [5, Example 100] gives the identity

$$\sum_{k=1}^{\infty} \frac{k(112k^2 - 114k + 25)\binom{4k}{k}}{(2k - 1)(4k - 1)(4k - 3)(-256)^k} = -\frac{\sqrt{2}}{12},$$

which is equivalent to the identity

$$\sum_{k=0}^{\infty} \frac{(14k - 3)\binom{4k}{k}}{(2k - 1)(4k - 1)(4k - 3)(-256)^k} = \frac{2}{3}\sqrt{2}$$

as pointed out in [13, Remark 3.1]. In the spirit of Theorem 1.4 and its proof, the last identity is also equivalent to any of the following three identities:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(224k^2 - 86k + 1) \binom{4k}{k}}{(-256)^k} &= -\frac{5}{8}\sqrt{2}, \\ \sum_{k=0}^{\infty} \frac{(504k^2 + 310k - 191) \binom{4k}{k}}{(k+1)(-256)^k} &= -192 - \frac{5}{2}\sqrt{2}, \\ \sum_{k=0}^{\infty} \frac{(112k^2 + 110k + 23) \binom{4k}{k}}{(3k+1)(3k+2)(-256)^k} &= 8\sqrt{2}. \end{aligned}$$

**Conjecture 4.15** (2025-02-17). (i) *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{(95k^2 + 84k + 16) \binom{4k}{k}}{(3k+1)(3k+2)(9/8)^k} \equiv 6 \binom{p}{3} \pmod{p}$$

and

$$\sum_{k=0}^{p-1} \frac{(95k^2 + 84k + 16) \binom{4k}{k}}{(3k+1)(3k+2)(9/8)^k} \equiv 8 \binom{p}{3} - \frac{20}{3} p^2 B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3}.$$

Moreover, for any  $n \in \mathbb{Z}^+$  we have

$$\frac{1}{(pn)^2} \left( \sum_{k=0}^{pn-1} \frac{(95k^2 + 84k + 16) \binom{4k}{k}}{(3k+1)(3k+2)(9/8)^k} - \binom{p}{3} \sum_{k=0}^{n-1} \frac{(95k^2 + 84k + 16) \binom{4k}{k}}{(3k+1)(3k+2)(9/8)^k} \right) \in \mathbb{Z}_p.$$

(ii) *Let  $P(k) = 95k^2 - 84k + 16$ . Then*

$$\sum_{k=1}^{\infty} \frac{(P(k)(H_{4k-1} - H_{k-1}) - 25k + 12)}{k(3k-1)(3k-2) \binom{4k}{k}} \left( \frac{9}{8} \right)^{k-1} = \frac{\log 3}{\sqrt{3}} \pi + \frac{15}{4} K. \quad (4.19)$$

**Remark 4.16.** This is motivated by (1.3).

**Conjecture 4.17** (2025-02-06). (i) *We have*

$$\sum_{k=1}^{\infty} \frac{(13k^2 - 15k + 2) 9^{k-1}}{k(3k-1)(3k-2) \binom{4k}{k}} = \frac{\pi}{\sqrt{3}}. \quad (4.20)$$

(ii) *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{(13k^2 + 15k + 2) \binom{4k}{k}}{(3k+1)(3k+2) 9^k} \equiv 3 \binom{p}{3} \pmod{p}$$

and

$$\sum_{k=0}^{p-1} \frac{(13k^2 + 15k + 2) \binom{4k}{k}}{(3k+1)(3k+2) 9^k} \equiv \binom{p}{3} + \frac{2}{3} p^2 B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3}.$$

Moreover, for any  $n \in \mathbb{Z}^+$  we have

$$\frac{1}{(pn)^2} \left( \sum_{k=0}^{pn-1} \frac{(13k^2 + 15k + 2) \binom{4k}{k}}{(3k+1)(3k+2)9^k} - \left(\frac{p}{3}\right) \sum_{k=0}^{n-1} \frac{(13k^2 + 15k + 2) \binom{4k}{k}}{(3k+1)(3k+2)9^k} \right) \in \mathbb{Z}_p.$$

**Remark 4.18.** In the spirit of our proof of Theorem 1.1 in Section 2, we can get (4.20) if we have

$$\sum_{k=1}^{\infty} (221k^2 - 6411k + 844) \frac{9^{k-1}}{k \binom{4k}{k}} = 243 - \frac{20\pi}{\sqrt{3}}$$

or

$$\sum_{k=1}^{\infty} \frac{(65k^2 - 1829k - 580)9^{k-1}}{(4k+1) \binom{4k}{k}} = 52 + \frac{8\pi}{3\sqrt{3}}.$$

However, we are unable to prove either of the last two identities by the integration method in Section 2.

**Conjecture 4.19** (2025-02-06). Let  $P(k) = 13k^2 - 15k + 2 = (k-1)(13k-2)$ . Then

$$\sum_{k=1}^{\infty} \frac{9^{k-1} (P(k)(3H_{2k-1} - 4H_{k-1}) + 12(4k-1))}{k(3k-1)(3k-2) \binom{4k}{k}} = \frac{27}{4} K \quad (4.21)$$

and

$$\sum_{k=1}^{\infty} \frac{9^{k-1} (P(k)(8H_{4k-1} - 9H_{2k-1}) - 160k + 84)}{k(3k-1)(3k-2) \binom{4k}{k}} = \frac{4 \log 3}{\sqrt{3}} \pi - \frac{93}{4} K. \quad (4.22)$$

**Remark 4.20.** This is motivated by the identity (4.20).

**Conjecture 4.21** (2025-02-17). Let  $P(k) = 77k^2 - 53k + 10$ . Then

$$\sum_{k=1}^{\infty} \frac{P(k)(H_{2k-1} - H_{k-1}) - (209k^2 - 113k + 18)/(2k)}{k(3k-1)(3k-2)(-2)^k \binom{4k}{k}} = 3 \log^2 2, \quad (4.23)$$

and

$$\sum_{k=1}^{\infty} \frac{P(k)(2H_{4k-1} - H_{2k-1} - H_{k-1}) + 11k - 27 + 6/k}{k(3k-1)(3k-2)(-2)^k \binom{4k}{k}} = -\frac{\pi^2}{2}. \quad (4.24)$$

**Remark 4.22.** This is motivated by the identity (1.5).

**Conjecture 4.23** (2025-02-17). Let  $P(k) = 415k^2 - 343k + 62$ . Then

$$\sum_{k=1}^{\infty} \frac{P(k)(H_{4k-1} - H_{k-1}) - (581k^2 - 229k - 6)/(4k)}{k(3k-1)(3k-2)(-8)^k \binom{4k}{k}} = -\frac{\pi^2}{4}. \quad (4.25)$$

**Remark 4.24.** This is motivated by the identity (1.6).

**Conjecture 4.25** (2025-02-12). Let  $P(k) = 88k^3 + 108k^2 + 36k + 3$ . Then

$$\sum_{k=0}^{\infty} \frac{(P(k)(2H_{2k} - 3H_k) + 40k^2 + 56k + 14) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)1024^k} = \frac{32 \log 2}{\pi} \quad (4.26)$$

and

$$\sum_{k=0}^{\infty} \frac{(P(k)(2H_{4k} - 3H_{2k}) + 40k^2 + 32k + 8) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)1024^k} = \frac{16 \log 2}{\pi}. \quad (4.27)$$

**Remark 4.26.** [5, Example 110] gives the identity

$$\sum_{k=0}^{\infty} \frac{88k^3 + 108k^2 + 36k + 3}{(3k+1)(3k+2)1024^k} \binom{3k}{k} \binom{4k}{k}^2 = \frac{8}{\pi}.$$

**Conjecture 4.27** (2025-02-12). Let  $P(k) = 368k^3 + 400k^2 + 118k + 9$ . Then

$$\sum_{k=0}^{\infty} \frac{(P(k)(H_{2k} + H_k) - 128k^2 - 136k - 31) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)4096^k} = -\frac{64 \log 2}{\pi} \quad (4.28)$$

and

$$\sum_{k=0}^{\infty} \frac{(P(k)(2H_{4k} - 5H_{2k}) + 192k^2 + 212k + 46) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)4096^k} = \frac{96 \log 2}{\pi}. \quad (4.29)$$

**Remark 4.28.** [5, Example 106] gives the identity

$$\sum_{k=0}^{\infty} \frac{368k^3 + 400k^2 + 118k + 9}{(3k+1)(3k+2)4096^k} \binom{3k}{k} \binom{4k}{k}^2 = \frac{16}{\pi}.$$

**Conjecture 4.29** (2025-02-12). Let  $P(k) = 896k^3 + 992k^2 + 296k + 21$ . Then

$$\sum_{k=0}^{\infty} \frac{(P(k)(4H_{4k} - 4H_{2k} + 3H_k) - 576k^2 - 544k - 110) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)(-2^{14})^k} = -\frac{256 \log 2}{\pi}. \quad (4.30)$$

**Remark 4.30.** [5, Example 97] gives the identity

$$\sum_{k=0}^{\infty} \frac{896k^3 + 992k^2 + 296k + 21}{(3k+1)(3k+2)(-2^{14})^k} \binom{3k}{k} \binom{4k}{k}^2 = \frac{32}{\pi}.$$

**Conjecture 4.31** (2025-02-17). (i) Let  $P(k) = 74k^3 - 84k^2 + 29k - 3$ . Then

$$\sum_{k=1}^{\infty} \frac{(P(k)(3H_{2k-1} - 4H_{k-1}) - 57k^2 + 33k - 5)256^k}{k^3(3k-1)(3k-2) \binom{3k}{k} \binom{4k}{k}^2} = 24\pi^2 \log 2 - 84\zeta(3), \quad (4.31)$$

$$\sum_{k=1}^{\infty} \frac{(P(k)(4H_{4k-1} - 7H_{2k-1}) + 51k^2 - 15k - 1)256^k}{k^3(3k-1)(3k-2) \binom{3k}{k} \binom{4k}{k}^2} = 24\pi^2 \log 2 - 84\zeta(3), \quad (4.32)$$

and

$$\sum_{k=1}^{\infty} \frac{(P(k)(4H_{4k-1}^{(2)} - H_{2k-1}^{(2)} - 4H_{k-1}^{(2)}) + 4k - 4)256^k}{k^3(3k-1)(3k-2) \binom{3k}{k} \binom{4k}{k}^2} = 2\pi^4. \quad (4.33)$$

(ii) Let  $p$  be an odd prime, and set  $P^*(k) = 74k^3 + 84k^2 + 29k + 3$ . Then

$$\sum_{k=0}^{(p-1)/2} \frac{P^*(k) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)256^k} \equiv p + 2p^2 \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1} \frac{P^*(k) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)256^k} \equiv \frac{3}{2}p + \frac{21}{4}p^4 B_{p-3} \pmod{p^5}.$$

Moreover, for any  $n \in \mathbb{Z}^+$  we have

$$\frac{1}{(pn)^4} \left( \sum_{k=0}^{pn-1} \frac{P^*(k) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)256^k} - p \sum_{k=0}^{n-1} \frac{P^*(k) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)256^k} \right) \in \mathbb{Z}_p.$$

**Remark 4.32.** This is motivated by the identity

$$\sum_{k=1}^{\infty} \frac{(74k^3 - 84k^2 + 29k - 3)256^k}{k^3(3k-1)(3k-2) \binom{3k}{k} \binom{4k}{k}^2} = 6\pi^2$$

given by [5, Example 51].

**Conjecture 4.33** (2025-02-28). (i) Let  $P(k) = 77k^3 - 86k^2 + 29k - 3$ . Then

$$\sum_{k=1}^{\infty} \frac{P(k)(-32)^k}{k^3(3k-1)(3k-2) \binom{3k}{k} \binom{4k}{k}^2} = -6G \quad (4.34)$$

and

$$\sum_{k=1}^{\infty} \frac{(P(k)(4H_{4k-1}^{(2)} - H_{2k-1}^{(2)} - 4H_{k-1}^{(2)} - 8k + 4)(-32)^{k-1})}{k^3(3k-1)(3k-2) \binom{3k}{k} \binom{4k}{k}^2} = \frac{3}{4}\beta(4), \quad (4.35)$$

where  $\beta(4) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^4}$ .

(ii) Let  $p$  be an odd prime, and set  $P^*(k) = 77k^3 + 86k^2 + 29k + 3$ . Then

$$\sum_{k=0}^{(p-1)/2} \frac{P^*(k) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)(-32)^k} \equiv \left(\frac{-1}{p}\right) p \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{P^*(k) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)(-32)^k} \equiv \left(\frac{-1}{p}\right) \frac{3}{2}p + \frac{9}{2}p^3 E_{p-3} \pmod{p^4}.$$

Moreover, for any  $n \in \mathbb{Z}^+$  we have

$$\frac{1}{(pn)^3} \left( \sum_{k=0}^{pn-1} \frac{P^*(k) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)(-32)^k} - \left(\frac{-1}{p}\right) p \sum_{k=0}^{n-1} \frac{P^*(k) \binom{3k}{k} \binom{4k}{k}^2}{(3k+1)(3k+2)(-32)^k} \right) \in \mathbb{Z}_p.$$

**Remark 4.34.** For the series in (4.34), its rate of convergence is  $-27/512$ .

## 5. OTHER CONJECTURAL SERIES INVOLVING BINOMIAL COEFFICIENTS

In this section, we propose other new conjectures on series whose summands involve binomial coefficients and generalized harmonic numbers.

**Conjecture 5.1** (2023-10-15). (i) *We have*

$$\sum_{k=1}^{\infty} \frac{10k-3}{(2k-1)k^2 3^k \binom{2k}{k} \binom{3k}{k}} = \frac{K}{2} \quad (5.1)$$

and

$$\sum_{k=1}^{\infty} \frac{(4k-1)3^k}{(2k-1)k^2 \binom{2k}{k} \binom{3k}{k}} = 2K. \quad (5.2)$$

Also,

$$\sum_{k=1}^{\infty} \frac{(10k-3)(3H_{3k-1} + 4H_{2k-1} - 6H_{k-1}) - 12}{(2k-1)k^2 3^k \binom{2k}{k} \binom{3k}{k}} = \frac{4\pi^3}{27\sqrt{3}} \quad (5.3)$$

and

$$\sum_{k=1}^{\infty} \frac{3^k ((4k-1)(3H_{3k-1} - 3H_{2k-1} - H_{k-1}) + 6k/(2k-1))}{(2k-1)k^2 \binom{2k}{k} \binom{3k}{k}} = \frac{8\pi^3}{27\sqrt{3}}. \quad (5.4)$$

(ii) *Let  $p$  be a prime. For any  $n \in \mathbb{Z}^+$ , we have*

$$\frac{1}{(pn)^2} \left( \sum_{k=0}^{pn-1} \frac{(10k+3)3^k \binom{2k}{k} \binom{3k}{k}}{2k+1} - \left(\frac{p}{3}\right) \sum_{k=0}^{n-1} \frac{(10k+3)3^k \binom{2k}{k} \binom{3k}{k}}{2k+1} \right) \in \mathbb{Z}_p$$

where  $\mathbb{Z}_p$  is the ring of  $p$ -adic integers. When  $p > 3$ , for any  $n \in \mathbb{Z}^+$  we also have

$$\frac{1}{(pn)^2} \left( \sum_{k=0}^{pn-1} \frac{(4k+1) \binom{2k}{k} \binom{3k}{k}}{(2k+1)3^k} - \left(\frac{p}{3}\right) \sum_{k=0}^{n-1} \frac{(4k+1) \binom{2k}{k} \binom{3k}{k}}{(2k+1)3^k} \right) \in \mathbb{Z}_p.$$

Moreover, provided  $p > 3$  we have the following congruences:

$$\sum_{k=0}^{p-1} \frac{(10k+3)3^k \binom{2k}{k} \binom{3k}{k}}{2k+1} \equiv 3 \left(\frac{p}{3}\right) + p^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3},$$

$$\sum_{k=0}^{p-1} \frac{(4k+1) \binom{2k}{k} \binom{3k}{k}}{(2k+1)3^k} \equiv \left(\frac{p}{3}\right) + \frac{4}{9} p^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3},$$

$$p \sum_{k=0}^{p-1} \frac{3^k \binom{2k}{k} \binom{3k}{k}}{2k+1} ((10k+3)(3H_{3k} + 4H_{2k} - 6H_k) + 12)$$

$$\equiv 12 \left(\frac{p}{3}\right) + \frac{17}{2} p^2 B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3},$$

and

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k}}{(2k+1)3^k} \left( (4k+1)(3H_{3k} - 3H_{2k} - H_k) - \frac{6k}{2k+1} \right) \\ & \equiv \frac{2}{3} p B_{p-2} \left( \frac{1}{3} \right) \pmod{p^2}. \end{aligned}$$

**Remark 5.2.** For the series in (5.1), its rate of convergence is  $1/81$ . This series seems to be the fastest series for the constant  $K$ . Note also that

$$\frac{\binom{3k}{k}}{2k+1} = \binom{3k}{k} - 2 \binom{3k}{2k+1} \in \mathbb{Z} \quad \text{for all } k \in \mathbb{N}.$$

**Conjecture 5.3** (2023-08-21). (i) *We have*

$$\sum_{k=1}^{\infty} \frac{(-16)^k}{k^2 \binom{2k}{k} \binom{4k}{2k}} \left( \frac{5k-1}{2k-1} H_{k-1} - \frac{1}{8k} \right) = \pi^2 \log 2 - \frac{21}{4} \zeta(3), \quad (5.5)$$

$$\sum_{k=1}^{\infty} \frac{(-16)^k}{k^2 \binom{2k}{k} \binom{4k}{2k}} \left( \frac{5k-1}{2k-1} H_{2k-1} + \frac{1}{12k} \right) = \frac{\pi^2}{3} \log 2 - \frac{35}{6} \zeta(3), \quad (5.6)$$

and

$$\sum_{k=1}^{\infty} \frac{(-16)^k}{(2k-1)k^2 \binom{2k}{k} \binom{4k}{2k}} \left( (5k-1)H_{4k-1} - \frac{46k-5}{24k} \right) = \frac{\pi^2}{6} \log 2 - \frac{77}{12} \zeta(3). \quad (5.7)$$

(ii) *Let  $p$  be any odd prime. Then*

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k} \binom{4k}{2k}}{(-16)^k} \left( \frac{5k+1}{2k+1} H_k + \frac{1}{8k} \right) \equiv -\frac{3}{2} q_p(2) - \frac{5}{4} p q_p(2)^2 \pmod{p^2}$$

and

$$\sum_{p/2 < k < p} \frac{\binom{2k}{k} \binom{4k}{2k}}{(-16)^k} \left( \frac{5k+1}{2k+1} H_k + \frac{1}{8k} \right) \equiv 2p q_p(2)^2 \pmod{p^2}.$$

Provided  $p > 3$ , we have

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k} \binom{4k}{2k}}{(-16)^k} \left( \frac{5k+1}{2k+1} H_{2k} - \frac{1}{12k} \right) \equiv -\frac{p}{3} q_p(2)^2 \pmod{p^2}$$

and

$$\sum_{p/2 < k < p} \frac{\binom{2k}{k} \binom{4k}{2k}}{(-16)^k} \left( \frac{5k+1}{2k+1} H_{2k} - \frac{1}{12k} \right) \equiv -q_p(2) + \frac{5}{6} p q_p(2)^2 \pmod{p^2}.$$

**Remark 5.4.** Theorem 9 of [5] with  $a = e = 1$  and  $b = c = d = 1/2$  yields the identity

$$\sum_{k=1}^{\infty} \frac{(-16)^k (5k-1)}{(2k-1)k^2 \binom{2k}{k} \binom{4k}{2k}} = -\frac{\pi^2}{2}.$$

**Conjecture 5.5** (2023-10-28). (i) *We have*

$$\sum_{k=1}^{\infty} \frac{(-16)^k ((5k-1)H_{k-1}^{(2)} - 4(4k-1)/(2k-1)^2)}{(2k-1)k^2 \binom{2k}{k} \binom{4k}{2k}} = \frac{\pi^4}{6} \quad (5.8)$$

and

$$\sum_{k=1}^{\infty} \frac{(-16)^k ((5k-1)H_{k-1}^{(3)} + 12(4k-1)/(2k-1)^3)}{(2k-1)k^2 \binom{2k}{k} \binom{4k}{2k}} = -4\pi^2 \zeta(3). \quad (5.9)$$

(ii) *For any odd prime  $p$ , we have*

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} \binom{4k}{2k}}{(2k+1)(-16)^k} \left( (5k+1)H_k^{(2)} + \frac{4(4k+1)}{(2k+1)^2} \right) \equiv -\frac{p}{6} B_{p-3} \pmod{p^2}$$

and

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k} \binom{4k}{2k}}{(2k+1)(-16)^k} \left( (5k+1)H_k^{(3)} - \frac{12(4k+1)}{(2k+1)^3} \right) \equiv -6B_{p-3} \pmod{p}.$$

**Remark 5.6.** We have not found any identity involving the fourth harmonic numbers similar to (5.8) and (5.9).

**Conjecture 5.7** (2023-11-18). *We have*

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{(-256)^k (P(k)(H_{2k-1} - 2H_{k-1}) - (4k-1)(58k^2 + 181k - 66)/(26k))}{k^2(2k-1)(4k-1)(4k-3) \binom{3k}{k} \binom{6k}{3k}} \\ &= \frac{1680}{13} \zeta(3) - 32\pi^2 \log 2 \end{aligned} \quad (5.10)$$

and

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(-256)^k (P(k)(2H_{6k-1} - H_{3k-1} - 3H_{k-1}) - Q(k)/(13k))}{k^2(2k-1)(4k-1)(4k-3) \binom{3k}{k} \binom{6k}{3k}} \\ &= \frac{2464}{13} \zeta(3) - 64\pi^2 \log 2, \end{aligned} \quad (5.11)$$

where

$$P(k) = 344k^3 - 386k^2 + 115k - 9 \text{ and } Q(k) = 1952k^3 - 1732k^2 + 315k + 12.$$

**Remark 5.8.** This is motivated by the identity

$$\sum_{k=1}^{\infty} \frac{(-256)^k (344k^3 - 386k^2 + 115k - 9)}{k^2(2k-1)(4k-1)(4k-3) \binom{3k}{k} \binom{6k}{3k}} = -8\pi^2$$

given by [5, Example 44].

**Conjecture 5.9** (2023-10-16). (i) *We have*

$$\sum_{k=1}^{\infty} \frac{(8k-3) \binom{4k}{2k}}{k(4k-1)9^k \binom{2k}{k}^2} = \frac{\sqrt{3}\pi}{18}. \quad (5.12)$$

Also,

$$\sum_{k=1}^{\infty} \frac{\binom{4k}{2k} ((8k-3)(5H_{2k-1} - 4H_{k-1}) - 6)}{k(4k-1)9^k \binom{2k}{k}^2} = \frac{3}{2}K$$

and

$$\sum_{k=1}^{\infty} \frac{\binom{4k}{2k} (8k-3) (2H_{2k-1}^{(2)} - 5H_{k-1}^{(2)})}{k(4k-1)9^k \binom{2k}{k}^2} = \frac{\pi^3}{36\sqrt{3}},$$

(ii) Let  $p > 3$  be a prime. Then

$$\sum_{k=1}^{(p-1)/2} \frac{(8k-3) \binom{4k}{2k}}{k(4k-1)9^k \binom{2k}{k}^2} \equiv -\frac{5}{36} p B_{p-2} \left( \frac{1}{3} \right) \pmod{p^2}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{4k}{2k} ((8k-3)(5H_{2k-1} - 4H_{k-1}) - 6)}{k(4k-1)9^k \binom{2k}{k}^2} \equiv \frac{1}{6} B_{p-2} \left( \frac{1}{3} \right) \pmod{p}.$$

**Remark 5.10.** Part (i) is related to [13, Conjecture 4.2]. For the series in (5.12), its rate of convergence is  $1/9$ .

**Conjecture 5.11** (2023-06-18). (i) We have

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3}{4096^k} \left( (42k+5)H_k^{(3)} - \frac{352}{(2k+1)^2} \right) = \frac{32}{7} \left( 335 \frac{\zeta(3)}{\pi} - 224G \right). \quad (5.13)$$

(ii) For any prime  $p > 3$  with  $p \neq 7$ , we have

$$\begin{aligned} & \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^3}{4096^k} \left( (42k+5)H_k^{(3)} - \frac{352}{(2k+1)^2} \right) \\ & \equiv -32 \left( \left( \frac{-1}{p} \right) B_{p-3} + 16E_{p-3} \right) \pmod{p}. \end{aligned}$$

**Remark 5.12.** The present author [16, (110)] conjectured that

$$\sum_{k=0}^{\infty} (42k+5) \frac{\binom{2k}{k}^3}{4096^k} \left( H_{2k}^{(3)} - \frac{43}{352} H_k^{(3)} \right) = \frac{555}{77} \cdot \frac{\zeta(3)}{\pi} - \frac{32}{11} G,$$

which is still open.

**Conjecture 5.13** (2023-06-18). (i) We have

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3}{4096^k} \left( 9(42k+5) \left( H_{2k}^{(4)} - \frac{H_k^{(4)}}{16} \right) + \frac{25}{(2k+1)^3} \right) = \frac{5}{6} \pi^3. \quad (5.14)$$

(ii) For any odd prime  $p \neq 5$ , we have

$$\begin{aligned} & \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^3}{4096^k} \left( 9(42k+5) \left( H_{2k}^{(4)} - \frac{H_k^{(4)}}{16} \right) + \frac{25}{(2k+1)^3} \right) \\ & \equiv -4 \left( \frac{-1}{p} \right) B_{p-3} \pmod{p}. \end{aligned}$$

**Remark 5.14.** This was motivated by (1.1) and Conjecture 5.11.

**Conjecture 5.15** (2023-08-17). (i) We have

$$\sum_{k=1}^{\infty} \frac{8^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} \left( (50k-15)H_{k-1}^{(2)} + \frac{4}{k} \right) = \frac{\pi^4}{24}. \quad (5.15)$$

(ii) For any odd prime  $p$ , we have

$$\begin{aligned} & \sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{8^k} \left( (50k+15)H_k^{(2)} - \frac{4}{k} \right) \\ & \equiv -12q_p(2) + 6p q_p(2)^2 - 4p^2 q_p(2)^3 + 3p^3 q_p(2)^4 \pmod{p^4}. \end{aligned}$$

**Remark 5.16.** The present author's conjectural identity

$$\sum_{k=1}^{\infty} \frac{8^k ((10k-3)(H_{2k-1} - H_{k-1}) - 1)}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} = \frac{7}{2} \zeta(3)$$

(cf. [16, (63)]) remains open.

**Conjecture 5.17** (2023-08-17). (i) We have

$$\sum_{k=1}^{\infty} \frac{64^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} \left( (55k-15)H_{k-1}^{(2)} + \frac{8}{k} \right) = \frac{8}{3} \pi^4. \quad (5.16)$$

(ii) For any odd prime  $p$ , we have

$$\begin{aligned} & \sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{64^k} \left( (55k+15)H_k^{(2)} - \frac{8}{k} \right) \\ & \equiv -48q_p(2) + 24p q_p(2)^2 - 16p^2 q_p(2)^3 + 12p^3 q_p(2)^4 \pmod{p^4}. \end{aligned}$$

**Remark 5.18.** The present author's conjectural identity

$$\sum_{k=1}^{\infty} \frac{64^{k-1} ((11k-3)(2H_{2k-1} + H_{k-1}) - 4)}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} = \frac{7}{2} \zeta(3)$$

(cf. [16, (71)]) remains open.

**Conjecture 5.19** (2023-08-17). (i) We have

$$\sum_{k=1}^{\infty} \frac{81^k}{k^3 \binom{2k}{k}^2 \binom{4k}{2k}} \left( (350k-80)H_{k-1}^{(2)} + \frac{27}{k} \right) = 4\pi^4. \quad (5.17)$$

(ii) For any prime  $p > 3$ , we have

$$\begin{aligned} & \sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{81^k} \left( (350k + 80)H_k^{(2)} - \frac{27}{k} \right) \\ & \equiv -108q_p(3) + 54p q_p(3)^2 - 36p^2 q_p(3)^3 + 27p^3 q_p(3)^4 \pmod{p^4}. \end{aligned}$$

**Remark 5.20.** The present author's conjectural identity

$$\sum_{k=1}^{\infty} \frac{81^k ((35k - 8)(H_{4k-1} - H_{k-1}) - 35/4)}{k^3 \binom{2k}{k}^2 \binom{4k}{2k}} = 12\pi^2 \log 3 + 39\zeta(3)$$

(cf. [16, (75)]) remains open.

**Conjecture 5.21** (2023-10-13). (i) We have

$$\sum_{k=1}^{\infty} \frac{17(145k^2 - 104k + 18)H_{k-1}^{(3)} + 28(2k - 1)/k^2}{k^3(2k - 1)\binom{2k}{k}\binom{3k}{k}^2} = 528\zeta(5) - 46\pi^2\zeta(3). \quad (5.18)$$

(ii) Let  $n \in \mathbb{Z}^+$ . Then

$$\frac{1}{6n(2n - 1)\binom{3n}{n}} \sum_{k=0}^{n-1} (145k^2 + 104k + 18) \frac{\binom{2k}{k}\binom{3k}{k}^2}{2k + 1} \in \mathbb{Z}.$$

Also,

$$\frac{1}{(pn)^4} \left( \sum_{k=0}^{pn-1} (145k^2 + 104k + 18) \frac{\binom{2k}{k}\binom{3k}{k}^2}{2k + 1} - p \sum_{k=0}^{n-1} (145k^2 + 104k + 18) \frac{\binom{2k}{k}\binom{3k}{k}^2}{2k + 1} \right)$$

is a  $p$ -adic integer for every prime  $p$ .

**Remark 5.22.** Let  $P(k) = 145k^2 - 104k + 18$ . In 2023, the present author [11, Remark 4.13] conjectured that

$$\sum_{k=1}^{\infty} \frac{P(k)}{(2k - 1)k^3 \binom{2k}{k} \binom{3k}{k}^2} = \frac{\pi^2}{3}.$$

This, and the present author's following four conjectural identities

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{6P(k)(H_{3k-1} - H_{k-1}) - 232k + 89}{k^3(2k - 1)\binom{2k}{k}\binom{3k}{k}^2} = 18\zeta(3), \\ & \sum_{k=1}^{\infty} \frac{P(k)(H_{2k-1} - H_{k-1}) - \frac{3(58k^2 - 40k + 7)}{2(2k-1)}}{k^3(2k - 1)\binom{2k}{k}\binom{3k}{k}^2} = \zeta(3), \\ & \sum_{k=1}^{\infty} \frac{P(k)(H_{3k-1}^{(2)} - 2H_{k-1}^{(2)}) - \frac{17k+32}{9k}}{k^3(2k - 1)\binom{2k}{k}\binom{3k}{k}^2} = \frac{7\pi^4}{180}, \end{aligned}$$

and

$$\sum_{k=1}^{\infty} \frac{P(k)(297H_{3k-1}^{(2)} - 192H_{2k-1}^{(2)} - 978H_{k-1}^{(2)}) + \frac{27(180k^2+12k-35)}{(2k-1)^2}}{k^3(2k-1)\binom{2k}{k}\binom{3k}{k}^2} = \frac{167}{20}\pi^4,$$

were confirmed by K. C. Au via the WZ method (cf. [12]).

**Conjecture 5.23** (2023-10-07). (i) *We have*

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^k(59(410k^2 - 197k + 24)H_{k-1}^{(3)} - 62(2k-1)/k^2)}{k^3(2k-1)\binom{2k}{k}\binom{4k}{2k}^2} \\ = 6(57\pi^2\zeta(3) - 652\zeta(5)). \end{aligned} \quad (5.19)$$

(ii) *For any prime  $p$  and positive integer  $n$ , we have*

$$\frac{1}{(pn)^4} \left( \sum_{k=0}^{pn-1} (-1)^k P(k) \frac{\binom{2k}{k}\binom{4k}{2k}^2}{2k+1} - p \sum_{k=0}^{n-1} (-1)^k P(k) \frac{\binom{2k}{k}\binom{4k}{2k}^2}{2k+1} \right) \in \mathbb{Z}_p,$$

where  $P(k) = 410k^2 + 197k + 24$ .

**Remark 5.24.** [5, Example 63] indicates that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(410k^2 - 197k + 24)}{k^3(2k-1)\binom{2k}{k}\binom{4k}{2k}^2} = \frac{\pi^2}{3}.$$

**Conjecture 5.25** (2023-11-13). *Let  $P(k) = 360k^3 + 612k^2 + 230k + 15$ .*

(i) *We have*

$$\sum_{k=0}^{\infty} \frac{P(k)\binom{3k}{k}\binom{6k}{3k}^2}{(3k+1)(3k+2)2^{15k}} = \frac{32\sqrt{2}}{\pi}. \quad (5.20)$$

Moreover, for any odd prime  $p$  we have

$$\sum_{k=0}^{(p-1)/2} \frac{P(k)\binom{3k}{k}\binom{6k}{3k}^2}{(3k+1)(3k+2)2^{15k}} \equiv 12p \left( \frac{-2}{p} \right) - 18p^2 \left( \frac{2}{p} \right) \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1} \frac{P(k)\binom{3k}{k}\binom{6k}{3k}^2}{(3k+1)(3k+2)2^{15k}} \equiv \frac{15}{2}p \left( \frac{-2}{p} \right) - \frac{225}{32}p^3 E_{p-3} \left( \frac{1}{4} \right) \pmod{p^4}.$$

(ii) *We have*

$$\sum_{k=0}^{\infty} \frac{\binom{3k}{k}\binom{6k}{3k}^2 (P(k)(H_{2k} - H_k) - 180k^2 + 36k + 23)}{(3k+1)(3k+2)2^{15k}} = 48\sqrt{2} \frac{\log 2}{\pi}, \quad (5.21)$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{3k}{k}\binom{6k}{3k}^2 (P(k)(4H_{6k} - 3H_{3k} - H_k) + f(k))}{(3k+1)(3k+2)2^{15k}} = 192\sqrt{2} \frac{\log 2}{\pi}, \quad (5.22)$$

where  $f(k) = (1296k^3 + 1980k^2 + 912k + 139)/((3k+1)(3k+2))$ .

**Remark 5.26.** For the series in (5.20), its rate of convergence is  $27/32$ .

**Conjecture 5.27** (2025-02-07). (i) *We have*

$$\sum_{k=1}^{\infty} \frac{(-1)^k (28k^2 - 8k + 1) \binom{2k}{k}^2}{k(2k-1)^2 \binom{6k}{3k} \binom{3k}{k}} = -2 \log 2. \quad (5.23)$$

(ii) *For any prime  $p > 3$ , we have*

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k (28k^2 - 8k + 1) \binom{2k}{k}^2}{k(2k-1)^2 \binom{6k}{3k} \binom{3k}{k}} \equiv -2q_p(2) + p q_p(2)^2 \pmod{p^2}.$$

**Remark 5.28.** For the series in (5.23), its rate of convergence is  $-1/27$ .

**Conjecture 5.29** (2023-10-06). *Let  $P(k) = 60k^2 - 26k + 3$ . Then*

$$\sum_{k=1}^{\infty} \frac{256^k (P(k)(5H_{2k-1} - 2H_{k-1}) - 9(4k-1)^2/(2k-1))}{(2k-1)k^4 \binom{2k}{k}^2 \binom{4k}{2k}^2} = 2\pi^4 \quad (5.24)$$

and

$$\sum_{k=1}^{\infty} \frac{256^k (P(k)(2H_{4k-1} - H_{2k-1}) - 4(2k^2 - 5k + 1)/(2k-1))}{(2k-1)k^4 \binom{2k}{k}^2 \binom{4k}{2k}^2} = 2\pi^4. \quad (5.25)$$

Also,

$$\sum_{k=1}^{\infty} \frac{256^k (P(k)H_{k-1}^{(2)} - (136k^3 - 76k^2 + 14k - 1)/(k(2k-1)^2))}{(2k-1)k^4 \binom{2k}{k}^2 \binom{4k}{2k}^2} = -124\zeta(5) \quad (5.26)$$

and

$$\sum_{k=1}^{\infty} \frac{256^k (P(k)(H_{4k-1}^{(2)} - H_{2k-1}^{(2)}/4) - Q(k)/(k(2k-1)^2))}{(2k-1)k^4 \binom{2k}{k}^2 \binom{4k}{2k}^2} = 0, \quad (5.27)$$

where  $Q(k) := 92k^3 - 64k^2 + 15k - 1$ .

**Remark 5.30.** [5, Example 12] has the following equivalent form:

$$\sum_{k=1}^{\infty} \frac{256^k (60k^2 - 26k + 3)}{(2k-1)k^4 \binom{2k}{k}^2 \binom{4k}{2k}^2} = 56\zeta(3).$$

**Conjecture 5.31** (2023-10-06). *Let  $P(k) = 364k^2 - 227k + 36$ . Then*

$$\sum_{k=1}^{\infty} \frac{P(k)(3H_{2k-1} - 2H_{k-1}) - (1276k^2 - 844k + 139)/(4k-2)}{(2k-1)k^4 \binom{2k}{k}^2 \binom{3k}{k}^2} = \frac{\pi^4}{15} \quad (5.28)$$

and

$$\sum_{k=1}^{\infty} \frac{P(k)(H_{3k-1} - H_{k-1}) - (728k^2 - 728k + 155)/(12k-6)}{(2k-1)k^4 \binom{2k}{k}^2 \binom{3k}{k}^2} = \frac{\pi^4}{15}. \quad (5.29)$$

Also,

$$\sum_{k=1}^{\infty} \frac{P(k)(99H_{3k-1}^{(2)} - 757H_{k-1}^{(2)}) + 18Q(k)/(2k-1)^2}{(2k-1)k^4 \binom{2k}{k}^2 \binom{3k}{k}^2} = 1316\zeta(5), \quad (5.30)$$

where  $Q(k) := 2952k^2 - 1572k + 163$ .

**Remark 5.32.** [5, Example 118] has the following equivalent form:

$$\sum_{k=1}^{\infty} \frac{364k^2 - 227k + 36}{(2k-1)k^4 \binom{2k}{k}^2 \binom{3k}{k}^2} = 4\zeta(3).$$

**Conjecture 5.33** (2025-06-26). *Set  $P(k) = 1344k^3 + 944k^2 + 156k + 9$ .*

(i) *We have*

$$\sum_{k=0}^{\infty} P(k) \frac{\binom{2k}{k}^2 \binom{4k}{2k}^3}{(2k+1)2^{22k}} = \frac{64\sqrt{2}}{\pi^2}. \quad (5.31)$$

Also,

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 \binom{4k}{2k}^3 ((3H_{4k} - 2H_{2k} - H_k)P(k) + 672k^2 + 404k + 34 + \frac{1}{4k+2})}{(2k+1)2^{22k}} \\ &= \frac{352\sqrt{2} \log 2}{\pi^2} \end{aligned} \quad (5.32)$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2 \binom{4k}{2k}^3}{(2k+1)2^{22k}} \left( (32H_{4k}^{(2)} - 8H_{2k}^{(2)} - H_k^{(2)})P(k) + 32(12k+5) \right) = \frac{352}{3}\sqrt{2}. \quad (5.33)$$

(ii) *Let  $p$  be any odd prime. Then*

$$\sum_{k=0}^{(p-1)/2} P(k) \frac{\binom{2k}{k}^2 \binom{4k}{2k}^3}{(2k+1)2^{22k}} \equiv p^2 \left( 1 + 8 \left( \frac{2}{p} \right) \right) \pmod{p^3}.$$

Moreover, for any  $n \in \mathbb{Z}^+$  we have

$$\frac{1}{(pn)^5} \left( \sum_{k=0}^{pn-1} P(k) \frac{\binom{2k}{k}^2 \binom{4k}{2k}^3}{(2k+1)2^{22k}} - p^2 \left( \frac{2}{p} \right) \sum_{k=0}^{n-1} P(k) \frac{\binom{2k}{k}^2 \binom{4k}{2k}^3}{(2k+1)2^{22k}} \right) \in \mathbb{Z}_p.$$

**Remark 5.34.** For the series in (5.31), its rate of convergence is  $1/64$ .

**Conjecture 5.35** (2023-11-15). (i) *We have*

$$\sum_{k=0}^{\infty} \frac{(28k^2 + 10k + 1) \binom{2k}{k}^5}{(6k+1)(-64)^k \binom{3k}{k} \binom{6k}{3k}} = \frac{3}{\pi}, \quad (5.34)$$

$$\sum_{k=0}^{\infty} \frac{((28k^2 + 10k + 1)(2H_{2k} - 3H_k) + 20k + 4) \binom{2k}{k}^5}{(6k+1)(-64)^k \binom{3k}{k} \binom{6k}{3k}} = \frac{18 \log 2}{\pi}, \quad (5.35)$$

and

$$\sum_{k=0}^{\infty} \frac{((28k^2 + 10k + 1)(2H_{6k} - H_{3k} - 3H_k) + f(k)) \binom{2k}{k}^5}{(6k + 1)(-64)^k \binom{3k}{k} \binom{6k}{3k}} = \frac{30 \log 2}{\pi}, \quad (5.36)$$

where  $f(k) = 4(138k^2 + 52k + 5)/(3(6k + 1))$ .

(ii) Let  $p$  be an odd prime. If  $p > 3$ , then

$$\sum_{k=0}^{(p-1)/2} \frac{(28k^2 + 10k + 1) \binom{2k}{k}^5}{(6k + 1)(-64)^k \binom{3k}{k} \binom{6k}{3k}} \equiv \left(\frac{-1}{p}\right) \left(p + \frac{7}{48} p^4 B_{p-3}\right) \pmod{p^5}.$$

When  $p \neq 5$ , we have

$$\sum_{k=0}^{p-1} \frac{(28k^2 + 10k + 1) \binom{2k}{k}^5}{(6k + 1)(-64)^k \binom{3k}{k} \binom{6k}{3k}} \equiv p \left(\frac{-1}{p}\right) + \frac{p^3}{5} E_{p-3} \pmod{p^4}.$$

**Remark 5.36.** For the series in (5.34), its rate of convergence is  $-1/27$ .

**Conjecture 5.37** (2023-11-15). (i) We have

$$\sum_{k=0}^{\infty} \frac{\left((28k^2 + 10k + 1)(10H_{2k}^{(2)} - 3H_k^{(2)}) + 2\right) \binom{2k}{k}^5}{(6k + 1)(-64)^k \binom{3k}{k} \binom{6k}{3k}} = \frac{\pi}{2}. \quad (5.37)$$

(ii) Let  $p > 3$  be a prime. Then

$$\begin{aligned} p \sum_{k=0}^{(p-1)/2} \frac{\left((28k^2 + 10k + 1)(10H_{2k}^{(2)} - 3H_k^{(2)}) + 2\right) \binom{2k}{k}^5}{(6k + 1)(-64)^k \binom{3k}{k} \binom{6k}{3k}} \\ \equiv \left(\frac{-1}{p}\right) \left(2 + \frac{35}{24} p^3 B_{p-3}\right) \pmod{p^4} \end{aligned}$$

and

$$\begin{aligned} p \sum_{k=0}^{p-1} \frac{\left((28k^2 + 10k + 1)(10H_{2k}^{(2)} - 3H_k^{(2)}) + 2\right) \binom{2k}{k}^5}{(6k + 1)(-64)^k \binom{3k}{k} \binom{6k}{3k}} \\ \equiv 2 \left(\frac{-1}{p}\right) + 2p^2 E_{p-3} \pmod{p^3}. \end{aligned}$$

**Remark 5.38.** Compare this with Conjecture 5.35.

**Conjecture 5.39** (2025-02-04). We have

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^5 \left((28k^2 + 10k + 1)(28H_{2k}^{(3)} - 3H_k^{(3)}) + 8/(2k + 1)\right)}{(6k + 1)(-64)^k \binom{3k}{k} \binom{6k}{3k}} = 75 \frac{\zeta(3)}{\pi} - 24G. \quad (5.38)$$

**Remark 5.40.** We haven't found identities similar to (5.38) involving harmonic numbers of order greater than three.

**Conjecture 5.41** (2025-01-24). (i) *We have the formula*

$$\sum_{k=1}^{\infty} \frac{256^k (21k^3 - 22k^2 + 8k - 1) (64H_{2k-1}^{(6)} - 65H_{k-1}^{(6)})}{k^7 \binom{2k}{k}^7} = \frac{31\pi^{10}}{3780}. \quad (5.39)$$

(ii) *For any prime  $p \geq 5$ , we have the congruence*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^7 (21k^3 + 22k^2 + 8k + 1) (64H_{2k}^{(6)} - 65H_k^{(6)})}{256^k} \equiv \frac{1488}{5} p^2 B_{p-5} \pmod{p^3}.$$

**Remark 5.42.** K. C. Au [2] used the WZ method to confirm the identity

$$\sum_{k=1}^{\infty} \frac{(21k^3 - 22k^2 + 8k - 1) 256^k}{k^7 \binom{2k}{k}^7} = \frac{\pi^4}{8}$$

conjectured by J. Guillera [7] in 2003. Let  $P(k) := 21k^3 - 22k^2 + 8k - 1$ . The present author conjectured the identities

$$\sum_{k=1}^{\infty} \frac{256^k}{k^7 \binom{2k}{k}^7} \left( P(k) (4H_{2k-1}^{(2)} - 5H_{k-1}^{(2)}) - 6k + 2 \right) = \frac{\pi^6}{24}$$

and

$$\sum_{k=1}^{\infty} \frac{256^k}{k^7 \binom{2k}{k}^7} \left( P(k) (16H_{2k-1}^{(4)} + 7H_{k-1}^{(4)}) + \frac{4}{k} \right) = \frac{31\pi^8}{1440}$$

in the papers [16, (178)] and [15, (5.20)], respectively.

**Conjecture 5.43** (2025-02-02). *Let*

$$P(k) = 5460k^4 - 8341k^3 + 4864k^2 - 1280k + 128.$$

*Then*

$$\sum_{k=1}^{\infty} \frac{\binom{4k}{2k} (P(k) (H_{2k-1}^{(2)} - 2H_{k-1}^{(2)}) - 453k^2 + 1349k/4 - 64)}{(4k-1)k^7 \binom{2k}{k}^8} = \frac{\pi^6}{189}. \quad (5.40)$$

*Also,*

$$\sum_{k=1}^{\infty} \frac{\binom{4k}{2k} (P(k) (H_{2k-1}^{(4)} + H_{k-1}^{(4)}) + (1796k - 705)/(16k))}{(4k-1)k^7 \binom{2k}{k}^8} = \frac{\pi^8}{1350}, \quad (5.41)$$

*and*

$$\sum_{k=1}^{\infty} \frac{\binom{4k}{2k} (P(k) (H_{2k-1}^{(6)} - 2H_{k-1}^{(6)}) - 1323(4k-1)/(64k^3))}{(4k-1)k^7 \binom{2k}{k}^8} = \frac{2\pi^{10}}{31185}. \quad (5.42)$$

**Remark 5.44.** This was announced in [18] first, motivated by the identity

$$\sum_{k=1}^{\infty} \frac{P(k) \binom{4k}{2k}}{(4k-1)k^7 \binom{2k}{k}^8} = \frac{\pi^4}{15}$$

conjectured by D. Chen [4]. Inspired by Conjecture 5.43 on MatheOverflow, Henri Cohen found the following conjectural identity (cf. [18]):

$$\sum_{k=1}^{\infty} \frac{\binom{4k}{2k} (P(k)(2H_{2k-1}^{(8)} + 17H_{k-1}^{(8)}) - 2091(4k-1)/(128k^5))}{(4k-1)k^7 \binom{2k}{k}^8} = \frac{529\pi^{12}}{38697750}.$$

**Conjecture 5.45** (2025-02-04). *Let  $Q(k) = 92k^3 - 84k^2 + 27k - 3$ . Then*

$$\sum_{k=1}^{\infty} \frac{4096^k (Q(k)(5H_{2k-1} - 2H_{k-1}) - 94k^2 + 57k - 9)}{k^7 \binom{2k}{k}^6 \binom{3k}{k}} = 2976\zeta(5), \quad (5.43)$$

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{4096^k (Q(k)(H_{3k-1} - 3H_{k-1}) - 88k^2/3 + 18k - 3)}{k^7 \binom{2k}{k}^6 \binom{3k}{k}} \\ = 16(2\pi^4 \log 2 - 93\zeta(5)). \end{aligned} \quad (5.44)$$

Also,

$$\sum_{k=1}^{\infty} \frac{4096^k (Q(k)(8H_{2k-1}^{(2)} - 7H_{k-1}^{(2)}) - 44k + 12)}{k^7 \binom{2k}{k}^6 \binom{3k}{k}} = \frac{16}{3}\pi^6, \quad (5.45)$$

$$\sum_{k=1}^{\infty} \frac{4096^k (Q(k)(64H_{2k-1}^{(4)} + 13H_{k-1}^{(4)}) + 32/k)}{k^7 \binom{2k}{k}^6 \binom{3k}{k}} = \frac{248}{45}\pi^8. \quad (5.46)$$

**Remark 5.46.** This was announced in [19] motivated by the identity

$$\sum_{k=1}^{\infty} \frac{Q(k)4096^k}{k^7 \binom{2k}{k}^6 \binom{3k}{k}} = 8\pi^4$$

conjectured by J. Guillera in 2003 and confirmed by K. C. Au [2, Example IV].

**Conjecture 5.47** (2025-03-01). *Let*

$$R(k) = 43680k^4 + 20632k^3 + 4340k^2 + 466k + 21.$$

(i) *We have*

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^8 \binom{4k}{2k}}{2^{32k}} \left( R(k)(17H_{2k}^{(4)} - H_k^{(4)}) + \frac{1796k + 193}{2k + 1} \right) = \frac{8704}{45}, \quad (5.47)$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^8 \binom{4k}{2k}}{2^{32k}} \left( R(k)(127H_{2k}^{(6)} - 2H_k^{(6)}) + \frac{1323(4k + 1)}{(2k + 1)^3} \right) = \frac{126976}{945}\pi^2 \quad (5.48)$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^8 \binom{4k}{2k}}{2^{32k}} \left( R(k)(4354H_{2k}^{(8)} - 17H_k^{(8)}) - \frac{4182(4k + 1)}{(2k + 1)^5} \right) = -\frac{197632}{4725}\pi^4. \quad (5.49)$$

(ii) Let  $p > 3$  be a prime. Then

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^8 \binom{4k}{2k}}{2^{32k}} \left( R(k)(17H_{2k}^{(4)} - H_k^{(4)}) + \frac{1796k + 193}{2k + 1} \right) \equiv \frac{51}{10} p^5 B_{p-5} \pmod{p^6}.$$

When  $p > 5$ , we also have

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^8 \binom{4k}{2k}}{2^{32k}} \left( R(k)(127H_{2k}^{(6)} - 2H_k^{(6)}) + \frac{1323(4k + 1)}{(2k + 1)^3} \right) \equiv \frac{6144}{5} p^3 B_{p-5} \pmod{p^4}$$

and

$$\begin{aligned} & \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^8 \binom{4k}{2k}}{2^{32k}} \left( R(k)(4354H_{2k}^{(8)} - 17H_k^{(8)}) - \frac{4182(4k + 1)}{(2k + 1)^5} \right) \\ & \equiv -\frac{12288}{5} p B_{p-5} \pmod{p^2}. \end{aligned}$$

**Remark 5.48.** This is motivated by the conjectural identities

$$\sum_{k=0}^{\infty} R(k) \frac{\binom{2k}{k}^8 \binom{4k}{2k}}{2^{32k}} = \frac{2048}{\pi^4}$$

and

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^8 \binom{4k}{2k}}{2^{32k}} \left( R(k)(7H_{2k}^{(2)} - 2H_k^{(2)}) + 3624k^2 + 926k + 69 \right) = \frac{2048}{3\pi^2}$$

(cf. Conjecture 70 and Remark 54 of [16]).

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